Most students attempted every question on the exam and the majority did reasonably well. There were few algebraic and calculus problems, but there was confusion about formulating boundary conditions (particularly in question 4) and index notation.

1 The second and third parts of this question were answered well, but the first part was not. In general, students find sketching displacement fields (or anything) difficult.

(i) The most common mistake was to assume that zero displacement of the corners $(0,0)$, $(1,0)$ and $(1,1)$ meant that the entire elastic body remained undeformed. The displacement of the corners **is** indeed zero, but there is a sinusoidal variation over all three edges of the triangle. Those that indicated curved boundaries of the correct sort of form were given most of the marks. Many graphs were poor and, in some cases, points were plotted on the wrong axes.

(ii) This was typically fine apart from a few silly algebraic slips and missing factors of a half. Remember that the strain tensor is symmetric (by construction) and the rotation tensor is anti-symmetric.

(iii) The vast majority of people correctly stated that the principal strains are the eigenvalues and correctly calculated the eigenvalues. The matrix is not diagonal, it is of the form

$$\begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix},$$

where $A(x_1, x_2)$, and so the eigenvalues are given by the solution of the equation $\lambda^2 - A^2 = 0 \Rightarrow \lambda = \pm A$. A few people managed to misplace factors and constants when performing the calculation. The eigenvectors did **not** need to be calculated.

For the last part the condition that the strain is zero was $\cos \pi x_1 = \cos \pi x_2$, which implies that (within the elastic body) $x_1 = x_2$. This was answered correctly by nearly everybody. The condition that the rotation is zero is $\cos \pi x_1 = -\cos \pi x_2$, which leads to the condition that $\cos \pi x_1 = \cos(\pi x_2 + \pi + 2n\pi)$, $n \in \mathbb{Z}$ from standard trigonometric identities and so $x_1 = x_2 + 1 + 2n$. Thus, within the elastic body, the condition is met when $x_1 = x_2 - 1$, corresponding to $n = -1$. This final step was missed by most.

2 Most people had the right general idea, but a surprising number assumed that the elastic body was two-dimensional, despite the question stating that it is **three-dimensional**. Recall that in a state of plane strain, the remaining (out-of-plane) strain components are zero.

(i) Here, you need only substitute the given values of the strain into the constitutive law. Some students derived the results $\tau_{kk} = (3\lambda + 2\mu)e_{kk}$, but this is not required (and didn't get any marks). The stress tensor is simply

$$\begin{pmatrix} \lambda(a+b) + 2\mu a & 2\mu c & 0 \\ 2\mu c & \lambda(a+b) + 2\mu b & 0 \\ 0 & 0 & \lambda(a+b) \end{pmatrix}.$$

Quite a few people wrote that $e_{kk} = a + b + c$, which suggests misreading of the question ... be sure to read carefully.

(ii) The stress components are all constant, so **any** derivative of **any** component will be zero. Thus, the body can only be in equilibrium if the body force is zero, $\boldsymbol{F} = \mathbf{0}$. A worrying number of people did not realise that the derivative of constants was zero — this may have been lack of confidence with index notation.

(iii) Here most people had the correct idea, but the most common error was neglecting the three-dimensional nature of the problem, which led to oversimplified solutions. The most general form of the displacement field is

$$u_1 = ax_1 + cx_2 + \gamma x_2, \quad u_2 = bx_1 + cx_1 - \gamma x_1,$$

and it is unique up to rigid-body motions represented by the constant γ.

3 This question was designed to test fundamental understanding of the principles from which we derived the governing equations and revealed that the vast majority were not comfortable with this material. This question was consistently the least well answered of the exam.

(i) was bookwork from the lecture notes (or simply reconstructed if you could remember the general idea). It is notable that very few students actually drew pictures to aid the argument. Those that did usually got full marks for this part!

(ii) was an exercise in index notation that was again essentially covered in the lecture notes, but here the explicit form of the quadratic term was required:

$$A_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}.$$

(iii) When the displacement gradient is small and the quadratic term is neglected then the standard result from the lecture notes is recovered.

