Two hours

THE UNIVERSITY OF MANCHESTER

ELASTICITY

20 January 2015

14:00 – 16:00

Answer **ALL FIVE** questions.

Electronic calculators may be used, provided that they cannot store text.

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1. An undeformed three-dimensional elastic body is a circular cylinder of radius 1 and length 2, whose axis is aligned with the x_3-axis of a Cartesian coordinate system. The base of the cylinder lies in the plane $x_3 = 0$. The body is subjected to the displacement field (u_1, u_2, u_3) where

$$u_1 = \epsilon\, 2x_1x_3, \qquad u_2 = \epsilon\, 2x_2x_3, \qquad u_3 = -\epsilon\, \left(x_1^2 + x_2^2\right),$$

and ϵ is a small positive constant.

(i) Write down the expression for the deformed position X_i and hence sketch the undeformed and deformed configurations in the plane $x_2 = 0$. Is the deformation axisymmetric?

(ii) Determine the strain tensor e_{ij} and rotation tensor ω_{ij} corresponding to this displacement field.

(iii) Find the principal strains and principal axes of strain and hence describe the deformation.

[16 marks]

2.

(i) Show that

$$\Phi(x, y) = -Fxy^2(3 - 2y),$$

is a valid Airy stress function, where F is a positive constant and x and y are Cartesian coordinates.

(ii) If Φ describes the stress field within an elastic body occupying the domain $y \in [0, 1]$ and $x \in [0, 1]$, find the corresponding tractions on boundaries of the domain.

(iii) Sketch the loads on each boundary and give a physical interpretation of the loading.

[12 marks]

3. A three-dimensional elastic body in a state of plane stress has

$$\tau_{11} = ax_1, \quad \tau_{12} = \tau_{21} = b, \quad \tau_{22} = c\sin x_2,$$

where a, b and c are constants and all other components of the stress tensor, τ_{ij}, are zero.

(i) Assuming that the body is in equilibrium, use Cauchy's equation

$$\tau_{ij,j} + F_i = 0,$$

to find components of the body force per unit volume, $\boldsymbol{F}$.

(ii) The constitutive law for a homogeneous, linearly elastic material is

$$\tau_{ij} = \lambda\delta_{ij}e_{kk} + 2\mu e_{ij},$$

where λ and μ are the Lamé coefficients. Use the constitutive law to find all components of the strain tensor e_{ij} corresponding to the given stress field.

(iii) Explain why the body is not in a state of plane strain.

[12 marks]

4. A uniform elastic cylinder of elliptical cross-section has its axis aligned with the z-axis of a Cartesian coordinate system (x, y, z). The displacement vector in the Cartesian coordinate system is given by the components (u, v, w) and the cylinder is twisted such that

$$u = -\theta z y, \quad v = \theta z x, \quad w = \theta \psi(x, y),$$

where θ is a given constant and $\psi(x, y)$ is an unknown function. There are no body forces acting.

(i) Find the strain tensor corresponding to the given deformation.

(ii) Find the corresponding stress tensor, assuming that the body is linearly elastic:

$$\tau_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij},$$

where μ and λ are the Lamé coefficients.

(iii) Show that the equations of equilibrium, $\tau_{ij,j} + F_i = 0$, are satisfied by the introduction of a stress function, $\phi(x, y)$, where

$$\tau_{xz} = \frac{\partial \phi}{\partial y}, \quad \tau_{yz} = -\frac{\partial \phi}{\partial x};$$

and use the results from (ii) to show that

$$\nabla^2 \phi = -2\mu\theta. \tag{1}$$

(iv) Explain why a traction-free boundary condition on the side of the cylinder is given by

$$\frac{\mathrm{d}\phi}{\mathrm{d}s} = 0,$$

where s is an arclength coordinate around the perimeter of the cylinder's cross-section.

(v) If the boundary of the elliptical cross-section is given by the equation $(x/a)^2 + (y/b)^2 = 1$, where $a > b > 0$, show that

$$\phi = C\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right),$$

satisfies the traction-free boundary condition and the equation (1) for a particular value of C, which is to be found.

