Two Hours

UNIVERSITY OF MANCHESTER

ELASTICITY

2005

Answer all four questions in **SECTION A** (40 marks in total)
and
all of the two questions in **SECTION B** (20 marks each).
The total for the paper is 80 marks. A further 20 marks are available from course work during the semester making a total of 100.
NOTE: All elastic bodies referred to in the following questions are linear, isotropic and homogeneous.

Electronic calculators may be used, provided that they cannot store text.

SECTION A

Answer all four questions

A1.

The displacement field in a 2D elastic body is given by

$$\begin{aligned} u_1 &= \epsilon\left(5x_1 + (x_2-2)^2\right), \\ u_2 &= \epsilon\left(4x_2 + (x_1-1)^2\right), \end{aligned}$$

where ϵ is a small positive constant.

(i) Determine the strain tensor e_{ij} and the rotation tensor ω_{ij}.

(ii) At the point $\mathbf{r} = (3, 2)$ find the extension $e_{\mathbf{n}}$ of the line element $\mathbf{n}\,ds$, where $\mathbf{n} = (3/5, 4/5)$.

[10 marks]

A2. The components of the stress tensor τ_{ij} at a certain point in an elastic body are given by

$$(\tau_{ij}) = \begin{pmatrix} 0 & -2 & 1 \\ -2 & 0 & 1 \\ 1 & 1 & \tau_{33} \end{pmatrix}.$$

You are given that the stress vector $\mathbf{t}$ in a plane through this point is given by $\mathbf{t} = (0, 0, T)$ where T is a given constant. From this information, determine a unit normal vector $\mathbf{n}$ on this plane, and the value of τ_{33}.

[10 marks]

A3. Write down (i) the definition of the strain tensor in terms of the displacements, (ii) the equations of stress equilibrium (for a body at rest but subject to a body force $\mathbf{F}$), and (iii) the constitutive equations for a linearly elastic, homogeneous, isotropic body. Starting from these equations, derive the Navier-Lamé equations of static equilibrium

$$(\lambda + 2\mu)\ \text{grad div}\ \mathbf{u} - \mu\ \text{curl curl}\ \mathbf{u} + \mathbf{F} = \mathbf{0}.$$

[Hint: You can use the identity $\nabla^2\mathbf{u} = \text{grad div}\ \mathbf{u} - \text{curl curl}\ \mathbf{u}$.]

[10 marks]

A4. The strain field in a 2D elastic body is given by

$$(e_{ij}) = \begin{pmatrix} 2x_1x_2 & \frac{1}{2}(x_1^2 + 2x_2 + 1) \\ \frac{1}{2}(x_1^2 + 2x_2 + 1) & 4x_2 \end{pmatrix}$$

(i) Determine a solution for the displacement field.

(ii) Explain briefly how your solution differs from the most general solution for the displacement field.

[10 marks]

SECTION B

Answer all two questions

B5. An infinitely long hollow cylinder of outer radius a and inner radius b is made of homogeneous, isotropic elastic material with Lamé constants λ, μ. The cylinder is forced into an infinite *rigid* hollow cylinder of inner radius $a - \epsilon$ where $0 < \epsilon \ll a$. The two cylinders are in smooth contact and body forces can be neglected. Determine the displacement field throughout the elastic cylinder and determine the locations at which the azimuthal stress $\tau_{\theta\theta}$ has its maximum and minimum (absolute) values.

[20 marks]

B6.

An elastic half-plane (located in $y \leq 0$) is subject to the applied traction

$$\mathbf{t} = \sigma_0 \begin{pmatrix} \sin(x) \\ \cos(x) \end{pmatrix} \qquad \text{along } y = 0,$$

and to gravity which acts vertically downwards so that the body force is given by $\mathbf{F} = -\rho g \, \mathbf{e}_y$, where $\mathbf{e}_y$ is the unit vector in the y-direction. You can assume that the material is in a state of plane strain. Determine the components of the stress tensor τ_{ij}, following these steps:

(i) Decompose the stress tensor τ_{ij} into two components such that $\tau_{ij} = \tau_{ij}^{(h)} + \tau_{ij}^{(p)}$, and determine $\tau_{ij}^{(p)}$ so that it balances the body force.

(ii) Use an Airy stress function ϕ to determine $\tau_{ij}^{(h)}$ [Hint: Write $\tau_{ij}^{(h)}(x, y)$ as $\tau_{ij}^{(h)}(x, y) = f(x)\, g(y)$ and use the stress boundary condition to determine the required functional form of $f(x)$.]

[20 marks]

Some Equations in Cylindrical Polar Coordinates

- $x_1 = x = r\cos\theta$, $x_2 = y = r\sin\theta$, $x_3 = z = z$.

$$\mathbf{u} = (u_r, u_\theta, u_z), \quad e = (e_{ij}), \quad \tau = (\tau_{ij}), \quad \text{where } i, j = r, \theta, z.$$

- Vector calculus:

$$\operatorname{grad} f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}, \qquad \operatorname{div}\mathbf{u} = \frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z},$$

$$\operatorname{curl}\mathbf{u} = \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\hat{\boldsymbol{\theta}} + \left(\frac{1}{r}\frac{\partial (r u_\theta)}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right)\hat{\mathbf{z}}.$$

- Stress-strain relations

$$\tau_{ij} = \lambda \delta_{ij} \operatorname{div}\mathbf{u} + 2\mu e_{ij}, \qquad i, j = r, \theta, z.$$

- Stress-displacement relations:

$$\tau_{rr} = \lambda \operatorname{div}\mathbf{u} + 2\mu\frac{\partial u_r}{\partial r}, \quad \tau_{\theta\theta} = \lambda \operatorname{div}\mathbf{u} + 2\mu\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right), \quad \tau_{zz} = \lambda \operatorname{div}\mathbf{u} + 2\mu\frac{\partial u_z}{\partial z},$$

$$\frac{\tau_{r\theta}}{\mu} = \frac{\tau_{\theta r}}{\mu} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}, \quad \frac{\tau_{rz}}{\mu} = \frac{\tau_{zr}}{\mu} = \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}, \quad \frac{\tau_{\theta z}}{\mu} = \frac{\tau_{z\theta}}{\mu} = \frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}.$$

- Strain-displacement relations:

$$e_{rr} = \frac{\partial u_r}{\partial r}, \qquad e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \qquad e_{zz} = \frac{\partial u_z}{\partial z},$$

$$2e_{r\theta} = 2e_{\theta r} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}, \quad 2e_{rz} = 2e_{zr} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}, \quad 2e_{z\theta} = 2e_{\theta z} = \frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}.$$

- Stress boundary conditions: Applied traction $\mathbf{T}$ is prescribed. We have, from $T_i = \hat{n}_j \tau_{ij}$,

$$\begin{aligned} T_r &= \hat{n}_r\tau_{rr} + \hat{n}_\theta\tau_{r\theta} + \hat{n}_z\tau_{rz} \\ T_\theta &= \hat{n}_r\tau_{r\theta} + \hat{n}_\theta\tau_{\theta\theta} + \hat{n}_z\tau_{\theta z} \\ T_z &= \hat{n}_r\tau_{rz} + \hat{n}_\theta\tau_{\theta z} + \hat{n}_z\tau_{zz} \end{aligned}$$