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Bicoprime Factor Stability Criteria and Uncertainty Characterisation ⋆

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Abstract: Bicoprime factorisations can be considered a generalisation of the well known coprime factorisations. This paper deals with the possible applications of such factorisations in control theory. The concept of minimal dimension bicoprime factorisations is introduced and shown to be especially beneficial in the case of normal rank deficient and redundant control systems. Some methods of obtaining a bicoprime factorisation for a plant are given based on state-space data, right or left coprime factorisations and left or right standard factorisations. Finally, bicoprime factor uncertainty is characterised and shown to have an interesting structure closely resembling the standard four-block uncertainty.

Keywords: Factorization methods, coprime factorizations, stability criteria, redundant control, uncertainty

1. INTRODUCTION

Coprime factorisations over $\mathscr{RH}_\infty$ have been extensively studied by the control community. They have many applications in control theory, forming the basis of many important results. For example, distance measures such as the gap, graph and ν-gap metrics can be posed as normalised coprime factor model matching problems as demonstrated in Georgiou and Smith (1990); Vidyasagar (1984) and Vinnicombe (2001). Additionally, solutions to the $\mathscr{H}_\infty$ loopshaping robust stabilisation problem use normalised coprime factorisations of the plant (Glover and McFarlane, 1989). Coprime factorisations have also been used to validate controllers for internal stability and robust performance using closed loop data (Dehghani et al., 2009; Patra and Lanzon, 2012).

Bicoprime factorisations (BCFs) were first introduced by Vidyasagar (2011) [a] as a generalisation of standard left or right coprime factorisations. This fact will be exemplified with the derivation of bicoprime (BC) factor stability results in Section 4. Briefly studied in the late 1980's, some results were derived including stability of the feedback interconnection of a plant given as a BC treble and a controller expressed as either a left coprime factorisation (LCF) or right coprime factorisation (RCF) (Desoer and Gündeş, 1988; Gündeş and Desoer, 1990). This was achieved by transforming the given BCF into a RCF or LCF and using existing results. However, those early results are far from a comprehensive study of the subject matter.

Two motivating points for the study of BCFs given in Vidyasagar (2011) are as follows. First, they naturally emerge in closed loop transfer matrices. Indeed, most proofs in this paper use this fact to establish their results. Second, any minimal state-space realisation of a plant is a BCF over $\mathscr{R}$. This follows directly from the well known Popov-Belevitch-Haututs tests. In fact, a number of factorisations can be related to BCFs over different spaces. Other examples include spectral and Wiener-Hopf factorisations which can be considered as BCFs over $\mathscr{RL}_\infty$ with some extra impositions on the factors. Further to these points, it can also be argued that BCFs, being a generalised version of RCFs and LCFs, provide a link between the two, unifying any results. The duality of LCFs and RCFs has been well established for some time now, with BCFs demonstrating how this duality arises.

BCFs received some attention in the study of decentralized control. Ünyelioğlu et al. (2000) shows that BC factors can be used to characterise fixed zeros in decentralized control and by extent deduce the existence of a decentralized stabilising controller for a given plant. Additionally, Baski et al. (1999) presents methods for the decentralized stabilisation of plants using BCFs.

Another possible application for BCFs that emerges from the results of this paper is in the area of redundant control systems. Using more actuators or sensors than needed for the purposes of fault tolerance can lead to a rank deficient system. The use of BCFs can be beneficial to the study of such systems. Using a BCF stability test leads to reduced dimension stability matrices. It will be shown by example that in some instances the stability of a closed loop transfer matrix can be established by the invertibility in $\mathscr{RH}_\infty$ of a single SISO transfer function or equivalently if it has any right half plane zeros.

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[a] This is a reprint of the 1985 original published by M.I.T. Press.

There is a considerable amount of mathematical richness associated with BCFs that so far remains untapped by the control community. Moreover, most results pertaining to BCFs exhibit a considerable amount of mathematical symmetry which makes them a very appealing subject of study. As mentioned above BCFs are a generalised version of coprime factorisations. This leads to more complex mathematical results than in the classical case. However, this complexity should not serve as a deterrent as the potential advancements to control theory necessitate further investigation.

2. PRELIMINARIES

This section defines the notation that will be used throughout this paper and recalls some standard, well-known results, mostly related to stability and coprime factorisations.

