The majority of students attempted every question, but it is clear that this was a difficult exam. General problems were lack of confidence with vector calculus and partial differential equations; not being able to formulate appropriate boundary conditions; and, surprisingly, plotting points on a graph.

A1 On the whole the first two parts of this question were answered correctly by the vast majority of students, but part (iii) was not.

(i) A common mistake here was to assume that $\epsilon = 1$, it should be small. If the assumption $\epsilon = 1$ was clearly stated then marks were still awarded. Many of the graphs were poor, with no labels and confusion about how to plot the points.

(ii) This was generally fine apart from a few silly algebraic slips and missing factors of a half. Remember that the strain tensor is symmetric (by construction) and the rotation tensor is anti-symmetric.

(iii) Most people attempted something with eigenvalues and eigenvectors, but it needed to be clearly stated that the maximum strain is the largest eigenvalue of the strain tensor. A standard calculation gives the eigenvalues to be

$$\lambda = \frac{\epsilon}{2}\left(1 \pm \sqrt{1 + 4(x_1 + x_2)^2}\right),$$

so that maximum extensional strain is given by taking the positive part of the $\pm$. (Note that there was some confusion about how to calculate the eigenvalues of a 2×2 matrix and also about how to use the quadratic formula.) The maximum within the body occurs when $|x_1 + x_2| = 1$, which occurs on two of the boundaries of the shape. The eigenvectors did **not** need to be calculated. An alternative approach was to argue that the maximum strain occurred when the entries of the strain tensor are maximum, which also gives the condition $|x_1 + x_2| = 1$.

A2 Almost everybody knew what to do with this question. The most common mistake was forgetting a factor of two in the definition of the biharmonic operator in Cartesians

$$\nabla^2\Phi = \frac{\partial^4\Phi}{\partial x^4} + 2\frac{\partial^4\Phi}{\partial x^2\partial y^2} + \frac{\partial^4\Phi}{\partial y^4}.$$

A number of people wrote at $g(y) = y^4$, rather than $g(y) = -y^4$ and there were some that wrote $g(y) = -x^2y^2$, which is of the wrong functional form (and gives the trivial Airy stress function $\Phi \equiv 0$). The **important point** here is that $g(y)$ is a function **only** of y so it cannot include any x's.

A3 Part (i) was bookwork from one of the example sheets, but was typically not answered well (probably because people thought it wouldn't come up). The process is simply to substitute the deformed lengths into the formula for the vector triple product and neglect quadratic terms.

Part (ii) relied on the using the conservation of mass to argue that $\rho\, \mathrm{d}V = \rho_0\, \mathrm{d}v$. Using the result in part (i) leads directly to the required result.

Part (iii) was made more difficult by the presence of a typographical error (really sorry). The equation should have been

$$P(\rho) = K(\rho - \rho_0)/\rho.$$

(You may be interested to know that the results are actually equivalent if we neglect quadratic terms, *i.e.* within the framework of linear elasticity.) The procedure was then simply to set $i = j = k$ in the constitutive law and use the result from part (ii). Everybody was given full marks for this part of the question.

A4 (i) Nearly everybody correctly deduced that

$$f(r) = Ar + B/r^2,$$

where A and B are constants. A worrying number of people argued that the displacements had to be bounded as $r \to \infty$ and set $A = 0$. The geometry in this question is a sphere (think of a rubber ball), so the domain does not extend to infinity. Instead, we want the displacement to remain finite as $r \to 0$, which means that $B = 0$. We then apply the given boundary condition to find that

$$\boldsymbol{u} = \frac{br}{a}\hat{\boldsymbol{r}}.$$

(ii) This question was answered correctly by the majority of students. Even if the displacement field was incorrect, marks were given for the correct method.

(iii) Many people did not actually calculate the surface pressure which is given by $p = -\tau_{rr}|_{r=a}$. Otherwise answers were generally good.

B5 This question revealed a fundamental lack of understanding of vector differential equations. The Navier–Lamé equation is a vector equation and can be resolved into different components. These components are independent.

