MATH45061 COURSEWORK: LIQUID CRYSTALS

Please submit your work to reception in the Alan Turing Building by
2:00pm on Friday 29th November 2019.

This coursework constitutes 10% of the final mark for the module and should take approximately 7 hours to complete, including assimilating the appropriate material from the lecture notes.

A liquid crystal can flow, but has preferred directions arising from its molecular structure. Nematic liquid crystals are modelled as suspensions of small rods within a fluid. The unit vector $\boldsymbol{A}(\boldsymbol{R}, t)$ describes the preferred direction of the rods. Moving the rods away from a uniformly aligned state builds up internal energy, which depends on $\boldsymbol{A}$, its gradient and its relative rotation quantified by $\boldsymbol{N} = D\boldsymbol{A}/Dt - \mathsf{W}\,\boldsymbol{A}$. The Eulerian velocity of the surrounding fluid is denoted by $\boldsymbol{V}$, as usual; and $\mathsf{W} = \frac{1}{2}(\mathsf{L} - \mathsf{L}^T)$ is the spin tensor; in Cartesian components $L_{IJ} = V_{I,J}$.

1.) The presence of the rods can induce a body moment per unit mass $\boldsymbol{K}$ and surface moment per unit area $\boldsymbol{M}$, known as the couple-stress vector. These moments do not lead to any net force. The material is also loaded by a body force per unit mass $\boldsymbol{F}$ and surface traction $\boldsymbol{T}$. Explain why the equation for balance of linear momentum will be unchanged by the introduction of couple-stresses and body moments. Show also that the balance of angular momentum in components in a general **Eulerian** coordinate system ξ^i becomes

$$\rho K^i + \epsilon^{ijk} T_{kj} + M^{ij}||_j = 0,$$

where T_{ij} is the Cauchy stress tensor **defined** so that the surface stress $T^i = T^{ij}\nu_j$; M_{ij} is a couple-stress tensor defined such that $M^i = M^{ij}\nu_j$; $\boldsymbol{\nu}$ is the outer unit normal to a surface; and $||_j$ indicates the covariant derivative with respect to the basis $\boldsymbol{G}_j = \partial\boldsymbol{R}/\partial\xi^j$. [6 marks]

2.) A simple(!) constitutive assumption is that

$$T_{IJ} = -P\delta_{IJ} - A_{K,I}A_{K,J} + \widetilde{T}_{IJ} \quad \text{and} \quad M_{IJ} = e_{IPQ}A_P A_{Q,J},$$

where

$$\widetilde{T}_{IJ} = \alpha_1 A_K D_{KP} A_P A_I A_J + \alpha_2 A_J N_I + \alpha_3 A_I N_J$$

Here, α_i are constants, e_{IJK} is the alternating symbol and $\mathsf{D} = \frac{1}{2}(L + L^T)$ is the rate of strain tensor. Explain why the expression for the stress is objective. You may assume that $\boldsymbol{A}$ and N are objective. [5 marks]

3.) Consider a stationary liquid crystal ($\boldsymbol{V} = 0$) between two parallel plates separated by a distance d with normal in the X_3 direction. Assuming that the $\boldsymbol{A}$ has no component in the X_3 direction, but varies only with X_3 and t, show that $A_1 = \cos\theta(X_3, t)$, $A_2 = \sin\theta(X_3, t)$, $A_3 = 0$ and define $\theta(X_3, t)$. Under the action of a magnetic field $\boldsymbol{H}$, the body couple is $\rho K_I = e_{IJK}A_J A_L H_L H_K$. If $\boldsymbol{H} = H\boldsymbol{e}_2$, use the balance of angular momentum to show that

$$\gamma\frac{\partial\theta}{\partial t} = C\frac{\partial^2\theta}{\partial X_3^2} + D\cos\theta\sin\theta, \tag{1}$$

where γ, $C > 0$ and D are constants to be found. [6 marks]

4.) Assuming that $\theta \ll 1$, $\theta(0, t) = \theta(d, t) = 0$; and the initial distribution is given by $\theta(X_3, 0) = \theta_0(X_3)$, show that a solution of equation (1) is given by

$$\theta = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi z}{d}\right) \mathrm{e}^{B_n t/\gamma},$$

where A_n and B_n are constants. You should find an explicit expression for B_n, but need not specify A_n. Hence find the dominant (slowest decaying) timescale for alignment of the liquid crystal. You may assume that the timescale for each exponential term is given by $\tau_n = \gamma/B_n$ and that $\gamma \geq 0$. [3 marks]