Most students attempted every question on the exam and the majority did well. The overall average was 61% and 65% of students taking the course achieved a first class or upper second mark. The questions that caused the most problems were question 2 on first principles and the algebra/calculus in the middle of question 4.

1 This question tested knowledge of strain and rotation tensors, as well as principal strains and axes of strain. For a change there was no sketching and (perhaps as a consequence?) people did very well. The vast majority got nearly all the marks available, but the most common error was not writing down the relationship between the **displacement fields** corresponding two equal strain tensors. The answer was not simply that the strains were equal, but that the displacement fields must be related by a rigid-body transformation.

2 This question tested knowledge of what strain and stress are in physical terms and was essentially the setup described in the reading week material (section 4.1.1).

(i) The described deformation will not induce any shear and so the strains are simply the changes in length over original length in the particular coordinate directions, giving

$$e_{ij} = \begin{pmatrix} \frac{D-d}{d} & 0 & 0 \\ 0 & \frac{D-d}{d} & 0 \\ 0 & 0 & \frac{L-l}{l} \end{pmatrix}.$$

(ii) The stress only acts in the axial direction and we are told that it is T so we have

$$\sigma_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & T \end{pmatrix}.$$

(iii) The idea is now to use the constitutive law corresponding to the diagonal entries to find expressions for E and ν. The case when $i = j = 3$ yields

$$E = \frac{Tl}{L-l},$$

and then the equation when $i = j = 1$ gives

$$\nu = -\frac{l(D-d)}{d(L-l)}.$$

(iv) Many people had a go at this part and because the Poisson ratio is negative, the cylinder expands in diameter when it is extended. You can see this explicitly from the formula above. When $L > l$ and $nu < 0$, then $D > d$.

3 This question tested understanding of the definition of an Airy stress function, how to use it to recover the boundary traction and, hence, total force on a boundary.

(i) Everybody got this correct; of course, $\nabla^4\Phi = 0$.

(ii) The most common error here was messing up the differentiation of Φ_2. In fact, both Φ_2 and Φ_3 are valid Airy stress functions, but Φ_1 is not.

(iii) The idea here is to use the relationship $t_i = \tau_{ij}n_j$ and the fact that $n = (-1, 0)$ at $x = 0$ to deduce that $t_1 = -\tau_{11}$ and $t_2 = -\tau_{12}$. Then we must simply use the definition of the Airy stress function to compute τ_{11} and τ_{12} at $x = 0$ and integrate from $y = -1$ to 1 to find the resultant force. You can show quite quickly that Φ_2 doesn't have the correct form and so Φ_3 is the required Airy stress function.

4 This question tested whether students understood how to derive a new type of stress function appropriate for when the body force is not zero. Unfortunately the algebra in the middle defeated the majority of people.

(i) This was correctly answered by all that attempted it: $f = \rho g y + C$, where the constant C can be set to zero.

(ii) This was the difficult part, but follows the same ideas as the derivation of equation for the Airy Stress function. If we let the displacement in the x direction be u and that in the y direction be v, then we have from the constitutive law:

$$Eu_x = (1+\nu)\,[f + \Psi_{yy}] - \nu\,[2f + \Psi_{xx} + \Psi_{yy}] = (1-\nu)f + \Psi_{yy} - \nu\Psi_{xx}, \tag{1}$$

$$Ev_y = (1+\nu)\,[f + \Psi_{xx}] - \nu\,[2f + \Psi_{xx} + \Psi_{yy}] = (1-\nu)f + \Psi_{xx} - \nu\Psi_{yy}, \tag{2}$$

$$\frac{E}{2}\,(u_y + v_x) = -(1+\nu)\Psi_{xy}. \tag{3}$$

Differentiate equation (1) twice with respect to y and equation (2) twice with respect to x to obtain

$$Eu_{xyy} = \Psi_{yyyy} - \nu\Psi_{xxyy} \quad \text{and} \quad Ev_{yxx} = \Psi_{xxxx} - \nu\Psi_{yyxx},$$

because $f(y) = \rho g y$. If we take the derivative with respect to x and then with respect to y of equation (3) we obtain

$$E(u_{xyy} + v_{yxx}) = -2(1+\nu)\Psi_{xxyy} = \Psi_{yyyy} + \Psi_{xxxx} - 2\nu\Psi_{xxyy},$$

by using the equation above and then, we obtain the desired result:

$$\Psi_{xxxx} + \Psi_{yyyy} + 2\Psi_{xxyy} = \nabla^4\Psi = 0.$$

(iii) Here the boundary conditions are that the traction at the top surface $y = 0$ must be zero. The outer unit normal is $(0,1)$, so that means we have $\tau_{xy} = \tau_{yy} = 0$ as our boundary conditions. As $y \to -\infty$, we must have decay of all non-hydrostatic stresses, i.e. those associated with Ψ.

(iv) If $\Psi(x,y)$ is a function only of y, then integrating four times gives a cubic in y, which many people correctly deduced. Ignoring the physically irrelevant linear and constant terms, we should expect to decay to hydrostatic stress meaning that $\Psi = 0$, so the stress field is simply $\tau_{ij} = \rho g y \delta_{ij}$ — a hydrostatic solution.

5 Most students scored at least 50% on this question. It is related to question 4 from the 2014 paper, but in a different geometry.

(i) Apart from a few algebraic slips, everybody correctly deduced that the displacement has the form $u_r(r) = Ar + B/r^2$, with different constants in the different materials.

(ii) Note that because this is a shell the elastic body does not include $r = 0$ nor does it include $r \to \infty$, so we cannot immediately remove any of the constants. The required boundary conditions are that: $u_i = u_o$ at $r = r_i$ (continuity of displacement), τ_{rr} must be continuous at $r = r_i$, which leads to two algebraic equations.

(iii) The boundary conditions here are that $\tau_{rr} = 0$ at $r = 1$ and $\tau_{rr} = -P$ at $r = r_0$, which were correctly written down by the majority. This leads to the remaining two equations. Combining these equations with those from (ii) gives the required four simultaneous equations.

(iv) At $r = r_i$, the stress is continuous, but because

$$\tau_{rr} = \lambda \frac{1}{r^2}\frac{\partial(r^2 u_r)}{\partial r} + 2\mu\frac{\partial u_r}{\partial r}$$

and the values of λ and μ are different in the different materials, this means that the radial derivative of $\frac{\partial u_r}{\partial r}$ is not continuous. The physical interpretation is that different physical properties leads to different strains (displacement gradients) for the same imposed stresses.