## FORMULA SHEET

- For a general (Lagrangian) coordinate system $\xi^i$:

$$\boldsymbol{g}_i = \frac{\partial \boldsymbol{r}}{\partial \xi^i}, \quad \boldsymbol{g}_i \boldsymbol{\cdot} \boldsymbol{g}^j = \delta_i^j, \quad g_{ij} = \boldsymbol{g}_i \boldsymbol{\cdot} \boldsymbol{g}_j, \quad g = \det(g_{ij}).$$

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- For a scalar field $f(\boldsymbol{x})$ and vector field $\boldsymbol{u}(\boldsymbol{x})$

$$\boldsymbol{\nabla} f = \boldsymbol{g}^i \frac{\partial f}{\partial \xi^i}, \quad \operatorname{div} \boldsymbol{u} = \frac{1}{\sqrt{g}} \frac{\partial \left(u^i \sqrt{g}\right)}{\partial \xi^i}, \quad \operatorname{curl} \boldsymbol{u} = \epsilon^{ijk} u_j|_i \boldsymbol{g}_k.$$

- The material derivative in general coordinates is

$$\frac{DU^i}{Dt} = \frac{\partial U^i}{\partial t} + V^j U^i||_j,$$

where $\boldsymbol{V}$ is the velocity of the continuum and

$$U^i||_j = U^{i,j} + \Gamma^i_{jk} U^k,$$

where $\Gamma^i_{jk}$ are the Christoffel symbols for the chosen coordinate system in the deformed configuration.

- The deformation gradient tensor $\mathsf{F} = \boldsymbol{\nabla}_{\boldsymbol{r}} \boldsymbol{R}$ has components Cartesian coordinates given by

$$F_{IJ} = \frac{\partial X_I}{\partial x_j}.$$

The determinant of $\mathsf{F}$ is denoted by $J$.

- The Eulerian velocity gradient tensor, $\mathsf{L}$, has components in Cartesian coordinates given by

$$L_{IJ} = \frac{\partial V_I}{\partial X_J}.$$

- The deformation rate tensor, $\mathsf{D}$ and spin tensor, $\mathsf{W}$ are defined by

$$\mathsf{D} = \frac{1}{2}\left(\mathsf{L} + \mathsf{L}^T\right), \quad \mathsf{W} = \frac{1}{2}\left(\mathsf{L} - \mathsf{L}^T\right).$$

- Cauchy's equation in the usual notation in components in general coordinates $\xi^i$ is

$$T^{ji}||_j + \rho F^i = \rho \ddot{U}^i = \rho \frac{DV^i}{Dt}, \quad \text{where} \quad T^{ji}||_j = T^{ji}_{,j} + \Gamma^j_{jr} T^{ri} + \Gamma^i_{jr} T^{jr}.$$

- The material derivative of the determinant of the deformation gradient tensor is

$$\frac{DJ}{Dt} = J \boldsymbol{\nabla}_{\boldsymbol{R}} \boldsymbol{\cdot} \boldsymbol{V}.$$

- The Reynolds Transport theorem states that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t}\phi\,\mathrm{d}\mathcal{V}_t = \int_{\Omega_t}\left(\frac{D\phi}{Dt} + \phi\boldsymbol{\nabla}_{\boldsymbol{R}}\boldsymbol{\cdot}\boldsymbol{V}\right)\mathrm{d}\mathcal{V}_t,$$

where $\Omega_t$ is a material volume, $\phi$ is a scalar field and $\boldsymbol{V}$ is the velocity of the continuum.

- For a Cartesian line element $\mathrm{d}X_I$ in the deformed configuration

$$\frac{D\mathrm{d}X_I}{Dt} = V_{I,K}\mathrm{d}X_K,$$

where $V_I$ is the $I$-th Cartesian component of the velocity.

