4.1 Experimental determination of elastic constants

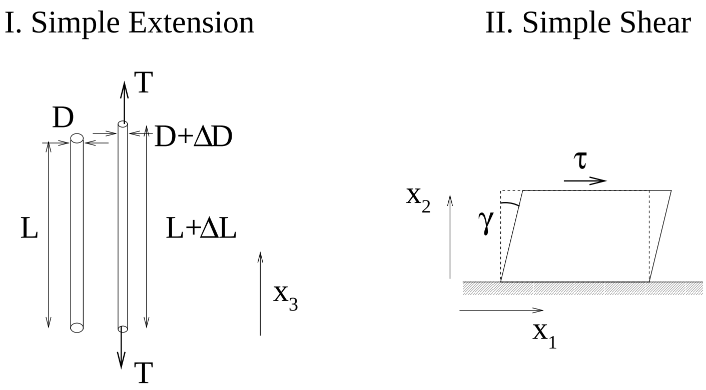

Figure 4.1: Sketch illustrating the two fundamental experiments for the determination of the elastic constants.

4.1.1 Experiment I: Simple extension of a thin cylinder

A circular cylinder of diameter D, cross-sectional area A and initial length L is subject to an applied tension T along its axis which is chosen to lie in the x_3 direction. After the load has been applied the cylinder has length $L + \Delta L$ and diameter $D + \Delta D$.

The extension is in the x_3 direction and so $e_{33} = \Delta L/L \ll 1$. The change in diameter is in the x_1 and x_2 directions, so $e_{11} = e_{22} = \Delta D/D$. Moreover, the stress is uniaxial (along one axis, x_3) so $\tau_{33} = T/A$, and $\tau_{ij} = 0$ otherwise.

- Observations:
  - Rod extends in the x_3 direction in proportion to the applied tension:

$$\tau_{33} = E e_{33}, \quad \text{where } E \text{ is the Elastic (or Young's) modulus.} \tag{4.1}$$

  - Rod cross-section decreases (contracts in plane) in proportion to its extension in the axial direction:

$$e_{11} = e_{22} = -\nu\, e_{33}, \quad \text{where } \nu \text{ is the Poisson ratio.} \tag{4.2}$$

4.1.2 Experiment II: Simple shear

A block in the x_1-x_2 plane is subject to an applied traction, τ, on its upper surface in the positive x_1 direction that induces a change in angle $\gamma = 2e_{12}$ between two sides. The body is in a state of simple shear so $\tau_{12} = \tau$ and $\tau_{ij} = 0$ otherwise.

- Observation:
  - The change in angle is proportional to the applied traction:

$$\tau = G\gamma \Rightarrow \tau_{12} = G\, 2e_{12}, \quad \text{where } G \text{ is the shear modulus of the material.} \tag{4.3}$$

4.2 Relating experiments (E, ν, G) and theory (λ, μ)

We shall now use the constitutive equations to relate the experimentally measured constants E, ν and G to the (theoretical) Lamé coefficients λ and μ. The material is homogeneous, isotropic and elastic so that

$$\tau_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu\, e_{ij}, \tag{4.4}$$

and, alternatively,

$$e_{ij} = \frac{1}{2\mu}\tau_{ij} - \frac{\lambda}{2\mu(3\lambda + 2\mu)}\delta_{ij}\tau_{kk}. \tag{4.5}$$

Experiment I

In this experiment, $\tau_{33} = Ee_{33}$ and $\Theta = \tau_{kk} = \tau_{33}$, because $\tau_{11} = \tau_{22} = 0$. If we set $i = j = 3$ in equation (4.5) we have

$$e_{33} = \frac{1}{2\mu}\tau_{33} - \frac{\lambda}{2\mu(3\lambda+2\mu)}\tau_{kk} = \left[\frac{1}{2\mu} - \frac{\lambda}{2\mu(3\lambda+2\mu)}\right]\tau_{33} = \left[\frac{3\lambda+2\mu-\lambda}{2\mu(3\lambda+2\mu)}\right]\tau_{33} = \frac{\lambda+\mu}{\mu(3\lambda+2\mu)}\tau_{33}.$$