2.1 Notation

$\mathscr{R}$	Set of proper real-rational transfer matrices
$\mathscr{RH}_\infty$	Set of proper real-rational stable transfer matrices
$\mathscr{GH}_\infty$	$\{Q \in \mathscr{RH}_\infty : Q^{-1} \in \mathscr{RH}_\infty\}$
$\mathrm{r}\,(A)$	Rank of the matrix $A \in \mathbb{C}^{m\times n}$
$\mathrm{nr}\,(P)$	Normal rank of the transfer matrix $P \in \mathscr{R}$
$\mathrm{diag}\,(\cdot)$	Block diagonal matrix starting from the top left
$\mathrm{adiag}\,(\cdot)$	Block anti-diagonal matrix starting from the top right
$A^\dagger$	Pseudo-inverse of $A \in \mathbb{C}^{m\times n}$
A^*	Complex conjugate transpose of $A \in \mathbb{C}^{m\times n}$
$\bar{s}$	Complex conjugate of $s \in \mathbb{C}$
$\mathbb{C}_{<0}$	$\{s : s \in \mathbb{C}, \Re(s) < 0\}$
$\mathbb{C}_{\leq 0}$	$\{s : s \in \mathbb{C}, \Re(s) \leq 0\}$
$\mathbb{C}_{>0}$	$\{s : s \in \mathbb{C}, \Re(s) > 0\}$
$\mathbb{C}_{\geq 0}$	$\{s : s \in \mathbb{C}, \Re(s) \geq 0\}$
$\mathcal{F}_l(H,\Delta)$	Lower LFT of H with respect to Δ
$\mathcal{F}_u(H,\Delta)$	Upper LFT of H with respect to Δ

2.2 Stability

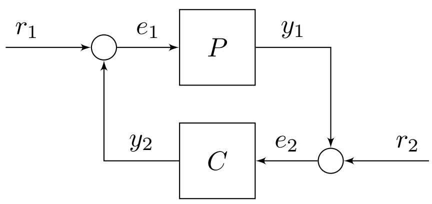

Fig. 1. Standard feedback interconnection of P and C.

The feedback interconnection of a plant $P \in \mathscr{R}$ and controller $C \in \mathscr{R}$, shown in Figure 1 and denoted by $[P, C]$, is well posed if all transfer matrices from (r_1, r_2) to (e_1, e_2) are well-defined and proper. A necessary and sufficient condition for this to be true is $\det(I - PC)(\infty) \neq 0$ (Zhou et al., 1996, Lemma 5.1). The transfer matrix from $(-r_2, r_1)$ to (y_1, e_1) is denoted by $H(P, C)$.

Lemma 1. (Zhou et al. (1996) Lemma 5.3). The feedback interconnection of a plant $P \in \mathscr{R}$ and controller $C \in \mathscr{R}$ is internally stable if and only if

$$\begin{bmatrix} I & -C \\ -P & I \end{bmatrix}^{-1} \in \mathscr{RH}_\infty$$

or equivalently

$$H(P, C) = \begin{bmatrix} P \\ I \end{bmatrix} (I - CP)^{-1} \begin{bmatrix} -C & I \end{bmatrix} \in \mathscr{RH}_\infty.$$

Lemma 2. Let $A = \left[\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{smallmatrix}\right] \in \mathscr{RH}_\infty$ and suppose that $A_{22} \in \mathscr{GH}_\infty$. Then

$$A \in \mathscr{GH}_\infty \Leftrightarrow A_{11} - A_{12}A_{22}^{-1}A_{21} \in \mathscr{GH}_\infty.$$

Proof. From the Schur complement decomposition

$$A = \begin{bmatrix} I & A_{12}A_{22}^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & 0 \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ A_{22}^{-1}A_{21} & I \end{bmatrix}.$$

Since by supposition $A_{22} \in \mathscr{GH}_\infty$,

$$\begin{bmatrix} I & A_{12}A_{22}^{-1} \\ 0 & I \end{bmatrix}, \begin{bmatrix} I & 0 \\ A_{22}^{-1}A_{21} & I \end{bmatrix} \in \mathscr{GH}_\infty$$

and the result follows. □

2.3 Coprime factorisations

Definition 1. (Zhou et al. (1996) Definition 5.3). The pair $\{N, M\}$ is right coprime (RC) in $\mathscr{RH}_\infty$ if $N, M \in \mathscr{RH}_\infty$ and there exist $Y_r, Z_r \in \mathscr{RH}_\infty$ such that $Z_rM + Y_rN = I$. Furthermore, the pair is a RCF of a plant P over $\mathscr{RH}_\infty$ if M is square, $\det M(\infty) \neq 0$ and $P = NM^{-1}$.

Definition 2. (Zhou et al. (1996) Definition 5.3). The pair $\{L, M\}$ is left coprime (LC) in $\mathscr{RH}_\infty$ if $M, L \in \mathscr{RH}_\infty$ and there exist $Y_l, Z_l \in \mathscr{RH}_\infty$ such that $MZ_l + LY_l = I$. Furthermore, the pair is a LCF of a plant P over $\mathscr{RH}_\infty$ if M is square, $\det M(\infty) \neq 0$ and $P = M^{-1}L$.

