Two hours

THE UNIVERSITY OF MANCHESTER

ELASTICITY

13 January 2014

09:45 – 11:45

Answer **ALL FIVE** questions.

Electronic calculators may be used, provided that they cannot store text.

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1. A two-dimensional elastic body has an undeformed configuration given by a triangle defined by the vertices $(x_1, x_2) = (0, 0)$, $(1, 0)$ and $(1, 1)$. The body is subjected to the displacement field (u_1, u_2) where

$$u_1 = \epsilon \sin(\pi x_2), \qquad u_2 = -\epsilon \sin(\pi x_1),$$

and ϵ is a small positive constant.

(i) By considering each side of the triangle, or otherwise, sketch the deformed body.

(ii) Determine the strain tensor e_{ij} and the rotation tensor ω_{ij} corresponding to this displacement field.

(iii) Find the principal strains as functions of position (x_1, x_2) and find all points within the body at which the strain is zero, as well as those where the rotation is zero.

[12 marks]

2. A three-dimensional elastic body is in a state of plane strain such that $e_{11} = a$, $e_{22} = b$ and $e_{12} = c$, where a, b and c are constants.

(i) Using the constitutive law for a homogeneous, linearly elastic material

$$\tau_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij},$$

calculate all components of the stress tensor.

(ii) Find the body force $\boldsymbol{F}$ required to ensure that the body is in equilibrium, *i.e.* that the equations $\tau_{ij,j} + \rho F_i = 0$ are satisfied.

(iii) Find a displacement field corresponding to the given strain. Is this displacement field unique up to rigid-body transformations?

[16 marks]

3. An elastic body undergoes a deformation described by the displacement field $\boldsymbol{u}(\boldsymbol{r})$, so that the position vector to the deformed configuration is $\boldsymbol{R} = \boldsymbol{r} + \boldsymbol{u}(\boldsymbol{r})$, where $\boldsymbol{r}$ is a position vector to the undeformed position.

(i) Starting from first principles, derive the relationship

$$\mathrm{d}R_i = \mathrm{d}r_i + \left.\frac{\partial u_i}{\partial x_j}\right|_{\boldsymbol{r}} \mathrm{d}r_j,$$

between an undeformed, infinitesimal line element $\mathrm{d}r_i$ and its counterpart after deformation, $\mathrm{d}R_i$.

(ii) Show that

$$\mathrm{d}R_i\mathrm{d}R_i = (\delta_{ij} + A_{ij})\mathrm{d}r_i\mathrm{d}r_j,$$

where A_{ij} is a quantity to be found.
Do not assume that the displacement gradient remains small.

(iii) Now consider the limit when the displacement gradient is small and, if quadratic terms are neglected, show that

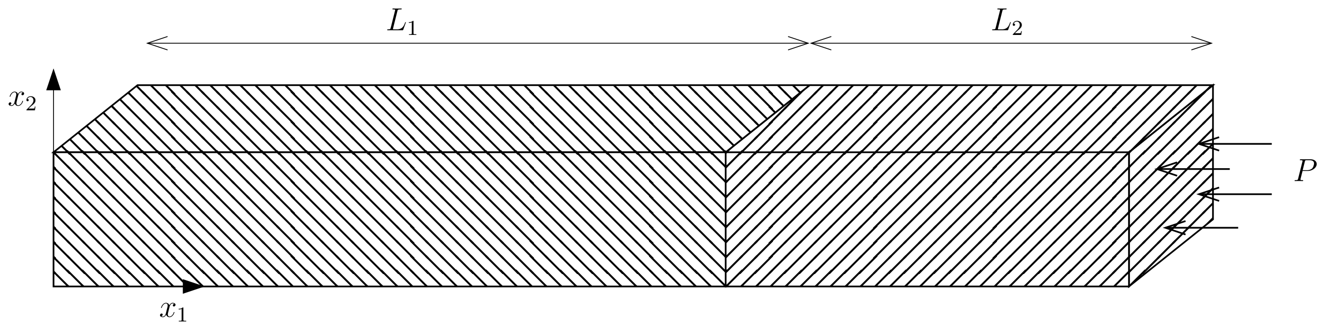
$$\mathrm{d}R_i\mathrm{d}R_i = (\delta_{ij} + 2e_{ij})\mathrm{d}r_i\mathrm{d}r_j,$$

where e_{ij} is the infinitesimal strain tensor.

(iv) Compute the difference between A_{ij} and $2e_{ij}$ for the displacement field $u_1 = \alpha x_1$, $u_2 = \beta x_2$, $u_3 = \gamma x_3$, where α, β and γ are constants.