Hence,

$$\frac{1}{E} = \frac{\lambda+\mu}{\mu(3\lambda+2\mu)} \quad \Rightarrow \quad \boxed{E = \tfrac{\mu(3\lambda+2\mu)}{\lambda+\mu}.} \tag{4.6}$$

If we substitute $i = j = 1$ into equation (4.5), we obtain

$$e_{11} = \frac{1}{2\mu}\tau_{11} - \frac{\lambda}{2\mu(3\lambda+2\mu)}\tau_{33} = -\frac{\lambda}{2\mu(3\lambda+2\mu)}\tau_{33}, \qquad \text{because } \tau_{11} = 0.$$

Using the first experimental observation, $\tau_{33} = Ee_{33}$, and the expression (4.6) for E gives

$$e_{11} = -\frac{\lambda}{2\mu(3\lambda+2\mu)}\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}e_{33} = -\frac{\lambda}{2(\lambda+\mu)}\,e_{33}.$$

The second experimental observation is $e_{11} = -\nu\, e_{33}$, so direct comparison yields

$$\boxed{\nu = \tfrac{\lambda}{2(\lambda+\mu)}.} \tag{4.7}$$

Note that an identical result is obtained on substitution of $i = j = 2$ into equation (4.5).

Experiment II

In this experiment, $\tau_{12} = 2Ge_{12}$, and using equation (4.4) with $i = 1$ and $j = 2$, we have

$$\tau_{12} = \lambda\delta_{12}e_{kk} + 2\mu\, e_{12} = 2\mu, e_{12} \quad \Rightarrow \quad \boxed{\mu = G.} \tag{4.8}$$

4.3 Relations between the elastic constants

Note that only two elastic constants are independent, see equations (4.4, 4.5). The following table lists the relationships between elastic coefficients for the most commonly used independent pairs.

Independent pair	$\lambda =$	$\mu = G =$	$E =$	$\nu =$
λ, μ	λ	μ	$\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$	$\frac{\lambda}{2(\lambda+\mu)}$
λ, ν	λ	$\frac{\lambda(1-2\nu)}{2\nu}$	$\frac{(1+\nu)(1-2\nu)\lambda}{\nu}$	ν
μ, E	$\frac{\mu(E-2\mu)}{3\mu-E}$	μ	E	$\frac{E-2\mu}{2\mu}$
E, ν	$\frac{E\nu}{(1+\nu)(1-2\nu)}$	$\frac{E}{2(1+\nu)}$	E	ν

4.3.1 Constitutive equations in terms of the experimentally measurable constants

Using the table above, we can rewrite the equation (4.4) and (4.5) using the Young's modulus E and Poisson ratio ν

$$\tau_{ij} = \frac{E}{1+\nu}\left(e_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\underbrace{e_{kk}}_{d}\right). \tag{4.9}$$

and

$$e_{ij} = \frac{1}{E}\left((1+\nu)\tau_{ij} - \nu\,\delta_{ij}\,\underbrace{\tau_{kk}}_{\Theta}\right). \tag{4.10}$$

For physically realisable materials,

$$E \geq 0, \quad -1 < \nu \leq \frac{1}{2}, \quad \mu \geq 0, \quad 3\lambda + 2\mu > 0.$$

Note that equation (4.10) implies that

$$d = e_{kk} = \frac{\tau_{kk}}{E}\left((1+\nu) - 3\nu\right) = \frac{\Theta}{E}\left(1 - 2\nu\right),$$

and so $d = 0$, when $\nu = 1/2$. Materials for which $\nu = 1/2$ are thus said to be *incompressible*. Care must be taken when considering the limit $\nu \to 1/2$ in equation (4.9). In fact, the distinguished limit in question is $\nu \to 1/2$ **and** $d \to 0$, in such a way that the term $d/(1-2\nu)$ remains finite.

