MATH30271

Two hours

UNIVERSITY OF MANCHESTER

ELASTICITY

16 January 2007

14.00 – 16.00

Answer **ALL** questions in Section A and **ALL** question in Section B.

Electronic calculators may be used, provided that they cannot store text.

SECTION A

Answer all **THREE** questions

A1. The displacement field in a two-dimensional elastic body is given by

$$\begin{aligned} u_1 &= \epsilon\left(3+(x_2-2)^2+\frac{1}{2}(x_1-1)^2\right), \\ u_2 &= \epsilon\left(4+(x_1-1)^2+\frac{1}{2}(x_2-2)^2\right), \end{aligned}$$

where ϵ is a small positive constant.

1. Determine the strain tensor e_{ij} and the rotation tensor ω_{ij}.
2. Show that there exists one point in the body at which the displacement field does not create any deformation. Determine the coordinates of this point.
3. At the point $\mathbf{r} = (2,3)$ find the principal strains and the directions (characterised by unit vectors $\mathbf{n}$) in which they occur.

[12 marks]

A2. The shelf of non-uniform thickness, shown in Fig. 1, is loaded by a linearly increasing pressure $p = p_0 x_1$ on its upper face which is located at $x_2 = 0,\ x_1 \geq 0$. The lower face of the shelf (inclined at an angle α against the upper face) is stress free. The weight of the shelf can be neglected, so that there are no body forces acting and p_0 is given.

You are given that the components of the stress tensor have the form

$$\begin{aligned}\tau_{11} &= a x_1 + b x_2\\ \tau_{22} &= c x_1\\ \tau_{12} &= \tau_{21} = -a x_2\end{aligned}$$

where a, b and c are constants.

1. Verify that the equilibrium equations are satisfied.
2. Determine the constants a, b and c from the stress boundary conditions, applied along the upper horizontal face and the lower inclined face of the shelf. [**Hint:** Along the lower, inclined face we have $x_2 = -x_1 \tan\alpha$.]

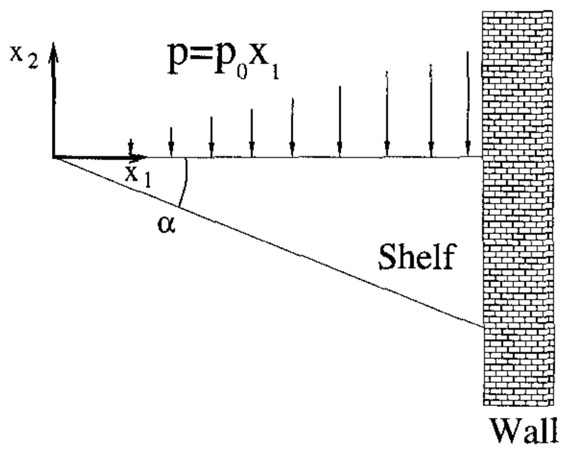

Figure 1: Sketch of a two-dimensional shelf, loaded by a linearly varying pressure $p = p_0\ x_1$.

[16 marks]

A3.

1. Show that
$$\phi(x,y) = \frac{3T}{4a}\left(\frac{x^3 y}{3a^2} - xy\right) + \frac{N}{4a}x^2$$
is a valid Airy stress function.

2. Now assume that ϕ describes the stress in the block occupying the region $-a \leq x \leq a$, $-h \leq y \leq 0$, sketched in Fig. 2. Calculate the stress field.

3. Determine the resultant force acting on the upper face of the block and hence give a physical meaning to the constants T and N.

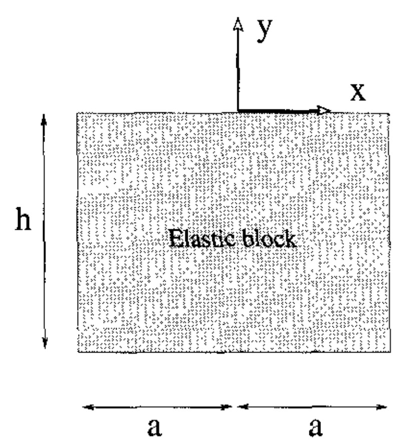

Figure 2: Sketch of a two-dimensional elastic block.

