MATH35021: EXAMPLE SHEET[1] IX

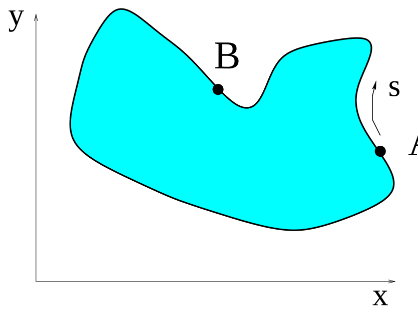

Figure 1: A rigid body in plane strain.

1.) From the lecture notes, if ϕ is the Airy stress function then on the boundary curve

$$t_x(s) = \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial \phi}{\partial y}\right), \quad \text{and} \quad t_y(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial \phi}{\partial x}\right),$$

where s is the arc length which parameterises the boundary, see Figure 1.

The resultant force is given by the integral of the traction over the boundary section

$$\boldsymbol{F} = \int_A^B \boldsymbol{t}\,\mathrm{d}s = \int_A^B \left(\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial \phi}{\partial y}\right), -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial \phi}{\partial x}\right)\right)\mathrm{d}s.$$

Both components are exact derivatives with respect to the arc length and so the integration is easy; hence,

$$\boldsymbol{F} = \left(\left.\frac{\partial \phi}{\partial y}\right|_A^B, -\left.\frac{\partial \phi}{\partial x}\right|_A^B\right),$$

or, in other words,

$$F_x = \left[\frac{\partial \phi}{\partial y}\right]_A^B, \qquad F_y = \left[-\frac{\partial \phi}{\partial x}\right]_A^B.$$

The resultant moment about the origin is the integral over the boundary section of the cross-product of the vector distance from the origin and the traction. The cross-product formulation is the generalisation of the "force × perpendicular distance" definition of a moment.

$$\boldsymbol{M}_0 = \int_A^B \boldsymbol{r} \times \boldsymbol{t}\,\mathrm{d}s.$$

The two vectors $\boldsymbol{r}$ and $\boldsymbol{t}$ both lie in the x-y plane and so the cross product must be directed in the direction $\boldsymbol{e}_z$, the unit vector in the z (out-of-plane) direction. Then

$$\boldsymbol{M}_0 = \boldsymbol{e}_z \int_A^B |\boldsymbol{r} \times \boldsymbol{t}|\,\mathrm{d}s = M_0\,\boldsymbol{e}_z.$$

We must now evaluate the cross-product to find the magnitude of the moment. Now,

$$\boldsymbol{r} = \begin{pmatrix} x(s) \\ y(s) \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{t} = \begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial \phi}{\partial y}\right) \\ -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial \phi}{\partial x}\right) \\ 0 \end{pmatrix},$$

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so,

$$\boldsymbol{r} \times \boldsymbol{t} = \begin{pmatrix} 0 \\ 0 \\ -x(s)\,\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial \phi}{\partial x}\right) - y(s)\,\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial \phi}{\partial y}\right) \end{pmatrix}.$$

Then,

$$M_0 = -\int_A^B \left[x(s)\,\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial \phi}{\partial x}\right) + y(s)\,\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial \phi}{\partial y}\right)\right] \mathrm{d}s.$$

We now integrate by parts to obtain:

$$M_0 = -\left[x(s)\,\frac{\partial \phi}{\partial x} + y(s)\,\frac{\partial \phi}{\partial y}\right]_A^B + \int_A^B \left(\frac{\mathrm{d}x}{\mathrm{d}s}\frac{\partial \phi}{\partial x} + \frac{\mathrm{d}y}{\mathrm{d}s}\frac{\partial \phi}{\partial y}\right) \mathrm{d}s.$$

$$\Rightarrow M_0 = -\left[x(s)\,\frac{\partial \phi}{\partial x} + y(s)\,\frac{\partial \phi}{\partial y}\right]_A^B + \int_A^B \frac{\mathrm{d}\phi}{\mathrm{d}s}\,\mathrm{d}s.$$

$$\Rightarrow M_0 = \left[\phi - x(s)\,\frac{\partial \phi}{\partial x} - y(s)\,\frac{\partial \phi}{\partial y}\right]_A^B.$$

2.) a) The easiest way to see this is to use index notation

$$\nabla^2(\phi\psi) = (\phi\psi)_{,ii} = (\phi_{,i}\psi + \phi\psi_{,i})_{,i} = \phi_{,ii}\psi + \phi_{,i}\psi_{,i} + \phi_{,i}\psi_{,i} + \phi\psi_{,ii},$$

and re-expressing the result in dyadic notation gives:

$$\nabla^2(\phi\psi) = \phi\nabla^2\psi + \psi\nabla^2\phi + 2\nabla\psi \cdot \nabla\phi.$$

b) This is a little bit of an algebra-fest. By part (a)

$$\nabla^2 F_1 = \nabla^2(xH) = x\nabla^2 H + 2\nabla x \cdot \nabla H + H\nabla^2 x,$$

but $\nabla^2 H = 0$ because it is harmonic and $\nabla^2 x = 0$, so

$$\nabla^2 F_1 = 2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial y} \\ \frac{\partial H}{\partial z} \end{pmatrix} = 2\frac{\partial H}{\partial x}.$$

Then,

$$\nabla^4 F_1 = \nabla^2\left(2\frac{\partial H}{\partial x}\right) = 2\frac{\partial}{\partial x}\left(\nabla^2 H\right) = 0, \quad \text{because } H \text{ is harmonic.}$$

Thus F_1 is biharmonic.