Chapter 5

The governing equations of linear elasticity

5.1 Navier–Lamé equations — displacement formulation

The constitutive equation for an isotropic, homogeneous linear elastic body is

$$\tau_{ij} = \lambda\,\delta_{ij}\,e_{kk} + 2\mu\,e_{ij}, \tag{5.1}$$

where λ and μ are the (experimentally determined) Lamé coefficients, τ_{ij} is the stress tensor and e_{ij} is the strain tensor defined by

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right). \tag{5.2}$$

Here, u_i is the component of the displacement vector, $\boldsymbol{u}$, in the x_i direction. Using equation (5.2) in equation (5.1) we obtain the an expression for the stresses in terms of the displacements

$$\tau_{ij} = \lambda\,\delta_{ij}\,u_{k,k} + \mu\left(u_{i,j} + u_{j,i}\right). \tag{5.3}$$

The equations of motion for an elastic body are

$$\rho\ddot{u}_i = \tau_{ij,j} + F_i, \tag{5.4}$$

where ρ is the density of the body and $\boldsymbol{F}$ is the body force per unit volume.

Now, taking the derivative of equation (5.3) gives

$$\tau_{ij,j} = \lambda\,\delta_{ij}\,u_{k,kj} + \mu\left(u_{i,jj} + u_{j,ij}\right) = \lambda\,u_{k,ki} + \mu\,u_{i,jj} + \mu u_{j,ji} = (\lambda+\mu)u_{k,ki} + \mu\,u_{i,jj},$$

by using the symmetry of the stress tensor and the index-switching property of the Kronecker delta. Using the last result in equation (5.4), we obtain the Navier–Lamé equations

$$\boxed{\rho\ddot{u}_i = (\lambda+\mu)\,u_{k,ki} + \mu\,u_{i,jj} + F_i.} \tag{5.5}$$

The Navier–Lamé equations may be written in the equivalent form

$$\rho\ddot{\boldsymbol{u}} = (\lambda+\mu)\,\mathrm{grad}(\mathrm{div}\,\boldsymbol{u}) + \mu\,\nabla^2\boldsymbol{u} + \boldsymbol{F},$$

or even

$$\rho\ddot{\boldsymbol{u}} = (\lambda+2\mu)\,\mathrm{grad}(\mathrm{div}\,\boldsymbol{u}) - \mu\,\mathrm{curl}(\mathrm{curl}\,\boldsymbol{u}) + \boldsymbol{F},$$

by using the vector identity

$$\nabla^2\boldsymbol{u} = \mathrm{grad}(\mathrm{div}\,\boldsymbol{u}) - \mathrm{curl}(\mathrm{curl}\,\boldsymbol{u}).$$

Note

- The general case of a non-zero body force is not very different from the "special" case $\boldsymbol{F} = \boldsymbol{0}$ because the Navier–Lamé equations are linear. $\boldsymbol{F}$ acts as an inhomogeneity that can be removed if a suitable particular solution is found. We seek a function $\hat{\boldsymbol{u}}$ such that

$$(\lambda + \mu)\, \boldsymbol{\nabla}\boldsymbol{\nabla} \cdot \hat{\boldsymbol{u}} + \mu\, \nabla^2 \hat{\boldsymbol{u}} + \boldsymbol{F} = 0,$$

irrespective of the boundary conditions. If we then define the displacement field to be $\boldsymbol{u} = \tilde{\boldsymbol{u}} + \hat{\boldsymbol{u}}$, then $\tilde{\boldsymbol{u}}$ satisfies the homogeneous equation

$$(\lambda + \mu)\, \boldsymbol{\nabla}\boldsymbol{\nabla} \cdot \tilde{\boldsymbol{u}} + \mu\, \nabla^2 \tilde{\boldsymbol{u}} = 0,$$

but the boundary conditions may have changed from the original problem.