(vi) Find the location and value of the maximum absolute stress exerted on the cylinder.

[20 marks]

5. An infinite linear elastic body with Lamé coefficients λ and μ contains a spherical hole of undeformed radius a. The hole has an internal pressure given by $p = p_0$.

(i) Explain why the displacement field may be assumed to be of the form

$$\boldsymbol{u} = u_\rho(\rho)\,\hat{\boldsymbol{\rho}},$$

where ρ is the distance from the centre of the hole; and $\hat{\boldsymbol{\rho}}$ is a unit vector directed away from the centre of the hole.

(ii) Solve the Navier–Lamé equations, in the usual notation,

$$(\lambda + 2\mu)\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) - \mu\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{u} + \boldsymbol{F} = \boldsymbol{0},$$

stating any boundary conditions clearly, to find the solution for $u_\rho(\rho)$ in terms of the Lamé constants λ, μ and the values p_0 and a.

(iii) Another infinite body of the same material contains a cylindrical hole of infinite length that is also of radius a and loaded by internal pressure p_0. Determine whether the cylindrical or spherical hole has a greater deformed radius.

(iv) If instead the undeformed cylinder radius is αa, find the value of α such that the deformed cylinder and sphere have the same radius.

[20 marks]

You may use the results that in spherical polar coordinates (ρ, ϑ, ϕ)

$$\text{grad}\, f = \frac{\partial f}{\partial \rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial f}{\partial \vartheta}\,\hat{\boldsymbol{\vartheta}} + \frac{1}{\rho \sin\vartheta}\frac{\partial f}{\partial \phi}\,\hat{\boldsymbol{\phi}},$$

$$\text{div}\,\mathbf{u} = \frac{1}{\rho^2 \sin\vartheta}\left\{\frac{\partial}{\partial \rho}(\rho^2 \sin\vartheta\, u_\rho) + \frac{\partial}{\partial \vartheta}(\rho \sin\vartheta\, u_\vartheta) + \frac{\partial}{\partial \phi}(\rho u_\phi)\right\},$$

$$\text{curl}\,\mathbf{u} = \frac{1}{\rho\sin\vartheta}\left[\frac{\partial}{\partial\vartheta}(u_\phi \sin\vartheta) - \frac{\partial u_\vartheta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{\rho}\left[\frac{1}{\sin\vartheta}\frac{\partial u_\rho}{\partial\phi} - \frac{\partial}{\partial\rho}(\rho u_\phi)\right]\hat{\boldsymbol{\vartheta}} + \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho u_\vartheta) - \frac{\partial u_\rho}{\partial\vartheta}\right]\hat{\boldsymbol{\phi}}.$$

$$\tau_{\rho\rho} = \lambda \text{div}\,\mathbf{u} + 2\mu\frac{\partial u_\rho}{\partial \rho}.$$

In cylindrical polar coordinates (r, θ, z)

$$\text{grad}\, f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\,\hat{\boldsymbol{\theta}} + \frac{\partial f}{\partial z}\,\hat{\mathbf{z}}, \qquad \text{div}\,\mathbf{u} = \frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z},$$

$$\text{curl}\,\mathbf{u} = \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\hat{\boldsymbol{\theta}} + \left(\frac{1}{r}\frac{\partial (r u_\theta)}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right)\hat{\mathbf{z}}.$$

$$\tau_{rr} = \lambda\,\text{div}\,\mathbf{u} + 2\mu\frac{\partial u_r}{\partial r}, \quad \tau_{\theta\theta} = \lambda\,\text{div}\,\mathbf{u} + 2\mu\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right), \quad \tau_{zz} = \lambda\,\text{div}\,\mathbf{u} + 2\mu\frac{\partial u_z}{\partial z},$$

$$\frac{\tau_{r\theta}}{\mu} = \frac{\tau_{\theta r}}{\mu} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}, \quad \frac{\tau_{rz}}{\mu} = \frac{\tau_{zr}}{\mu} = \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}, \quad \frac{\tau_{\theta z}}{\mu} = \frac{\tau_{z\theta}}{\mu} = \frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}.$$

END OF EXAMINATION PAPER