- Nanson's relation states that

$$\mathrm{d}A_{\bar{i}} = J\frac{\partial\xi^j}{\partial\chi^{\bar{i}}}\mathrm{d}a_j,$$

where $\xi^j$ are the Lagrangian coordinates, $\chi^{\bar{i}}$ are the Eulerian coordinates, $J$ is the determinant of the deformation gradient tensor, $\mathrm{d}\boldsymbol{A}$ is an area element in the deformed configuration and $\mathrm{d}\boldsymbol{a}$ is an area element in the undeformed configuration.

- The Green–Lagrange strain tensor is defined by

$$\gamma_{ij} = \frac{1}{2}\left(G_{ij} - g_{ij}\right).$$

- The strain invariants are defined by

$$I_1 = g^{ij}G_{ji}, \quad I_2 = \frac{1}{2}\left(I_1^2 - g^{ir}g^{js}G_{ij}G_{rs}\right), \quad I_3 = G/g,$$

where $g = det(g_{ij})$ and $G = det(G_{ij})$

- A hyperelastic material is described by a strain energy function $\mathcal{W}(I_1, I_2, I_3)$ such that

$$T^{ij} = PG^{ij} + Ag^{ij} + BB^{ij},$$

where

$$A = \frac{2}{\sqrt{I_3}}\frac{\partial\mathcal{W}}{\partial I_1}, \quad B = \frac{2}{\sqrt{I_3}}\frac{\partial\mathcal{W}}{\partial I_2}, \quad P = 2\sqrt{I_3}\frac{\partial\mathcal{W}}{\partial I_3},$$

$$\text{and} \quad B^{ij} = \left[I_1 g^{ij} - g^{ir}g^{js}G_{rs}\right].$$

- The physical components of the stress tensor are given by $\sigma_{(ij)} = T^{ij}\sqrt{G_{jj}/G^{ii}}$ (no summation).

- The body stress tensor $T^{ij}$ and second Piola–Kirchhoff stress tensor $s^{ij}$ are related by the expression $JT^{ij} = s^{ij}$.

- The first law of thermodynamics can be written as

$$\rho\frac{D\Phi}{Dt} = \mathsf{T} : \mathsf{D} + \rho B - \boldsymbol{\nabla}_{\boldsymbol{R}}\boldsymbol{\cdot}\boldsymbol{Q} + \mathcal{W}_e,$$

where $\mathcal{W}_e$ is any additional non-thermomechanical rates of work.

- The second law of thermodynamics for continuum mechanics can be written as

$$\rho\dot{\eta} \geq -\boldsymbol{\nabla}_{\boldsymbol{R}}\boldsymbol{\cdot}\left(\frac{\boldsymbol{Q}}{\Theta}\right) + \rho\frac{B}{\Theta}.$$

- The Clausius–Duhem inequality is

$$-\rho\dot{\Psi} - \rho\eta\dot{\Theta} - \frac{1}{\Theta}\boldsymbol{Q}\boldsymbol{\cdot}\boldsymbol{\nabla}_{\boldsymbol{R}}\Theta + \mathsf{T} : \mathsf{D} \geq 0,$$

where $\Psi = \Phi - \eta\Theta$; or (in the Lagrangian viewpoint)

$$-\rho_0\dot{\psi} - \rho_0\eta_0\dot{\theta} - \frac{1}{\theta}\boldsymbol{q}\boldsymbol{\cdot}\boldsymbol{\nabla}_{\boldsymbol{r}}\theta + s^{ij} : \dot{\gamma}_{ij} \geq 0,$$

where $\psi = \Psi$.

- The most general transformation of position and time between observers in Euclidean space is

$$\boldsymbol{R}^*(t^*) = \mathsf{Q}(t)\boldsymbol{R}(t) + \boldsymbol{C}(t), \quad t^* = t - a,$$

where $\mathsf{Q}$ is an orthogonal matrix, $\boldsymbol{C}$ is a translation vector and $a$ is a constant time shift.

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**END OF EXAMINATION PAPER**