MATH35021: SOLUTION SHEET VIII[1]

1.) If $\phi = Ax_1^2 x_2^3 + Bx_2^5$ is to serve as an Airy stress function it must be biharmonic:

$$\nabla^4 \phi = \frac{\partial^4 \phi}{\partial x_1^4} + 2\frac{\partial^4 \phi}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 \phi}{\partial x_2^4} = 0.$$

Considering each term separately

$$\frac{\partial^4 \phi}{\partial x_1^4} = 0, \quad \frac{\partial^4 \phi}{\partial x_1^2 \partial x_2^2} = 12Ax_2, \quad \frac{\partial^4 \phi}{\partial x_2^4} = 120Bx_2.$$

In order for the biharmonic equation to be satisfied

$$24Ax_2 + 120Bx_2 = 0, \quad \text{for all values of } x_2.$$

Hence, the required relationship between the constants A and B is

$$24A + 120B = 0 \quad \Rightarrow \quad A + 5B = 0 \quad \Rightarrow \quad \boxed{A = -5B}.$$

2.) a) The function

$$\phi = \frac{3F}{4c}\left(x_1 x_2 - \frac{x_1 x_2^3}{3c^2}\right) + \frac{P}{4c}x_2^2.$$

Then,

$$\frac{\partial^4 \phi}{\partial x_1^4} = \frac{\partial^4 \phi}{\partial x_1^2 \partial x_2^2} = \frac{\partial^4 \phi}{\partial x_2^4} = 0 \quad \Rightarrow \nabla^4 \phi = 0,$$

and so the function is trivially biharmonic.

b) The stress field is determined from the second derivatives of the Airy stress function:

$$\tau_{11} = \frac{\partial^2 \phi}{\partial x_2^2} = -\frac{3F}{4c}\frac{6x_1 x_2}{3c^2} + \frac{2P}{4c} = \frac{P}{2c} - \frac{3F}{2c^3}x_1 x_2.$$

$$\tau_{22} = \frac{\partial^2 \phi}{\partial x_1^2} = 0.$$

$$\tau_{12} = -\frac{\partial^2 \phi}{\partial x_1 \partial x_2} = -\frac{3F}{4c} + \frac{3F}{4c}\frac{3x_2}{3c^2} = \frac{3F}{4c}\left[\left(\frac{x_2}{c}\right)^2 - 1\right].$$

In an attempt to understand the physical meaning of the constants, we shall consider the applied tractions at the edges of the beam.

At left end: $x_1 = 0$ and $\boldsymbol{n} = (-1, 0)$. Then,

$$t_i = \tau_{ij} n_j = -\tau_{i1} \quad \Rightarrow \quad \boldsymbol{t} = (-\tau_{11}|_{x_1=0}, -\tau_{21}|_{x_1=0}),$$

so

$$t_1 = -\frac{P}{2c} \quad \text{and} \quad t_2 = \frac{3F}{4c}\left[1 - \left(\frac{x_2}{c}\right)^2\right].$$

[1] Any feedback to: *Andrew.Hazel@manchester.ac.uk*

The resultant applied force, $\boldsymbol{T}$, is given by integrating the traction vector (forces per unit area) over the face of the beam:

$$\boldsymbol{T} = \int \boldsymbol{t}\,\mathrm{d}A.$$

Then,

$$T_1 = \int_{-c}^{c} t_1\,\mathrm{d}x_2 = \int_{-c}^{c} -\frac{P}{2c}\,\mathrm{d}x_2 = -2c\frac{P}{2c} = -P.$$

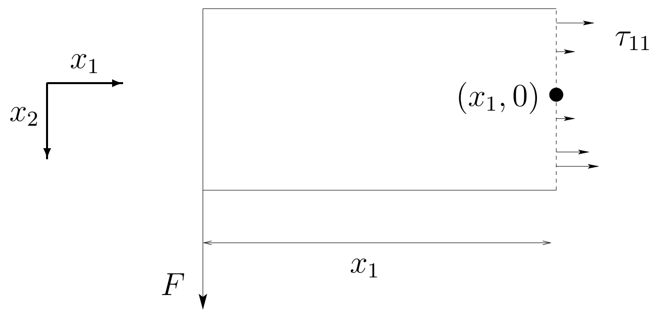
$$T_2 = \int_{-c}^{c} t_2\,\mathrm{d}x_2 = \int_{-c}^{c} \frac{3F}{4c}\left[1-\left(\frac{x_2}{c}\right)^2\right]\mathrm{d}x_2 = \frac{3F}{4c}\left[x_2 - \frac{x_2^3}{3c^2}\right]_{-c}^{c}$$

$$= \frac{3F}{4c}\left[2c - 2\frac{c^3}{3c^2}\right] = \frac{3F}{4c}\frac{4}{3}c = F.$$

Hence, the resultant forces applied to the "free" end of the cantilever beam are P in the negative x_1 direction and F in the x_2 direction.

It follows that the stress field described by ϕ corresponds to the application of two forces P "pulling" to the left of the beam and F acting downwards at the end of the beam, or, alternatively, a single force vector $(-P, F)$ acting at the left-hand face. Note that the body is in equilibrium, because any Airy stress function satisfies the Navier–Lamé equations, so an equal and opposite force must be applied (by the wall) at the right-hand end of the beam. The last statement is only true because the resultant forces on the upper and lower faces of the beam are both zero. In general, there could be surface tractions applied on these surfaces that contribute to the overall equilibrium of the beam. If the local distribution of surface tractions is exactly the same as $\boldsymbol{t}(x_2)$, then the solution is exact. If the local distribution gives the same resultant forces, but differs from $\boldsymbol{t}(x_2)$ then the solution is a St. Venant solution to the problem.

c) The are no body forces applied to the beam, so the sum of all applied tractions and their moments must vanish at the point $(x_1, 0)$.



Consider the resultant moment about $(x_1, 0)$, the downward resultant force F at the end of the beam acts on a lever arm of distance x_1. This must be balanced by the sum (integral) of the internal tractions acting in the x-direction τ_{11} on levers of distance x_2, so

$$M = Fx_1 + \int_{-c}^{c} \tau_{11} x_2\,\mathrm{d}x_2,$$

$$= Fx_1 + \int_{-c}^{c}\left(\frac{P}{2c}x_2 - \frac{3F}{2c^3}x_1 x_2^2\right)\mathrm{d}x_2$$

$$= Fx_1 + \left[-\frac{F}{2c^3} x_1 x_2^3 \right]_{-c}^{c} = Fx_1 - Fx_1 = 0,$$

as expected.

We deduce that τ_{11} must increase linearly in x_1 in order to balance the bending moment generated by the resultant force F.