Definition 3. The set of all RC (resp. LC) pairs in $\mathscr{RH}_\infty$ is defined as $\mathscr{C}_r$ (resp. $\mathscr{C}_l$). Similarly, the set of all RCFs (resp. LCFs) of a plant P over $\mathscr{RH}_\infty$ is defined as $\mathscr{C}_r(P)$ (resp. $\mathscr{C}_l(P)$).

Definition 4. Let $\{N, M\} \in \mathscr{C}_r$ and $\{L, M\} \in \mathscr{C}_l$. The associated Bézout factor sets are defined as [b]

$$\mathscr{C}^\dagger \left[\begin{smallmatrix} M \\ N \end{smallmatrix}\right] = \{\{Y_r, Z_r\} : Y_r, Z_r \in \mathscr{RH}_\infty, Z_rM + Y_rN = I\},$$
$$\mathscr{C}^\dagger \left[\begin{smallmatrix} M & L \end{smallmatrix}\right] = \{\{Y_l, Z_l\} : Y_l, Z_l \in \mathscr{RH}_\infty, MZ_l + LY_l = I\}.$$

Some well known coprime factor stability results are listed in following lemma.

Lemma 3. (Zhou et al. (1996) Lemma 5.2). Let $\{N, M\} \in \mathscr{C}_r(P)$, $\{L, \tilde{M}\} \in \mathscr{C}_l(P)$, $\{U, V\} \in \mathscr{C}_r(C)$ and $\{W, \tilde{V}\} \in \mathscr{C}_l(C)$. Then the following statements are equivalent.

(1) $[P, C]$ is internally stable
(2) $\tilde{M}V - LU \in \mathscr{GH}_\infty$
(3) $\tilde{V}M - WN \in \mathscr{GH}_\infty$
(4) $\begin{bmatrix} M & U \\ N & V \end{bmatrix} \in \mathscr{GH}_\infty$
(5) $\begin{bmatrix} \tilde{M} & -L \\ -W & \tilde{V} \end{bmatrix} \in \mathscr{GH}_\infty$

3. BICOPRIME FACTORISATIONS

In their original definition, BCFs of a plant were presented as a quad of objects in $\mathscr{RH}_\infty$.

[b] The use of the pseudo-inverse symbol (†) is appropriate in the above definition since $[Z_r\ Y_r]$ is the left inverse of $[M^*\ N^*]^*$ and $[Z_l^*\ Y_l^*]^*$ is the right inverse of $[M\ L]$.

Definition 5. (Vidyasagar (2011) Definition 4.3.1).
The quad $\{N, M, L, K\}$ is BC in $\mathscr{RH}_\infty$ if $\{L, M\} \in \mathscr{C}_l$, $\{N, M\} \in \mathscr{C}_r$ and the additive term $K \in \mathscr{RH}_\infty$. The set of all BC quads in $\mathscr{RH}_\infty$ is defined as $\mathscr{B}$. Furthermore, the quad is a BCF of a plant P over $\mathscr{RH}_\infty$ if M is square, $\det M(\infty) \neq 0$ and $P = NM^{-1}L + K$. The set of all BCFs of a plant P is defined as $\mathscr{B}(P)$.

In most instances in this paper, the additive term is assumed to be zero as it is not a necessary part of the factorisation and complicates results. This is also the approach taken in Desoer and Gündeş (1988) and Gündeş and Desoer (1990).

Lemma 4. Let $\{N, M, L, K\} \in \mathscr{B}$ and $Q, R, S, T \in \mathscr{RH}_\infty$. Then $\{N - QM, M - LSN, L - MR, K + T\} \in \mathscr{B}$ if $[Q, LS]$ and $[SN, R]$ are internally stable.

Proof.

$$\begin{aligned}
&\{N - QM, M - LSN\} \in \mathscr{C}_r \\
&\Leftrightarrow \exists \tilde{Y}_r, \tilde{Z}_r \in \mathscr{RH}_\infty : \tilde{Z}_r(M - LSN) + \tilde{Y}_r(N - QM) = I \\
&\Leftrightarrow \exists \tilde{Y}_r, \tilde{Z}_r \in \mathscr{RH}_\infty : \begin{bmatrix}\tilde{Z}_r & \tilde{Y}_r\end{bmatrix} \begin{bmatrix} I & -LS \\ -Q & I \end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix} = I \\
&\Leftrightarrow \exists \tilde{Y}_r, \tilde{Z}_r \in \mathscr{RH}_\infty : \begin{bmatrix}\tilde{Z}_r & \tilde{Y}_r\end{bmatrix} \begin{bmatrix} I & -LS \\ -Q & I \end{bmatrix} \in \mathscr{C}^\dagger \left[\begin{smallmatrix} M \\ N \end{smallmatrix}\right] \\
&\Leftarrow \begin{bmatrix} I & -LS \\ -Q & I \end{bmatrix} \in \mathscr{GH}_\infty \\
&\Leftrightarrow [Q, LS] \text{ is internally stable.}
\end{aligned}$$

An alternative proof is provided for the LC pair.

