Almost every student attempted all the questions on the exam and the majority did well. The questions that caused the most problems were question 3 on constitutive laws and the algebra towards the end of question 5.

1 This question, as usual, tested knowledge of strain and rotation tensors, principal strains and axes of strain and sketching. Almost everybody correctly computed the strain and rotation tensors, but fewer than 50% knew that the deformation was homogeneous — the displacement gradients were all constant.

Computation of the principal strains and axes of strains caused some problems, perhaps because it was a $3 \times 3$ matrix. This leads to a cubic characteristic equation and certain algebraic difficulties. However, the third row and column have only one non-zero entry on the diagonal, which means that it is an eigenvalue with corresponding eigenvector $(0, 0, 1)^T$. You can therefore extract a factor $(\lambda - 2\epsilon)$ from the characteristic equation, which leaves a simple quadratic yielding the other two eigenvalues $\lambda = \pm\epsilon/2$. Note that eigenvectors (principal axes of strain) do not need to be normalised.

The final question was least well answered. The deformation consists of a rigid-body rotation about the $x_3$-axis (non-zero rotation tensor); and three mutually orthogonal stretches $2\epsilon$ along the $x_3$ axis, $\epsilon/2$ along the line $x_1 = x_3$ and $-\epsilon/2$ along $x_1 = -x_2$. You were asked to sketch the deformed configuration in the plane $x_3 = 0$ and it is an approximately elliptical shape with semi-major axis aligned with $x_1 = x_2$.

2 This question tested knowledge of strain compatibility and how to recover a displacement field. Most people correctly answered part (i). The most elegant way is to use the strain compatibility equation with $i = j = 1$, $k = 2$ and show that the equation is not satisfied.

For part (ii) finding the most general form of $f(x_1, x_2)$ seemed to cause a few problems. You can either work directly from the strain compatibility equation or use the integrated expressions for $e_{11}$ and $e_{22}$ in the definition of $e_{12} = f(x_1, x_2)$. Either way, you should find that the most general form is

$$f(x_1, x_2) = \frac{1}{2} x_1 \cos x_2 + g(x_1) + h(x_2),$$

where $g$ and $h$ are arbitrary functions. Many people wrote

$$f(x_1, x_2) = \frac{1}{2} x_1 \cos x_2 + \frac{1}{2} x_2 + g(x_1) + h(x_2),$$

which is, of course, equivalent and was given full marks. I note that some people used the symbol $f$ for a function of integration which caused confusion.

When $f(x_1, x_2) = \frac{1}{2} x_1 \cos x_2$, the integration proceeds as usual, but the arbitrary functions cannot be zero, which threw some people. One possible displacement field is

$$u_1 = x_1 \sin x_2 - \frac{1}{2} x_2^2, \qquad u_2 = x_1 x_2,$$

where the second term in $u_1$ is required to ensure that the value of $e_{12}$ is correct. An arbitrary rigid-body motion can be added to the displacement field above.

3 This question tested whether students had understood the meaning of a constitutive law and the assumptions made in deriving a linear constitutive law. It also tested manipulations with index notation. Overall, this was rather poorly answered suggesting that people hadn't paid attention to this aspect of the course. Parts (ii) and (iii) revealed weaknesses in using index notation in a number of students.

(i) The answer was that a constitutive law relates stress to strain within an elastic body (or some suitable variation on that). The assumptions required to obtain the given linear constitutive law are that: stress depends only on instantaneous value of strain; and that the strain is small (or that stress is linearly related to strain). Most people correctly defined the pre-stress $\tau_{ij}^0$, stress $\tau_{ij}$ and strain $e_{ij}$ tensors. It is not necessary for a material to be isotropic nor homogeneous for it to have linear constitutive behaviour.

(ii) The most common mistakes here were to assume that $E_{ijkl}$ was a constant and then not to differentiate it, or to forget that the product rule must apply

$$\frac{\partial}{\partial x_j}\left(E_{ijkl}e_{kl}\right) = E_{ijkl,j}e_{kl} + E_{ijkl}e_{kl,j}.$$

A surprisingly large number of people wrote that

$$\frac{\partial}{\partial x_j}\left(E_{ijkl}e_{kl}\right) = E_{ijkl,j}e_{kl,j},$$

which is incorrect and is not even valid index notation because it contains three $j$'s.

(iii) This was generally well answered, but a few people still struggled to move between index notation and dyadic ($\nabla$) form.

4 This question tested whether students could solve a physical problem using the Navier–Lamé equations. We had been through a similar example towards the end of the lecture course and perhaps because of that, perhaps not, it was well answered by the majority.

(i) This was essentially the same as part of a question from the 2013 paper. A common mistake was not realising that the cross-terms vanish from the equations

$$\frac{\partial}{\partial r}\left(\frac{\partial u_z}{\partial z}\right) = 0 \quad \text{and} \quad \frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial(ru_r)}{\partial r}\right) = 0.$$

Another very common mistake was to forget to divide the body force term by $\lambda + 2\mu$.

