MATH35021: SOLUTION SHEET VII[1]

1.) It is natural to work in spherical polar coordinates. The geometry and the boundary conditions are independent of θ and ϕ so we pose a solution of the form

$$\boldsymbol{u} = u(r)\boldsymbol{e}_r.$$

It follows that curl $\boldsymbol{u} = \boldsymbol{0}$ and so the Navier–Lamé equations are

$$(\lambda + 2\mu)\text{grad}\,\text{div}\boldsymbol{u} = \boldsymbol{0},$$

because there are no body forces.

The only non-zero component is in the r-direction and is

$$(\lambda + 2\mu)\frac{\partial}{\partial r}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u\right)\right] = 0,$$

from which we obtain

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u\right) = \text{div}\,\boldsymbol{u} = A, \text{ a constant.}$$

Hence,

$$\frac{\partial}{\partial r}\left(r^2 u\right) = Ar^2 \quad \Rightarrow \quad r^2 u = \frac{A}{3}r^3 + B,$$

and

$$u = \frac{A}{3}r + \frac{B}{r^2}.$$

The displacements must remain finite throughout the elastic body, which extends to inifinity, so u must remain bounded as $r \to \infty$. Hence $A = 0$, and then

$$u = \frac{B}{r^2},$$

which tends to zero as $r \to \infty$.

The stress boundary condition at the wall of the cavity is that the internal pressure is a constant p_0. The pressure always acts onto the elastic body and acts in the radial direction so

$$\tau_{rr}\mid_{r=a} = -p_0.$$

In spherical polar coordinates

$$\tau_{rr} = \lambda\,\text{div}\,\boldsymbol{u} + 2\mu\frac{\partial u}{\partial r} = 2\mu(-2)Br^{-3} = -4\mu Br^{-3}.$$

Thus, at $r = a$,

$$-p_0 = -4\mu Ba^{-3} \quad \Rightarrow \quad B = \frac{p_0\, a^3}{4\mu},$$

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which gives

$$\boxed{u(r) = \frac{p_0\,a^3}{4\mu}\frac{1}{r^2}.}$$

The fractional increase in the radius of the cavity is given by

$$\frac{[a + u(a)] - a}{a} = \frac{u(a)}{a} = \frac{p_0}{4\mu},$$

so the cavity increases in size with increasing internal pressure and decreasing shear modulus. In other words, the harder you inflate or the more flexible the material the more the cavity radius increases.

2.) Assuming that we have a steady (elastostatic) problem and that there are no body forces, the Navier–Lamé equations are

$$(\lambda + \mu)u_{k,ki} + \mu u_{i,kk} = 0.$$

We are given that

$$u_i = \frac{\lambda + 2\mu}{\mu(\lambda + \mu)}\ \psi_{i,jj} - \frac{1}{\mu}\ \psi_{j,ji};$$

and inserting this into the Navier–Lamé equations gives

$$(\lambda + \mu)\left[\frac{\lambda + 2\mu}{\mu(\lambda + \mu)}\ \psi_{k,jj} - \frac{1}{\mu}\ \psi_{j,jk}\right]_{,ki} + \mu\left[\frac{\lambda + 2\mu}{\mu(\lambda + \mu)}\ \psi_{i,jj} - \frac{1}{\mu}\ \psi_{j,ji}\right]_{,kk} = 0.$$

Using the linearity of the partial differential operator, we find that

$$\frac{\lambda + 2\mu}{\mu}\ \psi_{k,jjki} - \frac{\lambda + \mu}{\mu}\ \psi_{j,jkki} + \frac{\lambda + 2\mu}{\lambda + \mu}\ \psi_{i,jjkk} - \ \psi_{j,jikk} = 0.$$

We are free to relabel the dummy indices in each term and so we swap the k's and j's in the first term. We are also free to permute the indices after the comma because the order in which we perform the partial differentiation does not affect the solution. It follows that

$$\frac{\lambda + 2\mu}{\mu}\ \psi_{j,kkji} - \frac{\lambda + \mu}{\mu}\ \psi_{j,kkji} + \frac{\lambda + 2\mu}{\lambda + \mu}\ \psi_{i,jjkk} - \ \psi_{j,kkji} = 0,$$

$$\Rightarrow \left[\frac{\lambda + 2\mu}{\mu} - \frac{\lambda + \mu}{\mu} - \frac{\mu}{\mu}\right]\ \psi_{j,kkji} + \frac{\lambda + 2\mu}{\lambda + \mu}\ \psi_{i,jjkk} = 0,$$

$$\Rightarrow \frac{\lambda + 2\mu}{\lambda + \mu}\ \psi_{i,jjkk} = 0 \quad \Rightarrow \quad \psi_{i,jjkk} = 0.$$

In other words, the condition on each component ψ_i is that it must be biharmonic:

$$\boxed{\nabla^4\psi_i = 0.}$$