$$\begin{aligned}
\{L - MR, M - LSN\} &\in \mathscr{C}_l \\
&\Leftrightarrow \mathscr{C}^\dagger \left[\begin{smallmatrix} M - LSN & L - MR \end{smallmatrix}\right] \neq \emptyset \\
&\Leftrightarrow \mathscr{C}^\dagger \left(\left[\begin{smallmatrix} M & L \end{smallmatrix}\right] \left[\begin{smallmatrix} I & -R \\ -SN & I \end{smallmatrix}\right]\right) \neq \emptyset \\
&\Leftarrow \begin{bmatrix} I & -R \\ -SN & I \end{bmatrix} \in \mathscr{GH}_\infty \\
&\Leftrightarrow [R, SN] \text{ is internally stable.}
\end{aligned}$$

Finally, since $K + T \in \mathscr{RH}_\infty$ the conclusion follows. □

In contrast to classical RCFs and LCFs, the BC factors of a plant can, by their definition, have different dimensions. This necessitates the following definition.

Definition 6. The internal dimension of $\{N, M, L, K\} \in \mathscr{B}$ is defined as the number of rows/columns of M. The set of all BCFs of a plant P of internal dimension n is defined as $\mathscr{B}_n(P)$.

Lemma 5. Suppose that $\{N, M, L, 0\} \in \mathscr{B}_r(P)$, then the BCFs internal dimension r can be arbitrarily large and satisfies $\mathrm{nr}(P) \leq r$.

Proof.
(r arbitrarily large)
Let $\{N, M, L, 0\} \in \mathscr{B}(P)$ and define

$$\hat{N} := [N \ 0], \hat{M} := \mathrm{diag}(M, I), \hat{L} := [L^* \ 0]^*.$$

Furthermore, let $\{Y_l, Z_l\} \in \mathscr{C}^\dagger \left[\begin{smallmatrix} M & L \end{smallmatrix}\right]$ and $\{Y_r, Z_r\} \in \mathscr{C}^\dagger \left[\begin{smallmatrix} M \\ N \end{smallmatrix}\right]$. Then

$$\left(\begin{bmatrix} Z_l^* & 0 & Y_l^* \\ 0 & I & 0 \end{bmatrix} \begin{bmatrix} \hat{M}^* \\ \hat{L}^* \end{bmatrix}\right)^* = I \text{ and } \begin{bmatrix} Z_r & 0 & Y_r \\ 0 & I & 0 \end{bmatrix} \begin{bmatrix} \hat{M} \\ \hat{N} \end{bmatrix} = I,$$

which confirms that $\{\hat{N}, \hat{M}, \hat{L}, 0\} \in \mathscr{B}(P)$. A BCF of P with arbitrary internal dimension r has therefore been constructed, which leads to the conclusion that there is no upper bound to the internal dimension of a plant BCF.

($\mathrm{nr}(P) \leq r$)
Suppose that $\{N, M, L, 0\} \in \mathscr{B}_r(P)$ where $r < \mathrm{nr}(P)$. Then $\mathrm{nr}\left(NM^{-1}L\right) \leq r < \mathrm{nr}(P)$ which is a contradiction since $P = NM^{-1}L$. Hence, r must be greater than or equal to the normal rank of the plant. □

3.1 Obtaining a BCF

Two simple methods for obtaining a BCF of a plant will now be presented. The first is based directly on state space data while the second uses a LCF or RCF of the plant.

Lemma 6. Let a plant $P \in \mathscr{R}^{p\times q}$ have the stabilisable and detectable state space realisation $P = C(sI - A)^{-1}B + D$, where $A \in \mathbb{R}^{n\times n}$ and the dimensions of B, C and D are compatible and dictated by those of the system. Furthermore define

$$M := (sI - \hat{A})^{-1}(sI - A), \; L := (sI - \hat{A})^{-1}B.$$

where $\hat{A} \in \mathbb{R}^{n\times n}$ is Hurwitz. Then $\{C, M, L, D\} \in \mathscr{B}_n(P)$.

Proof. The fact that $P = CM^{-1}L + D$ is trivial. The rest of the proof follows directly from the PBH tests and the fact that the state-space representation in stabilisable and detectable. First note that since $\hat{A}$ is Hurwitz, $\det(sI - \hat{A}) \neq 0$ for all $s \in \mathbb{C}_{\geq 0}$. Then, for all $s \in \mathbb{C}_{\geq 0}$,

$$\mathrm{r}([M \ L]) = \mathrm{r}\left((sI - \hat{A})^{-1}[sI - A \ B]\right) = \mathrm{r}([sI - A \ B])$$

which from the stabilisability of (A, B) has full row rank in $\mathbb{C}_{\geq 0}$. At the limiting case

$$\lim_{s\to\infty} \left((sI - \hat{A})^{-1}[sI - A \ B]\right) = [I \ 0]$$

which again has full row rank. This proves that $[M \ L]$ has a right inverse for all $s \in \mathbb{C}_{\geq 0} \cup \{\infty\}$ which implies $\{L, M\} \in \mathscr{C}_l$. The fact that $\{C, M\} \in \mathscr{C}_r$ follows by a similar argument based on the detectability of (C, A). □