5.2 Beltrami–Michell equations — stress formulation

The Navier–Lamé equations treat the displacements as the unknowns in elasticity problems; the strains and stresses are calculated indirectly, once the displacements are known. An alternative approach is to formulate the set of governing equations so that the stresses are the unknowns and are calculated directly. We can then use the constitutive law to determine the strains and hence the displacements. There is a potential problem, however, the displacements cannot be uniquely recovered from the strain field unless the strain compatibility conditions are satisfied.

$$e_{ij,kl} + e_{kl,ij} - e_{kj,il} - e_{il,kj} = 0. \tag{5.6}$$

The inverse form of the linear constitutive law (5.1) is

$$e_{ij} = \frac{1}{2\mu}\, \tau_{ij} - \frac{\lambda}{2\mu(3\lambda + 2\mu)} \delta_{ij} \tau_{kk},$$

and using the engineering (experimental) constants E and ν, we obtain a slightly neater expression

$$\frac{E}{1+\nu} e_{ij} = \tau_{ij} - \frac{\nu}{1+\nu} \delta_{ij} \Theta. \tag{5.7}$$

Substituting the expression (5.7) into the strain compatibility equations (5.6) yields

$$\tau_{ij,kl} + \tau_{kl,ij} - \tau_{kj,il} - \tau_{il,kj} - \frac{\nu}{1+\nu}\Big[\delta_{ij}\Theta_{,kl} + \delta_{kl}\Theta_{,ij} - \delta_{kj}\Theta_{,il} - \delta_{il}\Theta_{,kj}\Big] = 0.$$

All the terms in the above expression are second derivatives of τ_{ij}, so that the equations are all satisfied if the stress is a **linear** function of position, in which case every term is zero. Remember that only six of these 81 equations are linearly independent, so we are free to combine equations by setting $k = l$ and summing over the repeated index to obtain nine equations, of which six are still independent through the symmetry of the stress tensor,

$$\tau_{ij,kk} + \tau_{kk,ij} - \tau_{kj,ik} - \tau_{ik,kj} - \frac{\nu}{1+\nu}\left[\delta_{ij}\Theta_{,kk} + \delta_{kk}\Theta_{,ij} - \delta_{kj}\Theta_{,ik} - \delta_{ik}\Theta_{,kj}\right] = 0.$$

Using the fact that $\tau_{kk} = \Theta$, $\delta_{kk} = 3$ and the index-switching property of the Kronecker delta, the equations become

$$\tau_{ij,kk} + \Theta_{,ij} - \tau_{kj,ik} - \tau_{ik,kj} - \frac{\nu}{1+\nu}\left[\delta_{ij}\Theta_{,kk} + 3\Theta_{,ij} - \Theta_{,ij} - \Theta_{,ij}\right] = 0,$$

$$\Rightarrow \tau_{ij,kk} + \Theta_{,ij} - \tau_{kj,ik} - \tau_{ik,kj} - \frac{\nu}{1+\nu}\left[\delta_{ij}\Theta_{,kk} + \Theta_{,ij}\right] = 0,$$

$$\Rightarrow \tau_{ij,kk} + \frac{1}{1+\nu}\,\Theta_{,ij} - \frac{\nu}{1+\nu}\,\Theta_{,kk}\delta_{ij} - \tau_{kj,ik} - \tau_{ik,kj} = 0. \tag{5.8}$$

The equation (5.8) can be written in the alternative form

$$\nabla^2 \tau_{ij} + \frac{1}{1+\nu}\,\Theta_{,ij} - \frac{\nu}{1+\nu}\,\delta_{ij}\nabla^2\Theta = \tau_{kj,ik} + \tau_{ik,kj}. \tag{5.9}$$

Differentiating the equations of static equilibrium $\tau_{ij,j} + F_i = 0$, we obtain $\tau_{ij,jk} + F_{i,k} = 0$, which can be used to replace the terms on the right-hand side of equation (5.9).

