Almost every student attempted all the questions on the exam and many did very well. The question that caused the most problems was question 3, which was an extension of example 7.4 in the lecture notes. The later parts of question 5 were also not answered terribly well, perhaps because people were running out of time.

1 This question tested knowledge of strain and rotation tensors, principal strains and principal axes of strain as well as the distinction between deformation and rigid-body motion. Nearly everybody correctly computed the strain and rotation tensors and everybody that answered the question knew that the deformation was homogeneous. A number of students forgot the factor of 1/2 in the strain and rotation tensors, or wrote the factor down outside the matrix representation and then forgot to use it in (iii).

The only error in (ii) was that a number of student thought that rigid-body motion only occurs when the rotation tensor is zero. In fact, rigid-body motion occurs when there is **no deformation**, so that every entry of the strain tensor e_{ij} must be zero.

In part (iii), the majority of students correctly computed the principal strains, which were $\lambda = 0, \gamma/2 - 2$ and $2 - \gamma/2$. Although the strain tensor is a 3×3 matrix, only a 2×2 sub-matrix is non-zero, so the calculation is relatively straightforward. A few students forgot that there must be **three** principal strains; usually it was $\lambda = 0$ that was forgotten. The computation of the principal axes of strain proved to be more difficult than computing the principal strains. The most common mistake here was to write down that the principal axis corresponding to $\lambda = 0$ is a vector of all zeros. An eigenvector must be non-zero by definition. The appropriate axis is $(1, 0, 0)^T$.

2 This question tested knowledge of the properties of the strain tensor and how it relates to physical deformations, as well as how to use the constitutive law.

Everybody got part (i) correct. The given tensor cannot be a strain tensor because it is not symmetric. Part (ii) caused more problems, suggesting that students were not as familiar with the physical interpretation of the strain tensor as they should have been. If all material lines in the $x_1 - x_2$ planes are stretched by a factor of $(1 + \epsilon)$ then the deformed position is given by $X_1 = (1 + \epsilon)x_1$, $X_2 = (1 + \epsilon)x_2$, which means that the displacement field is $u_1 = \epsilon x_1$, $u_2 = \epsilon x_2$. This leads to a strain tensor of the form

$$e = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The most common error was to write $(1 + \epsilon)$ instead of ϵ in the strain tensor, and in many cases the off-diagonal terms $e_{12} = e_{21}$ were also set to $(1 + \epsilon)$ or ϵ, which introduces a shear rather than a uniform dilation. Part (iii) was generally attempted correctly and consisted of substituting the strain tensor into the given constitutive law to find the stress. Note that some students seemed to be unaware of the properties of the Kronecker delta, specifically $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$, which is inexcusable. If students had the wrong strain tensor from part (ii), but found the correct corresponding stress full marks were awarded.

3 This question tested whether students were able to formulate stress boundary conditions for a problem in plane strain and use the Airy stress function to find the corresponding stress field. Perhaps because (iii) was not divided into small sections, many students found this very difficult, which I found surprising given the similarity to the example 7.4 in the lecture notes.

(i) In the far field ($r \to \infty$) the stress field is $\tau_{xx} \to T$ and $\tau_{yy} \to T$, which can also be written as $\tau_{rr} \to T$ and $\tau_{\theta\theta} \to T$. This stress field is independent of θ. At the hole boundary we are told no information, so we should assume that is it stress free, i.e. at $r = a$, $\tau_{rr} = \tau_{r\theta} = 0$, because the outer unit normal to the hole is in the $-\boldsymbol{e}_r$ direction. Relatively few students got all of these conditions correct.

(ii) Everybody got this part correct. The Airy stress function satisfies the biharmonic equation.

(iii) This was the part that students found the most difficult, but, as with so many things, is straightforward if approached in the correct way. The boundary conditions do not depend on θ, which means that we do not expect the solution to depend on θ and so

$$\Phi = A_0 + B_0 r^2 + C_0 \ln r + D_0 r^2 \ln r.$$

Now we must use the boundary conditions to find the value of the unknown constants. Many students correctly wrote that the stress must remain finite as $r \to \infty$, which means that $D_0 = 0$. Matching to the conditions that $\tau_{rr} \to T$ and $\tau_{\theta\theta} \to T$ gives $B_0 = \frac{1}{2}T$. The constant A_0 does not affect the stress so we can set it to zero. From applying the boundary conditions at the hole, we obtain $C_0 = -Ta^2$ and so

$$\Phi = \frac{T}{2}r^2 - Ta^2 \ln r.$$

The corresponding stress fields are

$$\tau_{rr} = T\left(1 - \frac{a^2}{r^2}\right), \quad \tau_{\theta\theta} = T\left(1 + \frac{a^2}{r^2}\right) \quad \text{and} \quad \tau_{r\theta} = 0.$$

The maximum value of $\tau_{\theta\theta}$ occurs at the smallest value of r, which is a, so we obtain $\tau_{\theta\theta}|_{r=a} = 2T$.

4 This question tested the ability of students to find solutions of the Navier–Lamé equations in a spherical polar coordinate system where the only variation was in the r direction. The majority of students answered this question well.

(i) Under the given assumptions the general solution is

$$u(r) = A\,r + \frac{B}{r^2},$$

where A and B are constants. A worrying number of students did not cancel the $\sin\theta$ terms in the expression for $\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{u})$ and by some mysterious method ended up with a solution that was not independent of θ. A few included a body force (as in Example 6.2 of the lecture notes), suggesting that they had memorised that solution, but not really understood the methodology.