Lemma 7. Let $P \in \mathscr{R}^{p\times q}$, $\{N, M\} \in \mathscr{C}_r(P)$ and $\{L, M\} \in \mathscr{C}_l(P)$. Then for any $Q \in \mathscr{GH}_\infty$ of compatible dimensions $\{N, QM, Q, 0\} \in \mathscr{B}_q(P)$ and $\{Q, MQ, L, 0\} \in \mathscr{B}_p(P)$.

3.2 Minimal Dimension BCFs

Definition 7. A BCF of a plant is said to be of minimal dimension if it has the minimum possible internal dimension. The set of all minimal dimension BCFs of a plant P is defined as $\mathscr{B}_*(P)$.

As shown in the previous section, a lower bound for the minimal dimension of a BCF is given by the normal rank of the plant. As a consequence of Lemma 7, an upper bound for the minimal dimension of a BCF is given by the number of inputs or outputs for tall or fat plants respectively.

Obtaining a minimal dimension BCF of a plant involves first finding a factorisation based on its normal rank. One such factorisation is the left standard factorisation (LSF) given in Youla (1961, def. 5), which is analogous to full rank factorisation (Piziak and Odell, 1999) of constant matrices.

Definition 8. Every plant $P \in \mathscr{R}^{p\times q}$ with normal rank r has a left standard factorisation $P = W_+W_-$ where

- W_+ has dimensions $p \times r$ and is analytic together with its left inverse in $\mathbb{C}_{>0}$.
- W_+ and its inverse have no singularities on $j\mathbb{R}$.
- W_- has dimensions $r \times q$ and is analytic together with its right inverse in $\mathbb{C}_{<0}$.

This definition deviates slightly from the source material of Youla (1961), combining the plant's two unstable factors into one. A right standard factorisation (RSF) can also be defined where the properties of the factors are exchanged.

From a LSF or RSF of a plant in combination with RCF and LCF, a BCF of a plant with internal dimension equal to its normal rank can be constructed as follows.

(1) Obtain plant LSF $P = W_+W_-$;
(2) Generate RCF of $W_+ = NM_+^{-1}$;
(3) Generate LCF of $W_- = M_-^{-1}L$;
(4) BCF of P given by $\{N, M_-M_+, L, 0\}$.

One problem with the above procedure is the fact that the factors of a LSF or RSF are not necessarily proper which means that formulae such as those given in Nett et al. (1984) or Vidyasagar (1988) cannot be used to obtain coprime factorisations. Fortunately, Oară and Sabău (2009) provide methods using descriptor theory to generate the required factorisations for improper systems. Another more inhibitive problem is the fact that there is no closed form solution for obtaining a LSF or RSF of a plant, despite knowing that such a factorisation always exists (as shown in Youla, 1961).

An appealing feature of minimal dimension BCFs is the fact that both N and L are invertible. Furthermore, there exist two subsets of $\mathscr{B}_*(P)$ for every plant in which either N or L has a stable inverse. This follows from the above procedure by noting that N has no $\mathbb{C}_{\geq 0}$ transmission zeros and has full column normal rank which together imply that it has a stable left inverse. [c]

4. BCF STABILITY RESULTS

The first stability result pertaining to BCFs links the stability of a plant to the $\mathbb{C}_{\geq 0}$ transmission zeros of its BC factors. [d]

Lemma 8. Let $\{N, M, L, K\} \in \mathscr{B}(P)$, then

$$P \in \mathscr{RH}_\infty \Leftrightarrow M \in \mathscr{GH}_\infty.$$

Proof. This follows from Vidyasagar (2011, Theorem 4.3.12) which states that $p \in \mathbb{C}_{\geq 0}$ is a pole of P if and only if it is a transmission zero of M. □

Using the above theorem it will now be shown that closed loop transfer matrices naturally give rise to BCFs. Let G and $\tilde{K}$ be the right and inverse left graph symbols of the plant and controller respectively, as defined in Vinnicombe (2001) and Lanzon and Papageorgiou (2009). Then

$$H(P, C) = \begin{bmatrix} P \\ I \end{bmatrix} (I - CP)^{-1} \begin{bmatrix} -C & I \end{bmatrix} = G(\tilde{K}G)^{-1}\tilde{K}.$$

Lemma 9. Let G and $\tilde{K}$ be the right and inverse left graph symbols of a plant and controller respectively. Then $\{G, \tilde{K}G, \tilde{K}, 0\} \in \mathscr{B}(H(P, C))$.