$$\nabla^2 \tau_{ij} + \frac{1}{1+\nu}\,\Theta_{,ij} - \frac{\nu}{1+\nu}\,\delta_{ij}\nabla^2\Theta = -F_{j,i} - F_{i,j}. \tag{5.10}$$

Setting $i = j$ in equation (5.10) gives

$$\nabla^2 \tau_{ii} + \frac{1}{1+\nu}\,\Theta_{,ii} - \frac{\nu}{1+\nu}\,3\nabla^2\Theta = -2F_{i,i} \quad \Rightarrow \nabla^2\Theta + \frac{1}{1+\nu}\nabla^2\Theta - \frac{3\nu}{1+\nu}\nabla^2\Theta = -2F_{i,i}$$

$$\Rightarrow \frac{1-\nu}{1+\nu}\,\nabla^2\Theta = -F_{i,i}; \tag{5.11}$$

and on substitution of (5.11) into equation (5.10) we finally obtain the Beltrami–Michell (compatibility) equations

$$\boxed{\nabla^2 \tau_{ij} + \tfrac{1}{1+\nu}\,\Theta_{,ij} + \tfrac{\nu}{1-\nu}\,\delta_{ij}F_{k,k} + F_{i,j} + F_{j,i} = 0.} \tag{5.12}$$

These are stress compatibility equations and we note that in unsteady problems, we must add the inertia force $-\rho\ddot{u}_i$ to F_i.

The Beltrami–Michell equations are six linear partial differential equations for the six independent components of the stress tensor. As one might expect, these equations are (very) useful if the problem has stress boundary conditions. The equations are less useful if displacement boundary conditions are specified.

Notes

- The equations of equilibrium are automatically satisfied by construction.
- The strains e_{ij} can be recovered from the constitutive equation.
- Once the strain field is known, the displacements can be obtained by integrating

  $$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right).$$

  Remember that the constants of integration represent rigid-body motions.
- The above integration is possible because we started from the strain compatibility equations and, therefore, if the Beltrami–Michell equations are satisfied so are the strain compatibility equations; and hence, the body will be continuous.

5.3 The steady, constant-body-force case

If the body force is a constant and there are no unsteady terms, then all of the derivatives $F_{i,j}$ will be zero. The Beltrami–Michell equations become

$$\nabla^2 \tau_{ij} + \frac{1}{1+\nu}\Theta_{,ij} = 0,$$

and from equation (5.11) we have

$$\nabla^2\Theta = 0.$$

Thus, Θ is a harmonic function: a function that satisfies Laplace's equation. Recall that $\Theta = (3\lambda + 2\mu)\,d$, which implies that

$$\nabla^2 d = 0,$$

and so $d = e_{kk} = u_{k,k}$ is also a harmonic function. If we apply the operator ∇^2 to the (steady) Navier–Lamé equations (5.5) we obtain

$$(\lambda + \mu)\,u_{k,kijj} + \mu\,u_{i,kkjj} = 0.$$

The first term may be written as $(u_{k,kjj})_{,i}$ and because $u_{k,k}$ is harmonic then $u_{k,kjj} = 0$ and hence the first term is zero. Thus,

$$u_{i,kkjj} = 0, \quad \text{or rather} \quad \boxed{\nabla^4 \boldsymbol{u} = 0,}$$

and so the displacement field is a solution of the biharmonic equation. Furthermore, by using the definition of the strain tensor (5.2) and the expression for the stress tensor in terms of the displacements (5.3), it follows that

$$\boxed{\nabla^4 \tau_{ij} = 0, \quad \text{and} \quad \nabla^4 e_{ij} = 0;}$$

the stress and strain components are both biharmonic functions.