(ii) Here almost everyone got full marks for stating that at $r = a$, $\tau_{rr} = \tau_{r\theta} = \tau_{r\phi} = 0$, and at $r = b$, $\tau_{rr} = -P$ and $\tau_{r\theta} = \tau_{r\phi} = 0$.

(iii) Most students knew what to do here: use the two boundary conditions on τ_{rr} at $r = a$ and $r = b$ to find the unknown constants A and B in the general solution for the displacement. The only problems here were algebraic. Many students did not, however, complete the question by finding the stress field (all components of the stress tensor). Under this deformation the only non-zero stress components are τ_{rr} and $\tau_{\theta\theta} = \tau_{\phi\phi}$.

(iv) Most people had a go at this part of the question, but if students had not computed all stress components then they did not realise that the maximum stress actually occurs in the component

$$\tau_{\theta\theta} = \tau_{\phi\phi} = \frac{P\left(1 + \frac{1}{2}\left(\frac{a}{r}\right)^3\right)}{\left(\left(\frac{a}{b}\right)^3 - 1\right)}.$$

The maximum occurs when $r = a$ in which case the magnitude of $\tau_{\theta\theta}$ being less than σ corresponds to the constraint

$$\left|\frac{\frac{3}{2}P}{\left(\left(\frac{a}{b}\right)^3 - 1\right)}\right| < \sigma \quad \Rightarrow \quad P < \frac{\sigma\left(1 - \left(\frac{a}{b}\right)^3\right)}{3/2},$$

because $a < b$.

5 This question tested working with a different type of stress function and fundamental knowledge of strains, stresses and the equations of motion. The first two parts of this question were well attempted by most and then things got a bit more patchy, perhaps owing to it being near the end of the examination.

(i) If you use the displacement field to find the strain tensor and use that in the constitutive law to find the stress, you will find that the only non-zero terms are $\tau_{13} = \tau_{31}$ and $\tau_{23} = \tau_{32}$ as expected. For the second part of the question it is important to recognise that there are no body forces or acceleration present in which case we must satisfy the equation $\tau_{ij,j} = 0$. Substituting the given forms of τ_{13} and τ_{23} into this expression the only non-trivial terms are

$$\tau_{32,2} + \tau_{33,3} = -\mu\Omega\frac{\partial^2\Upsilon}{\partial x_1 \partial x_2} + \mu\Omega\frac{\partial^2\Upsilon}{\partial x_1 \partial x_2} = 0,$$

so the equations of motion are satisfied. Many people didn't consider the case $i = 3$ in the equations of motion.

(ii) Nearly everybody got this part correct. If you compare the components of the stress tensor involving Υ and those involving ψ derived from the displacement field you can find

$$\frac{\partial \Upsilon}{\partial x_2} = \frac{\partial \psi}{\partial x_1} - x_2 \quad \text{and} \quad \frac{\partial \Upsilon}{\partial x_1} = -\frac{\partial \psi}{\partial x_2} - x_1,$$

which lead directly to the equation $\nabla^2 \Upsilon = -2$.

(iii) There were all sorts of explanations here many of which did not use the important fact that $\tau_{ij} n_j = t_i = 0$. In other words, we do need to worry about the normal. Simply stating that $\Upsilon = 0$ on the boundary gives zero stress is not enough because a function can take the value of zero and have a non-zero gradient. The intention was to use the same ideas as in 7.2 to justify that Υ must be a constant on the boundary. The point is that for a circular boundary and the chosen form of the stress tensor the only non-trivial equation is

$$\tau_{3j} n_j = \tau_{31} n_1 + \tau_{32} n_2 = \mu\Omega \left[\cos\theta \frac{\partial \Upsilon}{\partial x_2} - \sin\theta \frac{\partial \Upsilon}{\partial x_1}\right] = 0 \quad \Rightarrow \quad \frac{\partial \Upsilon}{\partial \theta} = 0,$$

because

$$\frac{\partial}{\partial \theta} = -r\sin\theta \frac{\partial}{\partial x_1} + r\cos\theta \frac{\partial}{\partial x_2},$$

by the chain rule. Having established that Υ is constant on the boundary we can set $\Upsilon = 0$. Nobody used exactly this argument, but a number of students argued that Υ had to be constant on the boundary.

(iv) For this part, we solve the governing equation $\nabla^2 \Upsilon = -2$ under the assumption that Υ is a function only of r, in which case

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r \frac{\partial \Upsilon}{\partial r}\right] = -2 \quad \Rightarrow \quad \Upsilon = -\frac{r^2}{2} + C \ln r + D,$$

where C and D are constants. Note that a number of students used the wrong expression for the Laplacian because they forgot to work in polar coordinates, and used the Cartesian expression. We expect the stress to be non-singular at $r = 0$ so $C = 0$ and at $r = R$ we know that $\Upsilon = 0$. Thus the final expression is $\Upsilon = (R^2 - r^2)/2$.

(v) Almost everybody forgot that when integrating over an area in cylindrical polars you need to include an additional factor of r, *i.e.*

$$\int_0^{2\pi} \int_0^R 2\mu \Upsilon \, r \, \mathrm{d}r \, \mathrm{d}\theta.$$

The integral itself is straightforward and yields a torsional rigidity of $\mu\pi R^4/2$.