By part (a)

$$\nabla^2 F_2 = \nabla^2(yH) = y\nabla^2 H + 2\nabla y \cdot \nabla H + H\nabla^2 y,$$

but $\nabla^2 H = 0$ because it is harmonic and $\nabla^2 y = 0$, so

$$\nabla^2 F_2 = 2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial y} \\ \frac{\partial H}{\partial z} \end{pmatrix} = 2\frac{\partial H}{\partial y}.$$

Then,

$$\nabla^4 F_2 = \nabla^2\left(2\frac{\partial H}{\partial y}\right) = 2\frac{\partial}{\partial y}\left(\nabla^2 H\right) = 0, \quad \text{because } H \text{ is harmonic.}$$

Thus F_2 is biharmonic.

By part (a)

$$\nabla^2 F_3 = \nabla^2((x^2+y^2)H) = (x^2+y^2)\nabla^2 H + 2\nabla(x^2+y^2)\cdot\nabla H + H\nabla^2(x^2+y^2),$$

but $\nabla^2 H = 0$ because it is harmonic and $\nabla^2(x^2+y^2) = 4$, so

$$\nabla^2 F_3 = 2\begin{pmatrix} 2x \\ 2y \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial y} \\ \frac{\partial H}{\partial z} \end{pmatrix} + 4H = 4H + 4x\frac{\partial H}{\partial x} + 4y\frac{\partial H}{\partial y}.$$

Then,

$$\nabla^4 F_3 = \nabla^2\left[4H + 4x\frac{\partial H}{\partial x} + 4y\frac{\partial H}{\partial y}\right] = 4\nabla^2\left(x\frac{\partial H}{\partial x}\right) + 4\nabla^2\left(y\frac{\partial H}{\partial y}\right),$$

because H is harmonic. Now we can use the result from part (a) again to show that

$$\nabla^2\left(x\frac{\partial H}{\partial x}\right) = x\nabla^2\frac{\partial H}{\partial x} + 2\nabla x\cdot\nabla\frac{\partial H}{\partial x} + \frac{\partial H}{\partial x}\nabla^2 x = 2\nabla x\cdot\nabla\frac{\partial H}{\partial x},$$

because $\nabla^2 x = 0$ and H is harmonic; and so

$$\nabla^2\left(x\frac{\partial H}{\partial x}\right) = 2\frac{\partial^2 H}{\partial x^2}.$$

Similarly

$$\nabla^2\left(y\frac{\partial H}{\partial y}\right) = 2\frac{\partial^2 H}{\partial y^2}.$$

Then,

$$\nabla^4 F_3 = 8\left(\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2}\right) = 8\nabla^2 H = 0, \quad \text{because } H \text{ is harmonic.}$$

Thus, F_3 is biharmonic.

3.) The equation

$$\tilde{\nabla}^4\phi(r) = \phi_{,rrrr} + \frac{2}{r}\phi_{,rrr} - \frac{1}{r^2}\phi_{,rr} + \frac{1}{r^3}\phi_{,r} \tag{1}$$

is a linear Euler equation because, after multiplication by r^4, it has the form

$$\sum_n a_n r^n \frac{\partial^n \phi}{\partial r^n}.$$

In order for all terms to balance for all values of r, they must be of the form $\phi \sim r^m$, from which it follows that

$$\tilde{\nabla}^4\phi = [m(m-1)(m-2)(m-3) + 2m(m-1)(m-2) - m(m-1) + m]r = 0,$$

and because this must be true for all r we obtain the characteristic equation

$$\begin{aligned} m(m-1)(m-2)(m-3) + 2m(m-1)(m-2) - m(m-1) + m &= 0, \\ \Rightarrow m\left[(m-1)(m^2-5m+6) + 2m^2 - 6m + 4 - m + 1 + 1\right] &= 0, \\ \Rightarrow m\left[m^3 - m^2 - 5m^2 + 5m + 6m - 6 + 2m^2 - 6m + 6 - m\right] &= 0, \\ \Rightarrow m\left[m^3 - 4m^2 + 4m\right] &= 0, \\ \Rightarrow m^2\left(m^2 - 4m + 4\right) = m^2(m-2)^2 &= 0. \end{aligned}$$

There are two double roots, $m = 0, 2$. There must be four linearly independent solutions and so the double-roots require the introduction of logarithmic terms. The general solution is, therefore,

$$\phi = A_0 + B_0 r^2 + C_0 \ln r + D_0 r^2 \ln r,$$

where the terms multiplied by A_0 and C_0 correspond to the $m = 0$ roots and the terms multiplied by B_0 and D_0 correspond to the $m = 2$ roots.