Proof. The proof follows by noting that there exists a left inverse of G and a right inverse of $\tilde{K}$ both in $\mathscr{RH}_\infty$. □

From Lemma 8 and Lemma 9 it can be concluded that $H(P, C) \in \mathscr{RH}_\infty \Leftrightarrow \tilde{K}G \in \mathscr{GH}_\infty$. Interestingly, this leads to statement (2) of Lemma 3. Similarly, statement (3) of the same lemma is a simple dual consequence on noting that $H(P, C) = I - K(\tilde{G}K)^{-1}\tilde{G}$ where $\tilde{G}$ and K are the inverse left and right graph symbols of the plant and controller respectively. [e]

The following theorem deals with the internal stability of a standard feedback interconnection in terms of plant and controller BCFs.

Theorem 10. Consider the standard feedback interconnection of a plant $P \in \mathscr{R}$ and controller $C \in \mathscr{R}$. Let $\{N, M, L, 0\} \in \mathscr{B}(P)$ and $\{U, V, W, 0\} \in \mathscr{B}(C)$. Then

$$[P, C] \text{ is internally stable} \Leftrightarrow \begin{bmatrix} M & -LU \\ -WN & V \end{bmatrix} \in \mathscr{GH}_\infty.$$

Proof. For simplicity, first define $\tilde{M} := \operatorname{adiag}(M, V)$, $\tilde{N} := \operatorname{diag}(U, N)$ and $\tilde{L} := \operatorname{diag}(L, W)$ and note that $\{\tilde{N}, \tilde{M}, \tilde{L}, 0\} \in \mathscr{B}$. Then

$$\begin{aligned} &[P, C] \text{ is internally stable} \\ &\Leftrightarrow \begin{bmatrix} I & -C \\ -P & I \end{bmatrix}^{-1} \in \mathscr{RH}_\infty \\ &\Leftrightarrow (I - \tilde{N}\tilde{M}^{-1}\tilde{L})^{-1} \in \mathscr{RH}_\infty \\ &\Leftrightarrow I + \tilde{N}(\tilde{M} - \tilde{L}\tilde{N})^{-1}\tilde{L} \in \mathscr{RH}_\infty. \end{aligned}$$

Using the fact that $\{\tilde{N}, \tilde{M}, \tilde{L}, 0\} \in \mathscr{B}$ and Lemma 4 implies that $\{\tilde{N}, \tilde{M} - \tilde{L}\tilde{N}, \tilde{L}, I\} \in \mathscr{B}$. Then from Lemma 8 it follows that $[P, C]$ is internally stable if and only if $\tilde{M} - \tilde{L}\tilde{N} \in \mathscr{GH}_\infty$. The proof is concluded by post multiplying with the matrix $\left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] \in \mathscr{GH}_\infty$. □

The above theorem is a generalised version of the well known coprime factor results given in Lemma 3. An easy way of observing this is by considering the case where $N = U = I$ which forces $\{L, M\} \in \mathscr{C}_l(P)$ and $\{W, V\} \in \mathscr{C}_l(C)$. The result of Theorem 10 is then transformed to Lemma 3 statement (5). Similarly, forcing $L = W = -I$ leads to statement (4) of the same lemma. The rest of the lemma statements then follow using simple linear algebra.

When the plant or controller is known to be stable, the following two simplified lemmas can be used to establish the stability of the feedback interconnection.

Lemma 11. Consider the standard feedback interconnection of a plant $P \in \mathscr{R}$ and controller $C \in \mathscr{RH}_\infty$ and let $\{N, M, L, 0\} \in \mathscr{B}(P)$. Then

$$[P, C] \text{ is internally stable} \Leftrightarrow M - LCN \in \mathscr{GH}_\infty.$$

Proof. Let $\{U, V, W, 0\} \in \mathscr{B}(C)$. The proof follows by noting that $C \in \mathscr{RH}_\infty \Leftrightarrow V \in \mathscr{GH}_\infty$ (by Lemma 8) and then applying Lemma 2 to the result of Theorem 10. □

[c] Note that this inverse is not necessarily proper.
[d] A similar result holds for LCFs and RCFs.
[e] This follows from Vinnicombe (2001, Remark 1.2) and some algebra.

Lemma 12. Consider the standard feedback interconnection of a plant $P \in \mathscr{RH}_\infty$ and controller $C \in \mathscr{R}$ and let $\{U, V, W, 0\} \in \mathscr{B}(C)$. Then

$$[P, C] \text{ is internally stable} \Leftrightarrow V - WPU \in \mathscr{GH}_\infty.$$

Proof. Let $\{N, M, L, 0\} \in \mathscr{B}(P)$. The proof follows by noting that $P \in \mathscr{RH}_\infty \Leftrightarrow M \in \mathscr{GH}_\infty$ (by Lemma 8) and then applying Lemma 2 to the result of Theorem 10. □

4.1 Inclusion of the Additive Term

In most of the results in the previous section it is assumed that the BCF under consideration has no additive term. However, this restricts the set for which the stability results presented apply.

