Two hours

THE UNIVERSITY OF MANCHESTER

ELASTICITY

26 January 2016

09:45 – 11:45

Answer **ALL FIVE** questions.

Electronic calculators may be used, provided that they cannot store text.

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1. A three-dimensional elastic body is a unit cube when undeformed. The cube has been subjected to one of three possible displacement fields, A, B or C, where

A is	$u_1 = \epsilon\,(2x_1 + 5x_2 + 2x_3 + 5)\,,$	$u_2 = \epsilon\,(3x_2 - 5x_1)\,,$	$u_3 = -\epsilon\,(x_3 + 2x_1)\,,$
B is	$u_1 = 2\,(\epsilon\,x_1 + 1)\,,$	$u_2 = 3\,(\epsilon\,x_2 + 3)\,,$	$u_3 = -\epsilon\,x_3,$
C is	$u_1 = 2\epsilon\,x_1 + 1,$	$u_2 = 3\epsilon\,x_2 + 6,$	$u_3 = -\epsilon\,(x_1^2 + x_2^2)\,,$

where ϵ is a small positive constant, and u_i is the displacement component in direction of the Cartesian coordinate x_i.

(i) After deformation the diagonal entries of the strain tensor are

$$e_{11} = 2\epsilon, \quad e_{22} = 3\epsilon \quad , e_{33} = -\epsilon.$$

Determine which of the displacement fields A, B or C could have been applied to the body. Note that you should consider **all** three cases.

(ii) By determining all entries of the strain tensor find the relationship, if any, between the displacement fields found in part (i).

(iii) If every entry of the rotation tensor is zero, state the single displacement field A, B or C that has been applied to the body. Determine the principal strains associated with this displacement field and hence describe the deformation physically.

[16 marks]

2. A thin cylinder of linearly elastic material with undeformed length l and diameter d is subject to an applied tension, with corresponding stress T, along its length. After deformation the cylinder has length L and diameter D and the cross-sections are not twisted relative to one another. A Cartesian coordinate system is chosen so that the x_3 coordinate is aligned along the axis of the cylinder and the x_1 and x_2 directions are within its cross-section.

(i) Find the strain tensor corresponding to the deformation by working from first principles.

(ii) Find the stress tensor corresponding to the deformation by working from first principles.

(iii) Hence, find expressions for the Young's modulus, E, and Poisson ratio, ν, of the material in terms of T, L, l, D and d.

(iv) If $L > l$ and the Poisson ratio is negative, explain what happens to the cylinder's diameter during the deformation.

You may use without proof the constitutive relationship in its standard and inverse forms

$$\tau_{ij} = \lambda\delta_{ij}e_{kk} + 2\mu e_{ij},$$

$$Ee_{ij} = (1+\nu)\tau_{ij} - \nu\delta_{ij}\tau_{kk}.$$

[12 marks]

3.

(i) State the equation that must be satisfied by an Airy stress function, $\Phi(x, y)$, where (x, y) are Cartesian coordinates.

(ii) Which of the functions below are valid Airy stress functions?

$$\begin{aligned}\Phi_1(x,y) &= Tx^5 + Ny^5,\\ \Phi_2(x,y) &= \mathrm{e}^x(T\cos y + 3N\sin y),\\ \Phi_3(x,y) &= \tfrac{3T}{4}\left(xy - \tfrac{xy^3}{3}\right) - \tfrac{N}{4}y^2 + K,\end{aligned}$$

where T, N and K are constants

(iii) An elastic body occupies the domain $x \in [0, 1]$ and $y \in [-1, 1]$. The resultant force on the face at $x = 0$ is given by

$$\boldsymbol{F} = N\boldsymbol{e}_x + T\boldsymbol{e}_y,$$

where $\boldsymbol{e}_x$ is a unit vector in the x-direction; and $\boldsymbol{e}_y$ is a unit vector in the y-direction.

Determine the resultant force on the face $x = 0$ for each of the valid Airy stress functions given above. Hence determine the single stress function that is consistent with the boundary condition on $x = 0$.