[12 marks]

4. A linearly elastic material with Lamé coefficients λ_1, μ_1 occupies the region $0 \le x_1 \le L_1$, $0 \le x_2 \le 1$, $0 \le x_3 \le 1$. A different linearly elastic material with Lamé coefficients λ_2, μ_2 occupies the region $L_1 \le x_1 \le L_1 + L_2$, $0 \le x_2 \le 1$, $0 \le x_3 \le 1$. The first material is fixed at $x_1 = 0$ and the second is loaded by applying a pressure P at $x_1 = L_1 + L_2$. All other faces are unloaded, apart from the interface between the two materials at $x_1 = L_1$.



(i) Assuming that the displacement has the form $\boldsymbol{u} = u_1(x_1)\,\boldsymbol{e}_1$, where $\boldsymbol{e}_1$ is a unit vector in the positive x_1 direction, show that

$$u_1 = \begin{cases} A_1 x_1 + B_1, & 0 \le x_1 \le L_1, \\ A_2 x_1 + B_2, & L_1 \le x_1 \le L_1 + L_2, \end{cases}$$

where A_1, A_2, B_1 and B_2 are constants.

You may assume that the displacement in each material satisfies the Navier–Lamé equations

$$(\lambda + 2\mu)\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) - \mu \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{u} + \boldsymbol{F} = \boldsymbol{0},$$

where λ and μ are the appropriate Lamé coefficients.

(ii) Explain why the assumption in (i) will not be valid unless λ_1 and λ_2 are both zero.

Hint: Consider the stress boundary conditions.
You may assume that the stress in each material is given by

$$\tau_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij},$$

where λ and μ are the appropriate Lamé coefficients.

(iii) Assuming that $\lambda_1 = \lambda_2 = 0$ and the displacements and stresses are continuous between the two materials, write down the boundary conditions at $x_1 = 0$, $x_1 = L_1$ and $x_1 = L_1 + L_2$.

(iv) Hence, find the displacement field throughout the domain $0 \le x_1 \le L_1 + L_2$ and the change in total length of the entire block. What is the physical interpretation of the result?

(v) Consider the same load applied a single linearly elastic material with Lamé coefficients $\lambda_3 = 0$ and μ_3. The new material occupies the region $0 \le x_1 \le L_1 + L_2$, $0 \le x_2 \le 1$ and $0 \le x_3 \le 1$. Find an expression for μ_3 that will ensure that the block of single material exhibits the same change in length as the block composed of two materials.

[20 marks]

5. In cylindrical polar coordinates (r, θ, z), the constitutive law for a linearly elastic material is

$$\tau_{ij} = \lambda\delta_{ij}\ (e_{rr} + e_{\theta\theta} + e_{zz}) + 2\mu\, e_{ij},$$

where τ_{ij} is the stress tensor; e_{ij} is the strain tensor; and λ and μ are the Lamé coefficients.

(i) Show that

$$\tau_{rr} + \tau_{\theta\theta} + \tau_{zz} = (3\lambda + 2\mu)\,(e_{rr} + e_{\theta\theta} + e_{zz})\,.$$

(ii) Assuming a state of plane stress, $\tau_{zz} = \tau_{zr} = \tau_{z\theta} = 0$, show that

$$e_{rr} = \frac{1}{E}\left(\tau_{rr} - \nu\tau_{\theta\theta}\right), \quad e_{\theta\theta} = \frac{1}{E}\left(\tau_{\theta\theta} - \nu\tau_{rr}\right), \quad e_{r\theta} = \frac{1+\nu}{E}\tau_{r\theta},$$

where

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}, \quad \nu = \frac{\lambda}{2(\lambda + \mu)}.$$

(iii) Write down the equation satisfied by the Airy stress function. If the Airy stress function $\Phi(r)$ is a function only of r, show that the general solution is

$$\Phi = Ar^2 \ln r + Br^2 + C \ln r + D.$$

and write down the corresponding stresses τ_{rr}, $\tau_{\theta\theta}$, $\tau_{r\theta}$.

(iv) Hence, find the stress field τ_{ij} and radial displacement u_r throughout a linearly elastic infinite cylinder of undeformed radius a that is loaded only by an external pressure p.

You may use the results that in cylindrical polar coordinates:

$$e_{rr} = \frac{\partial u_r}{\partial r}, \quad e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \quad 2e_{r\theta} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}.$$

$$\nabla^4 f(r) = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial f}{\partial r}\right]\right)\right]$$

and for an Airy stress function, $\Phi(r, \theta)$

$$\tau_{rr} = \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r}, \quad \tau_{\theta\theta} = \frac{\partial^2 \Phi}{\partial r^2}, \quad \tau_{r\theta} = \frac{1}{r^2}\frac{\partial \Phi}{\partial \theta} - \frac{1}{r}\frac{\partial^2 \Phi}{\partial r \partial \theta}.$$

[20 marks]

END OF EXAMINATION PAPER