MATH35021: SOLUTION SHEET[1] VI

1.) a) The boundary conditions are:

Top: $\boldsymbol{n} = -\boldsymbol{e}_3$; air pressure is zero, $p = 0$:

$$\boldsymbol{t} = 0.$$

Curved sides: $\boldsymbol{n} = (n_1, n_2, 0)$; the water pressure increases linearly with depth $p = \rho g x_3$:

$$\boldsymbol{t} = -\rho g x_3 \boldsymbol{n}.$$

Bottom: $\boldsymbol{n} = \boldsymbol{e}_3$; the water pressure is now constant $p = \rho g h$, where h is the height of the cylinder:

$$\boldsymbol{t} = -\rho g h \boldsymbol{e}_3.$$

b) The body force inside the cylinder is simply that due to gravity

$$\boldsymbol{F} = \rho g \boldsymbol{e}_3.$$

c) The stress boundary conditions are obtained from $\tau_{ij} n_j = t_i$:

Top: $x_3 = 0$ and

$$\tau_{13} = \tau_{23} = \tau_{33} = 0. \tag{1}$$

Curved sides:

$$\tau_{11} n_1 + \tau_{12} n_2 = -\rho g x_3 n_1, \tag{2a}$$

$$\tau_{21} n_1 + \tau_{22} n_2 = -\rho g x_3 n_2, \tag{2b}$$

Bottom: $x_3 = h$ and

$$\tau_{13} = \tau_{23} = 0, \quad \text{and} \quad \tau_{33} = -\rho g h. \tag{3}$$

- Comparing coefficients of the normal components on the curved sides, there is no reason to expect a shear stress.
- The boundary conditions do not depend on x_1 or x_2, but depend linearly on x_3.

We, therefore, pose a solution of the form

$$\begin{aligned} \tau_{11} &= a_1 x_3 + a_2, \\ \tau_{22} &= b_1 x_3 + b_2, \\ \tau_{33} &= c_1 x_3 + c_2, \\ \tau_{ij} &= 0 \qquad i \neq j. \end{aligned}$$

The boundary conditions on the top (1) give

$$\tau_{33}|_{x_3=0} = c_2 = 0.$$

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The boundary conditions on the bottom (3) give

$$\tau_{33}|_{x_3=h} = c_1 h = -\rho g h \quad \Rightarrow c_1 = -\rho g.$$

The boundary conditions on the sides (2) give

$$(a_1 x_3 + a_2) n_1 = -\rho g x_3 n_1, \qquad (b_1 x_3 + b_2) n_2 = -\rho g x_3 n_2.$$

Equating coefficients we obtain

$$a_1 = -\rho g, \quad a_2 = 0, \quad b_1 = -\rho g, \quad b_2 = 0.$$

Hence

$$\tau_{11} = \tau_{22} = \tau_{33} = -\rho g x_3.$$

We must check that this does indeed satisfy the equilibrium equations $\tau_{ij,j} + F_i = 0$.

$$\tau_{11,1} = 0, \quad \tau_{22,2} = 0, \quad \tau_{33,3} + F_3 = -\rho g + \rho g = 0.$$

d) A linear stress field satisfies the Beltrami–Michell equations, which were derived from the strain compatibility equations. The strain compatibility equations are, therefore, satisfied and a continuous displacement field can be recovered from this strain field.

Now,

$$E e_{ij} = (1+\nu)\tau_{ij} - \nu\, \delta_{ij}\, \tau_{kk},$$

but $\tau_{(i)(i)} = -\rho g x_3$, so $\tau_{kk} = -3\rho g x_3$ and

$$\Rightarrow E e_{11} = E e_{22} = E e_{33} = -(1-2\nu)\rho g x_3,$$

and

$$e_{ij} = 0 \qquad i \neq j.$$

For ease of notation let $k = (1-2\nu)\rho g/E$ and then

$$e_{11} = u_{1,1} = -k x_3 \quad \Rightarrow u_1 = -k x_3 x_1 + f(x_2, x_3),$$

$$e_{22} = u_{2,2} = -k x_3 \quad \Rightarrow u_2 = -k x_3 x_2 + g(x_1, x_3),$$

$$e_{33} = u_{3,3} = -k x_3 \quad \Rightarrow u_2 = -k \frac{1}{2} x_3^2 + h(x_1, x_2).$$