5.4 Alternative coordinate systems:

5.4.1 Governing Equations in Cylindrical Polar Coordinates

- $x_1 = x = r\cos\theta$, $x_2 = y = r\sin\theta$, $x_3 = z = z$.

$$\mathbf{u} = (u_r, u_\theta, u_z), \quad e = (e_{ij}), \quad \tau = (\tau_{ij}), \quad \text{where } i, j = r, \theta, z.$$

- Vector calculus:

$$\operatorname{grad} f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}, \qquad \operatorname{div}\mathbf{u} = \frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z},$$

$$\operatorname{curl}\mathbf{u} = \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\hat{\boldsymbol{\theta}} + \left(\frac{1}{r}\frac{\partial (r u_\theta)}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right)\hat{\mathbf{z}}.$$

- Stress-strain relations have the same form as in Cartesian coordinates:

$$\tau_{ij} = \lambda\delta_{ij}\operatorname{div}\mathbf{u} + 2\mu e_{ij}, \qquad i, j = r, \theta, z.$$

- Stress-displacement relations:

$$\tau_{rr} = \lambda\operatorname{div}\mathbf{u} + 2\mu\frac{\partial u_r}{\partial r}, \quad \tau_{\theta\theta} = \lambda\operatorname{div}\mathbf{u} + 2\mu\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right), \quad \tau_{zz} = \lambda\operatorname{div}\mathbf{u} + 2\mu\frac{\partial u_z}{\partial z},$$

$$\frac{\tau_{r\theta}}{\mu} = \frac{\tau_{\theta r}}{\mu} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}, \quad \frac{\tau_{rz}}{\mu} = \frac{\tau_{zr}}{\mu} = \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}, \quad \frac{\tau_{\theta z}}{\mu} = \frac{\tau_{z\theta}}{\mu} = \frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}.$$

- Strain-displacement relations:

$$e_{rr} = \frac{\partial u_r}{\partial r}, \qquad e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \qquad e_{zz} = \frac{\partial u_z}{\partial z},$$

$$2e_{r\theta} = 2e_{\theta r} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}, \quad 2e_{rz} = 2e_{zr} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}, \quad 2e_{z\theta} = 2e_{\theta z} = \frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}.$$

- Equilibrium equations (statics): for the displacement formulation, use Navier's equation,

$$(\lambda + 2\mu)\operatorname{grad}\operatorname{div}\mathbf{u} - \mu\operatorname{curl}\operatorname{curl}\mathbf{u} + \mathbf{F} = \mathbf{0},$$

whereas for the stress formulation, use

$$\begin{aligned}
\frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} + F_r &= 0 \\
\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2}{r}\tau_{r\theta} + F_\theta &= 0 \\
\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z} + \frac{1}{r}\tau_{rz} + F_z &= 0.
\end{aligned}$$

- Stress boundary conditions: these are when $\mathbf{t}$ is prescribed. We have, from $t_i = \hat{n}_j\tau_{ij}$,

$$\begin{aligned}
t_r &= \hat{n}_r\tau_{rr} + \hat{n}_\theta\tau_{r\theta} + \hat{n}_z\tau_{rz} \\
t_\theta &= \hat{n}_r\tau_{r\theta} + \hat{n}_\theta\tau_{\theta\theta} + \hat{n}_z\tau_{\theta z} \\
t_z &= \hat{n}_r\tau_{rz} + \hat{n}_\theta\tau_{\theta z} + \hat{n}_z\tau_{zz}
\end{aligned}$$

5.4.2 Governing Equations in Spherical Polar Coordinates

- $x_1 = x = r\sin\theta\,\cos\phi$, $x_2 = y = r\sin\theta\,\sin\phi$, $x_3 = z = r\cos\theta$.

$$\mathbf{u} = (u_r, u_\theta, u_\phi), \quad e = (e_{ij}), \quad \tau = (\tau_{ij}), \quad \text{where } i,j = r,\theta,\phi.$$

- Vector calculus:

$$\operatorname{grad} f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}},$$

$$\operatorname{div}\mathbf{u} = \frac{1}{r^2\sin\theta}\left\{\frac{\partial}{\partial r}(r^2\sin\theta\, u_r) + \frac{\partial}{\partial \theta}(r\sin\theta\, u_\theta) + \frac{\partial}{\partial \phi}(r u_\phi)\right\},$$

$$\operatorname{curl}\mathbf{u} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\,\hat{\boldsymbol{\phi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ u_r & r u_\theta & r\sin\phi\, u_\phi \end{vmatrix}.$$

