MATH35021: SOLUTION SHEET II[1]

1.) a) The strain tensor is $e_{ij} = \epsilon \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, see Example Sheet I. Using the formula for the normal strain, $e_{\boldsymbol{n}} = e_{ij} n_i n_j$, we have

$$e_{\boldsymbol{n}_1} = \epsilon \begin{pmatrix} 3/5 & 4/5 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix} = \frac{49}{25}\,\epsilon.$$

Now, $\boldsymbol{n}_1 \cdot \boldsymbol{n}_2 = 0$ and so the angle between the undeformed vectors is $\pi/2$. The angle between the two vectors after deformation, Φ, obeys the rule $\cos\Phi = 2e_{ij} n_i^{(1)} n_j^{(2)}$. Thus, the change in angle is

$$\cos\Phi \approx \frac{\pi}{2} - \Phi = 2\epsilon \begin{pmatrix} 3/5 & 4/5 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -4/5 \\ 3/5 \end{pmatrix} = -\frac{14}{25}\,\epsilon.$$

b) The principal axes of strain are the eigenvectors of e_{ij} and the principal strains are the corresponding eigenvalues. We must solve

$$\det \begin{pmatrix} \epsilon - \lambda & \epsilon \\ \epsilon & \epsilon - \lambda \end{pmatrix} = (\epsilon - \lambda)^2 - \epsilon^2 = 0,$$

$$\Rightarrow \epsilon - \lambda = \pm\epsilon \Rightarrow \lambda = 0 \text{ or } 2\epsilon.$$

<u>$\lambda = 0$:</u>

$$\epsilon \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \epsilon \Rightarrow v_1 + v_2 = 0 \Rightarrow \boldsymbol{v} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

<u>$\lambda = 2$:</u>

$$\epsilon \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 2 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \epsilon \Rightarrow v_1 = v_2 \Rightarrow \boldsymbol{v} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Note that the principal axes of strain are orthogonal. Remember that we are working in the undeformed coordinate system, so the axes do not necessarily remain orthogonal during the deformation. For small deformations ($\epsilon \ll 1$), the body is extended along these two axes with extension ratios given by the two eigenvalues, 0 and 2ϵ.

We can verify the last statement by computing the change in length of line elements along the principal axes. It is easiest to choose the line elements to be the diagonals of the undeformed square. The initial lengths are $\Delta s = \sqrt{2}$. After the deformation the corners are located at

$$\boldsymbol{R}_{(0,0)} = \begin{pmatrix} 0 \\ 3\epsilon \end{pmatrix}, \quad \boldsymbol{R}_{(1,0)} = \begin{pmatrix} 1+\epsilon \\ 3\epsilon \end{pmatrix}, \quad \boldsymbol{R}_{(0,1)} = \begin{pmatrix} 2\epsilon \\ 1+4\epsilon \end{pmatrix}, \quad \boldsymbol{R}_{(1,1)} = \begin{pmatrix} 1+3\epsilon \\ 1+4\epsilon \end{pmatrix}.$$

Thus, the deformed lengths of the two diagonals are

$$\Delta S_1 = |\boldsymbol{R}_{(1,1)} - \boldsymbol{R}_{(0,0)}| = \left| \begin{pmatrix} 1+3\epsilon \\ 1+\epsilon \end{pmatrix} \right|.$$

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$$\Delta S_2 = |\boldsymbol{R}_{(0,1)} - \boldsymbol{R}_{(1,0)}| = \left| \begin{pmatrix} -1+\epsilon \\ 1+\epsilon \end{pmatrix} \right|.$$

Thus,

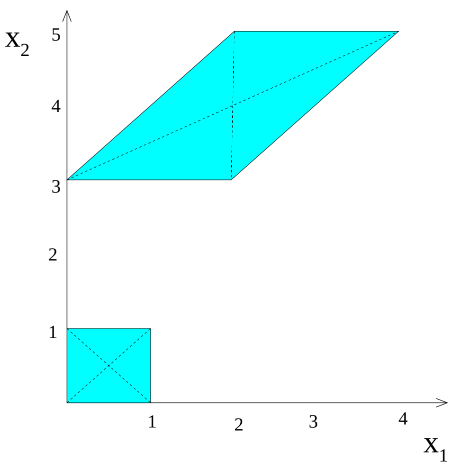
$$\Delta S_1 = \sqrt{(1+3\epsilon)^2 + (1+\epsilon)^2} = \sqrt{2+8\epsilon+10\epsilon^2} = \sqrt{2} + 2\sqrt{2}\epsilon + O(\epsilon^2),$$