The off-diagonal terms give

$$e_{12} = \frac{1}{2}(u_{1,2} + u_{2,1}) = 0, \quad \Rightarrow f_{,2} + g_{,1} = 0.$$

$$e_{23} = \frac{1}{2}(u_{2,3} + u_{3,2}) = 0, \quad \Rightarrow -k x_2 + g_{,3} + h_{,2} = 0.$$

$$e_{13} = \frac{1}{2}(u_{1,3} + u_{3,1}) = 0, \quad \Rightarrow -k x_1 + f_{,3} + h_{,1} = 0.$$

We are ignoring all rigid body motions so we can assume that $f \equiv g \equiv 0$ and then

$$-kx_2 + h_{,2} = 0, \quad \text{and} \quad -kx_1 + h_{,1} = 0;$$

from which we deduce

$$h(x_1, x_2) = k\frac{1}{2}x_2^2 + F(x_1), \quad \text{and} \quad h(x_1, x_2) = k\frac{1}{2}x_1^2 + G(x_2).$$

A consistent solution is thus

$$h(x_1, x_2) = \frac{1}{2}k(x_1^2 + x_2^2).$$

Finally, the displacement field is

$$\begin{aligned} u_1 &= -\frac{(1-2\nu)\rho g}{E} x_1 x_3, \\ u_2 &= -\frac{(1-2\nu)\rho g}{E} x_2 x_3, \\ u_3 &= \frac{(1-2\nu)\rho g}{2E} \left(x_1^2 + x_2^2 - x_3^2\right). \end{aligned}$$

2.) Given that $\mathbf{u} = u(r)\mathbf{e}_r$ then $\operatorname{curl}\mathbf{u} = \mathbf{0}$ and the Navier–Lamé equations become

$$(\lambda + 2\mu)\operatorname{grad}\operatorname{div}\mathbf{u} + \mathbf{F} = \mathbf{0}.$$

The only non-zero component is the radial component which is

$$(\lambda + 2\mu)\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(ru)\right] + \rho\omega^2 r = 0,$$

because the form of the body force was given as $\mathbf{F} = \rho\omega^2 r\mathbf{e}_r$.
Integrating once we obtain

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(ru) + \frac{\rho\omega^2}{\lambda + 2\mu}\frac{r^2}{2} - 2A = 0,$$

where the form of the constant has been carefully chosen for convenience. For ease of notation let $K = \frac{\rho\omega^2}{\lambda+2\mu}$ and integrating again gives

$$ru + K\frac{r^4}{8} - Ar^2 - B = 0.$$

Thus,

$$u = Ar + B\frac{1}{r} - K\frac{r^3}{8},$$

and the constants A and B must be determined from the boundary conditions.

- The displacement field must be non-singular at the origin $r \to 0$, so $B = 0$.
- The outer boundary is stress free so $\tau_{rr} = 0$ at $r = a$.

Now,

$$\tau_{rr} = \lambda \mathrm{div}\,\mathbf{u} + 2\mu e_{rr},$$

where

$$\mathrm{div}\,\mathbf{u} = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(ru) = 2A - \frac{1}{2}Kr^2,$$

and

$$e_{rr} = \frac{\partial u}{\partial r} = A - \frac{3}{8}Kr^2.$$

Therefore,

$$\tau_{rr} = \lambda\left(2A - \frac{1}{2}Kr^2\right) + 2\mu\left(A - \frac{3}{8}Kr^2\right) = 2A(\lambda+\mu) - Kr^2\left(\frac{1}{2}\lambda + \frac{3}{4}\mu\right).$$

At the outer boundary $r = a$ and $\tau_{rr} = 0$,

$$2A(\lambda+\mu) - Ka^2\left(\frac{1}{2}\lambda + \frac{3}{4}\mu\right) = 0,$$

$$\Rightarrow A = Ka^2\,\frac{\frac{1}{2}\lambda + \frac{3}{4}\mu}{2(\lambda+\mu)} = Ka^2\,\frac{2\lambda+3\mu}{8(\lambda+\mu)}.$$

Thus,

$$u(r) = Ka^2\,\frac{2\lambda+3\mu}{8(\lambda+\mu)}\,r - K\frac{r^3}{8} = \frac{\rho\omega^2\,r}{8(\lambda+2\mu)}\left(a^2\,\frac{2\lambda+3\mu}{(\lambda+\mu)} - r^2\right).$$