(iv) This is a straightforward calculation if you have the correct expression for A_{ij}. For the given displacement field $u_{1,1} = \alpha$, $u_{2,2} = \beta$ and $u_{3,3} = \gamma$ leads to

$$A_{ij} - 2e_{ij} = \begin{pmatrix} \alpha^2 & 0 & 0 \\ 0 & \beta^2 & 0 \\ 0 & 0 & \gamma^2 \end{pmatrix}.$$

4 This question proved to be quite testing, but the majority of students obtained at least half marks. The problems were in formulating the boundary conditions. Method marks were given if correct logic was followed after incorrect boundary conditions.

(i) This part was correctly answered by nearly everybody. The assumed form for the displacement with the absence of a body force means that the Navier–Lamé equations become:

$$\frac{\mathrm{d}^2 u_1}{\mathrm{d}x_1^2} = 0.$$

Integrating this simple ODE directly and assuming different properties in the two materials gives the solution.

(ii) This part was not answered terribly well, although a number of people had a good attempt. The assumption in part (i) leads to an non-zero value of e_{11}, and so, from the constitutive law,

$$\tau_{11} = (\lambda + 2\mu)e_{11}, \quad \tau_{22} = \lambda e_{11}, \quad \tau_{33} = \lambda e_{11}.$$

Non-zero stresses τ_{22} and τ_{33} are inconsistent with the stress-free boundary conditions on the sides of the block. Thus, we can only achieve consistency with the boundary conditions if $\lambda = 0$.

(iii) The majority of students correctly wrote down that at $x_1 = L_1 + L_2$, $\tau_{11} = -P$, which means that $2\mu_2 A_2 = -P$. The boundary condition at $x_1 = 0$ proved to be more difficult. The beam is fixed, so the condition is simply that $\boldsymbol{u} = \mathbf{0}$, and so $B_1 = 0$. A common mistake was to assume that fixed means that $\tau_{11} = 0$ — this is almost never the case: if an elastic body is fixed in place, a stress must usually be applied to keep it there. At the interface between the two materials continuity of displacement and stress means that:

$$A_1 L_1 + B_1 = A_2 L_1 + B_2 \quad [u_1(L_1^-) = u_1(L_1^+)],$$

and

$$2\mu_1 A_1 = 2\mu_2 A_2 \quad [\tau_{11}(L_1^-) = \tau_{11}(L_1^+)].$$

Many people correctly deduced the stress condition, but did not also apply the displacement condition.

(iv) Part (iii) leads to four equations for the four unknowns A_1, A_2, B_1, B_2, which then gives the solution for the displacement field. A few hardy students got through to the correct answer. The change in length of the entire block is given by

$$u_1(L_1 + L_2) = \frac{-P}{2}\left(\frac{L_1}{\mu_1} + \frac{L_2}{\mu_2}\right),$$

which is simply the sum of the changes in length of the two individual blocks. The displacements sum "in series" because the pressure load remains constant. A physical interpretation was not attempted by many students.

(v) This was attempted by only a few students. If the entire block is made of one material then the change in length is

$$\frac{-P}{2}\frac{L_1 + L_2}{\mu_3},$$

by setting $\mu_1 = \mu_2 = \mu_3$. Thus in order to find the value of μ_3, we simply rearrange the expression

$$\frac{-P}{2}\frac{L_1 + L_2}{\mu_3} = \frac{-P}{2}\left(\frac{L_1}{\mu_1} + \frac{L_2}{\mu_2}\right).$$

5 The majority of students obtained over half marks for this question typically from parts (i), (iii) and (iv).

(i) This is a simple exercise in index notation and is very similar to the Cartesian version seen in the lecture notes.

(ii) This question split the students. Those that had understood the method of recovering the strain from the constitutive law in Cartesian coordinates from the lecture notes were fine. Those that hadn't struggled.

(iii) Almost everybody correctly integrated the required fourth-order ODE. A few students took the given solution and confirmed that it satisfied the biharmonic equation by repeated differentiation; this approach **does not** show that the solution is the general solution and so a mark was deducted for this method.

(iv) Most students correctly found the expressions for the stress field. For the specific cylinder problem, both A and C must be zero in order for the stress field to remain finite at $r = 0$. Most also wrote down the correct boundary condition that $\tau_{rr}|_{r=a} = -P$. Thus, the stress field is given by $B = -P/2$ so that $\tau_{rr} = \tau_{\theta\theta} = -P$ (uniform pressure). The displacement field follows from direct integration of the expression for e_{rr} found using the expressions from (ii).