The following theorem considers the case where the plant BCF is allowed to have an additive component.

Theorem 13. Consider the standard feedback interconnection of a plant $P \in \mathscr{R}$ and controller $C \in \mathscr{R}$. Let $\{N, M, L, K\} \in \mathscr{B}(P)$ and $\{U, V, W, 0\} \in \mathscr{B}(C)$. Then

$$[P, C] \text{ is internally stable} \Leftrightarrow \begin{bmatrix} M & -LU \\ -WN & V - WKU \end{bmatrix} \in \mathscr{GH}_\infty.$$

Proof. The proof follows by inflating the plant factors to accommodate K, followed by some elementary row/column operations and application of Lemma 2.

A dual result to Theorem 13 can be trivially obtained for the case $\{N, M, L, 0\} \in \mathscr{B}(P)$ and $\{U, V, W, X\} \in \mathscr{B}(C)$.

4.2 Numerical Example

A numerical example will now be provided to demonstrate the stability results presented in the previous sections.

Consider the feedback interconnection of a plant $P \in \mathscr{R}$ and controller $C \in \mathscr{RH}_\infty$ given by

$$P = \begin{bmatrix} \frac{2}{(s+2)(s-1)} & \frac{4}{s+2} \\ \frac{2(s+1)}{(s+2)(s-1)} & \frac{4(s+1)}{(s+2)} \end{bmatrix}, \; C = -\begin{bmatrix} \frac{s+2}{s+1} & \frac{s+2}{4(s+1)} \\ \frac{s+2}{s+1} & \frac{s+2}{4(s+1)} \end{bmatrix},$$

both of which are rank deficient with $\mathrm{nr}(P) = 1$ and $\mathrm{nr}(C) = 1$.[f] Furthermore, define

$$N := \begin{bmatrix} \frac{2}{\bar{s}+1} & \frac{4(\bar{s}-1)}{\bar{s}+1} \end{bmatrix}^*, \; M := \frac{s-1}{s+1}, \; L := \begin{bmatrix} \frac{1}{s+2} & \frac{s+1}{s+2} \end{bmatrix},$$
$$U := \begin{bmatrix} 1 & 1 \end{bmatrix}^*, \; V := -\frac{s+1}{s+2}, \; W := \begin{bmatrix} 1 & \frac{1}{4} \end{bmatrix}$$

and note that $\{N, M, L, 0\} \in \mathscr{B}_*(P)$ and $\{U, V, W, 0\} \in \mathscr{B}_*(C)$. Then the stability criterion in Theorem 10 gives

$$\begin{bmatrix} M & -LU \\ -WN & V \end{bmatrix}^{-1} = \begin{bmatrix} \frac{s+1}{2s+1} & -\frac{s+2}{2s+1} \\ -\frac{s+2}{2s+1} & -\frac{(s+2)(s-1)}{(s+1)(2s+1)} \end{bmatrix} \in \mathscr{RH}_\infty,$$

hence it can be concluded that $[P, C]$ is internally stable. Since $C \in \mathscr{RH}_\infty$, Lemma 11 can also be used to establish the stability of $[P, C]$, leading to a simpler and more interesting condition

$$M - LCN = \frac{2s+1}{s+1} \in \mathscr{GH}_\infty$$

again demonstrating that $[P, C]$ is internally stable. The stability of a MIMO feedback interconnection has therefore been established by a simple SISO test in this example when using Lemma 11 due to the normal rank deficiency.

[f] Such rank deficient systems frequently occur in networked control systems, see for example Wang et al. (2015).

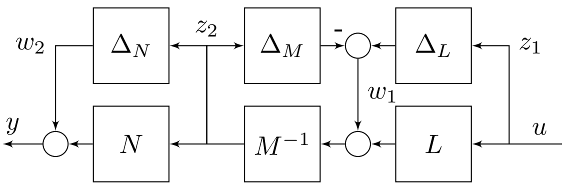

Fig. 2. Perturbed plant block diagram with BC factor uncertainty.

Even though the above is a special case where both the plant and controller are rank deficient and the controller is stable, it exemplifies the possible benefits of using BCFs, especially in the context of redundant or networked control systems.

5. BCF UNCERTAINTY

The case of a plant with additive perturbations to its BC factors will now be considered. It will be shown that this type of uncertainty encapsulates features from both LCF and RCF perturbations.

Let $P \in \mathscr{R}$ and $\{N, M, L, 0\} \in \mathscr{B}(P)$. A perturbed plant can be defined as

$$P_\Delta := (N + \Delta_N)(M + \Delta_M)^{-1}(L + \Delta_L),$$

which corresponds to the block diagram form shown in Figure 2.