(ii) Here surprisingly few people got full marks. Almost everyone correctly deduced that at $r = 1$, $\tau_{rr} = -P$, but most forgot to write that in addition $\tau_{r\theta} = \tau_{rz} = 0$.

(iii) This was mostly well answered. The correct answer is that $u_z \equiv 0$ throughout the cylinder. Some people tried to use stress boundary conditions, but the appropriate conditions are that $u_z = 0$ at both ends of the cylinder.

(iv) Everybody knew what to do here: use the stress boundary condition and the fact that the displacement is bounded at the centre of the cylinder to find the unknown constants in the general solution. The only problems were algebraic

(v) Here the pressure required is such that $u_r = 0$ at $r = 1$. Once this condition is used, you can find that $P = \rho\omega^2/4$, which means that the required pressure has to increase with the square of the rotation rate. With the chosen value of $P$ after a little algebra one finds that

$$u_r = \frac{\rho\omega^2(r - r^3)}{8(\lambda + 2\mu)}.$$

Most that attempted this question got the correct functional forms (or if they didn't were awarded marks for correct sketches and sensible arguments.)

5 This was probably the hardest question on the paper and it may also have suffered from being at the end when people were starting to suffer from exam fatigue. The question tested use of an Airy stress function to solve a problem in elasticity. Most people got full (or near full) marks for (i) and (ii), but then started having problems.

(i) Everybody got this correct. The answer is, of course, that $\nabla^4\Phi = 0$.

(ii) The important thing to notice is that the normals to the crack surfaces are in the $\pm\boldsymbol{e}_\theta$ directions. Using $t_i = \tau_{ij}n_j$ in polar coordinates and the condition that the crack surfaces are traction free yields the required result.

(iii) This caused all sorts of algebraic difficulties. Most people knew the method: use the ansatz $\Phi = r^{n+1}g(\theta)$ in the biharmonic equation (in plane polar coordinates), but then got tangled up in the algebra. The easiest way through is to apply the Laplacian twice:

$$\nabla^2\Phi = r^{n-1}\left[(n+1)^2 g + g''\right],$$

and then

$$\nabla^2\nabla^2\Phi = r^{n-3}\left[(n-1)^2(n+1)^2 g + (n-1)^2 g^{''} + (n+1)^2 g^{''} + g^{''''}\right] = 0.$$

Thus, we have a 4th-order constant coefficient ODE for $g$, which when you use the ansatz $g = \mathrm{e}^{\lambda\theta}$ yields the equation

$$\lambda^4 + \left[(n-1)^2 + (n+1)^2\right]\lambda^2 + (n-1)^2(n+1)^2 = \left[\lambda^2 + (n-1)^2\right]\left[\lambda^2 + (n+1)^2\right] = 0,$$

and so

$$\lambda = \pm\mathrm{i}(n-1), \pm\mathrm{i}(n+1),$$

which leads to the desired solution.

(iv) This was genuinely hard and I think threw nearly everybody. However, the initial parts should have been straightforward. On the crack surfaces, $\theta = \pm\pi$,

$$\tau_{\theta\theta} = \frac{\partial^2\Phi}{\partial r^2} = (n+1)nr^{n-1}g(\theta) = 0 \quad \text{and} \quad \tau_{r\theta} = \frac{1}{r^2}\frac{\partial\Phi}{\partial\theta} - \frac{1}{r}\frac{\partial^2\Phi}{\partial r\partial\theta} = -nr^{n-1}g'(\theta) = 0.$$

This is only possible for all $r$ iff $g(\pi) = g(-\pi) = g'(\pi) = g'(-\pi)$. Note that many people argued that $n = 0$, but this gives a trivial solution ($\tau_{\theta\theta} = \tau_{r\theta} = \tau_{rr} = 0$). I expected most people to get this far. The problem is then to determine which values of $n$ allow such a solution without forcing $A_1 = A_{-1} = B_1 = B_{-1} = 0$. The most general approach is to set the determinant of the 4x4 matrix representing the simultaneous set of linear equations to zero. Alternatively, the trick is to notice that the equations all have terms of the form $\cos[(n\pm1)\theta]$ and $\sin[(n\pm1)\theta]$ and we can ensure that some (but not all) of these terms vanish when $n = \pm\frac{1}{2}, \pm1, \pm\frac{3}{2}, \cdots$.

(v) For the slowest decay of the stress $n = \frac{1}{2}$ and then

$$\tau_{\theta\theta} = \frac{3}{4}r^{-\frac{1}{2}}\left\{A_1\cos\frac{3\theta}{2} + 3A_1\cos\frac{\theta}{2} + B_1\sin\frac{3\theta}{2} + B_1\sin\frac{\theta}{2}\right\}.$$