(i) Everybody correctly deduced that for the given form of the displacement field $\boldsymbol{\nabla} \times \boldsymbol{u} = \boldsymbol{0}$. The Navier–Lamé equations are then

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r u_r)\right]\hat{\boldsymbol{r}} + \frac{\partial^2 u_z}{\partial z^2}\hat{\boldsymbol{z}} + I r \hat{\boldsymbol{r}} = \boldsymbol{0},$$

where $I = \rho\omega^2/(\lambda + 2\mu)$ is a constant. The two components (the terms multiplying $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{z}}$) are independent and give two different ODEs. Formally, we can proceed by taking the dot product with $\hat{\boldsymbol{r}}$ to give

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r u_r)\right] = -Ir,$$

and with $\hat{\boldsymbol{z}}$ to give

$$\frac{\partial^2 u_z}{\partial z^2} = 0.$$

Both equations can be solved by direct integration.

(ii) There appears to have been some confusion in understanding the boundary conditions. The **ends** of the cylinder are subjected to a pressure P, but nothing is said about the curved side. Thus, we assume that the curved side is unloaded and the traction is zero. The boundary condition at $r = 1$ is then

$$\tau_{rr} = \tau_{rz} = \tau_{r\theta} = 0, \quad \text{at} \quad r = 1.$$

Given the possible misreading of the question, marks were still awarded for assuming that the pressure acted on all boundaries of the cylinder. A common confusion that was not awarded marks was that the stress has to be equal to the body force. The body force has already appeared in the equations, so it should **not** be present in the boundary conditions.

(iii) Most people correctly computed the stress tensor, but many stopped before finding the expression for u_r, perhaps because the incorrect boundary conditions made this seem more messy that it should have been. (It's still a bit messy though.)

(iv) Many people wrote sensible words about the Poisson effect here, but very few were able to write down the condition that the cylinder did not contract (namely that u_z = const). Using this condition, the solutions to (ii) and (iii) and the boundary condition that $\tau_{zz} = -P$ gives the result.

B6 This question caused a great deal of confusion, and was not answered well in general. Perhaps this is because people were pressed for time. Anyway, most people ran out of steam before getting to the end.

(i) The plausibility of the ansatz follows from the boundary conditions that

$$\tau_{yy} = \frac{\partial^2 \Phi}{\partial x^2} = -\beta \sin x, \quad \text{at} \quad y = h,$$

$$\tau_{yy} = \frac{\partial^2 \Phi}{\partial x^2} = -\alpha, \quad \text{at} \quad y = -h.$$

Integrating each twice with respect to x, suggests that a possible functional form for Φ is

$$\Phi = \sin x f(y) + \frac{1}{2}x^2 g(y).$$

Of course, this form is only correct if it satisfies the biharmonic equation and can satisfy all boundary conditions. Substitution into the biharmonic equation leads to two ODEs that can be solved to give the results for $f(y)$ and $g(y)$. (Some creative attempts were made to get the required forms.)

(ii) The boundary conditions cannot be satisfied exactly because if

$$\tau_{xy} = -\frac{\partial^2 \Phi}{\partial x}\partial y = -\cos x f'(y) - x g'(y),$$

is to be zero on both boundaries then $g'(y) = 0$ at $y = \pm h$. However, $g(y) = ay + b$, so $g'(y) = 0$ implies that $a = 0$ and $g(y) = b$.

This is not the problem, but if we consider the other boundary conditions, it follows that

$$\tau_{yy} = -\sin x f(y) + g(y) = -\sin x f(y) + b.$$

On $y = -h$, we need $\tau_{yy} = -\alpha$, so $f(y) = 0$ and $b = -\alpha$. This is still fine, but at the top boundary

$$\tau_{yy}(y = h) = -\sin x f(h) - \alpha,$$

which can never satisfy the desired boundary condition exactly.

If $f'(y) = 0$ then on both boundaries

$$\int_{-l}^{l} \tau_{xy}\,\mathrm{d}x = \int_{-l}^{l} -ax\,\mathrm{d}x = 0,$$

which means that we can satisfy the boundary condition in an average Saint-Venant sense for non-zero a.

If people made these arguments in words, marks were awarded.

(iii) Applying the boundary conditions, under the assumption of a Saint-Venant solution, gives

$$f(h) = \beta, \quad g(h) = 0, \quad f'(h) = 0,$$

$$f(-h) = 0, \quad g(-h) = -\alpha, \quad f'(-h) = 0,$$

which is the correct number of boundary conditions for the number of unknown constants. The rest is straightforward (but slightly tedious) algebra.

(iv) Most people correctly stated that the Saint-Venant solution is expected to be good far away from the boundaries. The added complication in this case is that we can only get “far away” from the boundaries if the strip thickness is sufficiently large (greater than the typical wavelength of the periodic load).