$$\Delta S_2 = \sqrt{(-1+\epsilon)^2 + (1+\epsilon)^2} = \sqrt{2+2\epsilon^2} = \sqrt{2} + O(\epsilon^2),$$

using the binomial expansion. Neglecting quadratic terms (consistent with the derivation of the strain tensor), the extension ratios, or normal strains, are

$$e_1 = \frac{\Delta S_1 - \Delta s}{\Delta s} = \frac{\sqrt{2} + 2\sqrt{2}\epsilon - \sqrt{2}}{\sqrt{2}} = 2\epsilon, \quad e_2 = \frac{\Delta S_2 - \Delta s}{\Delta s} = \frac{\sqrt{2} - \sqrt{2}}{\sqrt{2}} = 0,$$

as expected.



The principal axes of strain are marked as dashed lines in the above figure. Note that $\epsilon = 1$ as in the previous example sheet.

2.) a) From the definition in the lecture notes, $d = e_{ii}$. Given that $e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$, it follows that $e_{(i)(i)} = \frac{\partial u_{(i)}}{\partial x_{(i)}}$. Thus,

$$d = \frac{\partial u_i}{\partial x_i} = \text{div } \boldsymbol{u}.$$

Again from the lecture notes

$$\boldsymbol{\omega} = \begin{pmatrix} \omega_{32} \\ \omega_{13} \\ \omega_{21} \end{pmatrix}, \quad \text{where } \omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right).$$

Hence,

$$\boldsymbol{\omega} = \begin{pmatrix} \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) \end{pmatrix} = \frac{1}{2} \text{ curl } \boldsymbol{u}.$$

b) The volume of a parallelepiped formed by the three vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ is given by

$$V = (\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}.$$

If we consider the rectangular parallelepiped, formed by the three infinitesimal vectors $(dx_1, 0, 0)^T, (0, dx_2, 0)^T, (0, 0, dx_3)^T$, then before the deformation

$$dv = dx_1 \, dx_2 \, dx_3.$$

After deformation,

$$d\boldsymbol{R}^{(i)} = d\boldsymbol{r}^{(i)} + \frac{\partial \boldsymbol{u}}{\partial x_j} dr_j^{(i)},$$

giving

$$d\boldsymbol{R}^{(1)} = \begin{pmatrix} 1 + \frac{\partial u_1}{\partial x_1} \\ \frac{\partial u_2}{\partial x_1} \\ \frac{\partial u_3}{\partial x_1} \end{pmatrix} dx_1, \quad d\boldsymbol{R}^{(2)} = \begin{pmatrix} \frac{\partial u_1}{\partial x_2} \\ 1 + \frac{\partial u_2}{\partial x_2} \\ \frac{\partial u_3}{\partial x_2} \end{pmatrix} dx_2, \quad d\boldsymbol{R}^{(3)} = \begin{pmatrix} \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_3} \\ 1 + \frac{\partial u_3}{\partial x_3} \end{pmatrix} dx_3.$$

Then

$$dV = \left(d\boldsymbol{R}^{(1)} \times d\boldsymbol{R}^{(2)} \right) \cdot d\boldsymbol{R}^{(3)} = \left[1 + \frac{\partial u_i}{\partial x_i} + O\left(\left[\frac{\partial u_i}{\partial x_j} \right]^2 \right) \right] dx_1 \, dx_2 \, dx_3,$$

so that

$$\frac{dV - dv}{dv} = \frac{\partial u_i}{\partial x_i} = d.$$

3.) This is a straightforward exercise in index manipulation:

$$A = \frac{\partial \omega_{ik}}{\partial x_j} = \left[\frac{1}{2} (u_{i,k} - u_{k,i}) \right]_{,j} = \frac{1}{2} (u_{i,kj} - u_{k,ij}).$$

Now

$$B = \frac{\partial e_{ij}}{\partial x_k} - \frac{\partial e_{kj}}{\partial x_i} = \left[\frac{1}{2} (u_{i,j} + u_{j,i}) \right]_{,k} - \left[\frac{1}{2} (u_{k,j} + u_{j,k}) \right]_{,i} = \frac{1}{2} (u_{i,jk} + u_{j,ik} - u_{k,ji} - u_{j,ki}).$$

Using symmetry of partial differentiation, the second and fourth terms cancel and we have

$$B = \frac{1}{2} (u_{i,jk} - u_{k,ji}) = A.$$