From both the definition of P_Δ and its block diagram representation, it is apparent that BCF perturbations include features from RCF and LCF perturbations as presented in Zhou et al. (1996).

Let $z := (z_2^* \; z_1^*)^*$ and $w := (w_1^* \; w_2^*)^*$. Then a generalised plant $H : (w^* \; u^*)^* \mapsto (z^* \; y^*)^*$ and uncertainty matrix $\Delta : z \mapsto w$ can be defined as

$$H := \left[\begin{array}{cc|c} M^{-1} & 0 & M^{-1}L \\ 0 & 0 & I \\ \hline NM^{-1} & I & P \end{array}\right] \text{ and } \Delta := \begin{bmatrix} -\Delta_M & \Delta_L \\ \Delta_N & 0 \end{bmatrix}$$

respectively, satisfying $P_\Delta = \mathcal{F}_u(H, \Delta)$.

By inspection of the above definitions it is evident that ignoring the additive part of the factorisation leads to an uncertainty matrix that is naturally structured since its (2,2) block is zero. Such structured uncertainty can cause difficulties in deriving associated robust analysis results (Lanzon and Papageorgiou, 2009; Lanzon et al., 2012).

Including the previously omitted additive term provides an immediate solution to this problem. Let $\{N, M, L, K\} \in \mathscr{B}(P)$, then the resulting perturbed plant is given by

$$P_\Delta = (N + \Delta_N)(M + \Delta_M)^{-1}(L + \Delta_L) + (K + \Delta_K)$$

and the new unstructured uncertainty matrix by

$$\Delta = \begin{bmatrix} -\Delta_M & \Delta_L \\ \Delta_N & \Delta_K \end{bmatrix}.$$

Figure 3 shows the block diagram for the augmented perturbed plant. It is interesting to note how general this augmented BCF uncertainty structure is when compared to the standard uncertainty structures commonly studied in robust control.

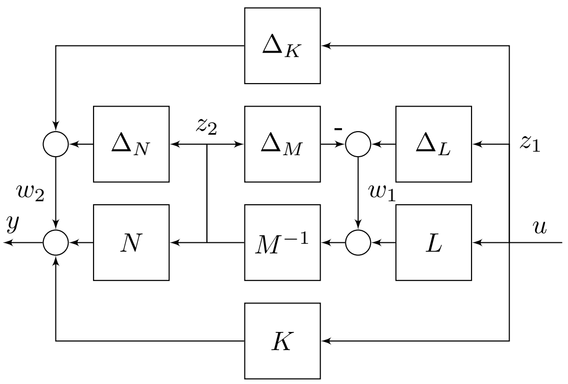

Fig. 3. Augmented perturbed plant block diagram with BC factor uncertainty.

As with any uncertainty structure, it is possible to define a robust stability condition or robust stability margin (Lanzon and Papageorgiou, 2009; Lanzon et al., 2012). This can then be used as an optimisation cost function for controller synthesis.

Theorem 14. Consider the standard feedback interconnection of a plant $P \in \mathscr{R}$ and controller $C \in \mathscr{R}$. Let $\{N, M, L, K\} \in \mathscr{B}(P)$ and $\{U, V, W, 0\} \in \mathscr{B}(C)$. Define $\Delta \in \mathscr{RH}_\infty$ as above and

$$P_\Delta := (N + \Delta_N)(M + \Delta_M)^{-1}(L + \Delta_L) + (K + \Delta_K).$$

Furthermore, suppose that $[P, C]$ is internally stable and $\{N + \Delta_N, M + \Delta_M, L + \Delta_L, K + \Delta_K\} \in \mathscr{B}(P_\Delta)$. [g] Then $[P_\Delta, C]$ is internally stable for all $\Delta \in \{\Delta \in \mathscr{RH}_\infty, \|\Delta\|_\infty < \gamma\}$ if and only if

$$\left\| \begin{bmatrix} I & 0 \\ 0 & U \end{bmatrix} \begin{bmatrix} M & -LU \\ -WN & V - WKU \end{bmatrix}^{-1} \begin{bmatrix} I & 0 \\ 0 & W \end{bmatrix} \right\|_\infty \le \frac{1}{\gamma}.$$

Proof. Proof will not be given here due to space limitations, but will be published elsewhere.

6. CONCLUSION

Bicoprime factorisations have been introduced and shown to be a generalised version of the well known left and right coprime factorisations. Stability results pertaining to BCFs were presented. Potential benefits include the study of redundant and networked control systems. Uncertainty on BC factors was presented and shown to have a structure that encompasses both RCF and LCF uncertainty and several different standard uncertainty structures.

The work presented in this paper naturally leads to a multitude of questions about BCFs and their possible applications in control theory. This paper aims to stimulate interest in this neglected field of robust control theory.

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[g] These are standard suppositions when dealing with coprime factor uncertainty. See for example (Zhou et al., 1996, Theorem 9.6).