[12 marks]

SECTION B

Answer **ALL** two questions

B4. Fig. 3 shows a sketch of two-dimensional semi-infinite, elastic body, whose boundary ∂B is given by the two straight line segments $\partial B_{\text{flat}\pm} = \{(x,y) \mid x = 0 \text{ and } \pm y > a\}$ and a semi-circular arc $\partial B_{\text{arc}} = \{(x,y) \mid x < 0 \text{ and } x^2 + y^2 = a^2\}$. (Note the orientation of the coordinate axes!) The body is in a state of plane strain and the cavity formed by the circular arc is filled with fluid that exerts a pressure of magnitude $p = P\,x$ onto the body (P is a known constant); the rest of the boundary is traction-free and there are no body forces. We wish to compute the stress distribution in the elastic body, using an Airy stress function.

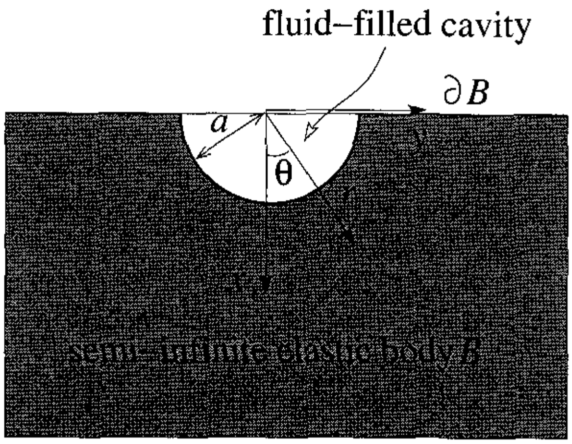

Figure 3: Sketch of a two-dimensional semi-infinite elastic body, loaded by a fluid pressure of magnitude $p = Px$ that acts inside semi-circular cavity of radius a.

1. State the PDE satisfied by the Airy stress function, Φ.
2. Determine the stress boundary conditions for the components of the stress tensor τ_{rr}, $\tau_{r\theta}$ and $\tau_{\theta\theta}$ along the boundary. What are the conditions on the components of the stress tensor as $r \to \infty$?
3. You are given that

$$\Phi = \frac{A}{2} r\theta \sin\theta + \left[Br + Cr^3 + \frac{D}{r} \right] \cos\theta,$$

where A, B, C and D are constants, satisfies the biharmonic equation. Use this ansatz to determine the stress field throughout the body.

Hint: You may use the following results:

$$\tau_{rr} = \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r}, \qquad \tau_{\theta\theta} = \frac{\partial^2\Phi}{\partial r^2}$$

$$\tau_{r\theta} = \frac{1}{r^2}\frac{\partial\Phi}{\partial\theta} - \frac{1}{r}\frac{\partial^2\Phi}{\partial r\partial\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right).$$

[20 marks]

B5. Fig. 4 shows a two-dimensional rectangular elastic body of width $2a$, height b and density ρ. The body is subject to gravity acting in the negative y-direction and is supported from below by a uniform pressure p_0. The material is linearly elastic and in a state of plane strain. The Lamé constant μ and Poisson's ratio ν are given.

1. Use the condition of overall equilibrium to determine p_0.

2. Show that the assumptions that $\tau_{xx} = \tau_{xy} = 0$ and $\tau_{yy} = f(y)$ are consistent with all stress boundary conditions and with Cauchy's equations, and thus determine $f(y)$.

   Hint: Because of symmetry you only have to check the boundary condition at the top (at $y = b$), at the bottom (at $y = 0$), and on one of the two sides (e.g. at $x = a$).

3. Determine the general solution for the displacement field and explain the physical meaning of any free parameters in that solution.

   Hint: The strain-stress relationship for a linear elastic body in a state of plane strain is given by

$$e_{ij} = \frac{1}{2\mu}\Big(\tau_{ij} - \nu\, \delta_{ij}\, \tau_{kk}\Big) \quad \text{for } i,j = 1,2.$$

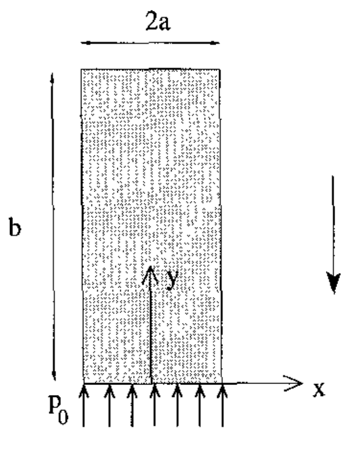

Figure 4: Sketch of a two-dimensional rectangular body loaded by gravity (acting vertically downwards) and supported from below by a constant pressure p_0.

[20 marks]

END OF EXAMINATION PAPER