[12 marks]

4. A semi-infinite elastic medium occupies the region $y < 0$ and is in a state of plane strain. The system is loaded by a body force $\boldsymbol{F} = -\rho g \boldsymbol{e}_y$, where ρ is the constant density of the elastic medium and g is the acceleration due to gravity.

(i) Show that a modified Airy stress function Ψ such that

$$\tau_{xx} = f(y) + \frac{\partial^2 \Psi}{\partial y^2}, \quad \tau_{xy} = -\frac{\partial^2 \Psi}{\partial x \partial y}, \quad \tau_{yy} = f(y) + \frac{\partial^2 \Psi}{\partial x^2},$$

satisfies the equation of equilibrium

$$\tau_{ij,j} + F_i = 0,$$

where $f(y)$ is a function to be found.

(ii) The material is linearly elastic with constitutive law

$$E e_{ij} = (1+\nu)\tau_{ij} - \nu \delta_{ij} \tau_{kk}.$$

By differentiating the components of the constitutive law, written in terms of displacement gradients and the modified Airy stress function, show that

$$\nabla^4 \Psi = 0.$$

(iii) State the boundary conditions that apply at $y = 0$ and as $y \to -\infty$.

(iv) Hence determine Ψ, assuming that it depends only on y and describe the resulting stress field.

[20 marks]

5. A spherical shell of linearly elastic material is completely surrounded by another concentric spherical shell composed of a different linearly elastic material. The inner shell has undeformed inner radius 1 and undeformed outer radius $r_i > 1$ with Lamé constants λ_i and μ_i. The outer shell has undeformed inner radius r_i and undeformed outer radius $r_o > r_i$ with Lamé constants λ_o and μ_o.

The governing equations in both shells are the Navier–Lamé equations

$$(\lambda + 2\mu)\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) - \mu\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{u} + \boldsymbol{F} = \mathbf{0},$$

where the Lamé coefficients, λ and μ take the appropriate values within each shell.

(i) Assume that the displacement can be written in the form

$$\boldsymbol{u} = u_r(r)\,\hat{\mathbf{r}},$$

where

$$u_r(r) = \begin{cases} u_i(r) & 1 \le r \le r_i, \\ u_o(r) & r_i \le r \le r_o, \end{cases}$$

and (r, θ, ϕ) are spherical polar coordinates with origin at the centre of the spheres. Under this assumption, solve the appropriate Navier–Lamé equations to find the general solution for the displacement field within each shell, assuming no body forces are acting.

(ii) Assuming that the stress τ_{rr} and radial displacement u_r must be continuous at the boundary between the two shells, write down the boundary conditions at $r = r_i$ in terms of u_i and u_o.

(iii) Given that the internal pressure is zero and the entire structure is loaded externally by a pressure P, find four simultaneous equations that relate the unknown constants in the general solution found in (i).

(iv) Explain what happens to the radial derivative of u_r at $r = r_i$ and give a physical interpretation.

[20 marks]

You may use the results that in spherical polar coordinates (r, θ, ϕ)

$$\operatorname{grad} f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\,\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\,\hat{\boldsymbol{\phi}},$$

$$\operatorname{div}\mathbf{u} = \frac{1}{r^2\sin\theta}\left\{\frac{\partial}{\partial r}(r^2\sin\theta\,u_r) + \frac{\partial}{\partial \theta}(r\sin\theta\,u_\theta) + \frac{\partial}{\partial \phi}(ru_\phi)\right\},$$

$$\operatorname{curl}\mathbf{u} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(u_\phi\sin\theta) - \frac{\partial u_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial u_r}{\partial \phi} - \frac{\partial}{\partial r}(ru_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(ru_\theta) - \frac{\partial u_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}.$$

$$\tau_{rr} = \lambda\operatorname{div}\mathbf{u} + 2\mu\frac{\partial u_r}{\partial r}.$$

END OF EXAMINATION PAPER