- Stress-strain relations have the same form as in Cartesian coordinates:

$$\tau_{ij} = \lambda\delta_{ij}\operatorname{div}\mathbf{u} + 2\mu e_{ij}, \qquad i,j = r,\theta,\phi.$$

- Stress-displacement relations:

$$\tau_{rr} = \lambda\operatorname{div}\mathbf{u} + 2\mu\frac{\partial u_r}{\partial r}, \quad \tau_{\theta\theta} = \lambda\operatorname{div}\mathbf{u} + \frac{2\mu}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right),$$

$$\tau_{\phi\phi} = \lambda\operatorname{div}\mathbf{u} + \frac{2\mu}{r}\left(\frac{1}{\sin\theta}\frac{\partial u_\phi}{\partial \phi} + u_r + u_\theta\cot\theta\right), \quad \frac{\tau_{r\theta}}{\mu} = \frac{\tau_{\theta r}}{\mu} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta},$$

$$\frac{\tau_{r\phi}}{\mu} = \frac{\tau_{\phi r}}{\mu} = \frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}, \quad \frac{\tau_{\theta\phi}}{\mu} = \frac{\tau_{\phi\theta}}{\mu} = \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi} + \frac{1}{r}\frac{\partial u_\phi}{\partial \theta} - \frac{u_\phi\cot\theta}{r}.$$

- Strain-displacement relations:

$$e_{rr} = \frac{\partial u_r}{\partial r}, \qquad e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \qquad e_{\phi\phi} = \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} + \frac{u_\theta\cot\theta}{r},$$

$$2e_{r\theta} = 2e_{\theta r} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}, \quad 2e_{r\phi} = 2e_{\phi r} = \frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r},$$

$$2e_{\phi\theta} = 2e_{\theta\phi} = \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi} + \frac{1}{r}\frac{\partial u_\phi}{\partial \theta} - \frac{u_\phi\cot\theta}{r}.$$

- Equilibrium equations (statics): for the displacement formulation, use Navier's equation,

$$(\lambda + 2\mu)\operatorname{grad}\operatorname{div}\mathbf{u} - \mu\operatorname{curl}\operatorname{curl}\mathbf{u} + \mathbf{F} = \mathbf{0},$$

whereas for the stress formulation, use

$$\begin{aligned}
\frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \tau_{r\phi}}{\partial \phi} + \frac{2\tau_{rr} - \tau_{\theta\theta} - \tau_{\phi\phi} + \cot\theta\,\tau_{r\theta}}{r} + F_r &= 0 \\
\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \tau_{\theta\phi}}{\partial \phi} + \frac{3\tau_{r\theta} + (\tau_{\theta\theta} - \tau_{\phi\phi})\cot\theta}{r} + F_\theta &= 0 \\
\frac{\partial \tau_{r\phi}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\phi}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \tau_{\phi\phi}}{\partial \phi} + \frac{3\tau_{r\phi} + 2\tau_{\theta\phi}\cot\theta}{r} + F_\phi &= 0.
\end{aligned}$$

- Stress boundary conditions: these are when $\mathbf{t}$ is prescribed. We have, from $t_i = \hat{n}_j\tau_{ij}$,

$$\begin{aligned}
t_r &= \hat{n}_r\tau_{rr} + \hat{n}_\theta\tau_{r\theta} + \hat{n}_\phi\tau_{r\phi} \\
t_\theta &= \hat{n}_r\tau_{r\theta} + \hat{n}_\theta\tau_{\theta\theta} + \hat{n}_\phi\tau_{\theta\phi} \\
t_\phi &= \hat{n}_r\tau_{r\phi} + \hat{n}_\theta\tau_{\theta\phi} + \hat{n}_\phi\tau_{\phi\phi}
\end{aligned}$$