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Frequency-domain dissipativity analysis for output negative imaginary systems allowing imaginary-axis poles

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***Abstract*— This brief addresses the frequency-domain dissipativity problem of a broader class of Output Negative Imaginary systems, termed as the time-domain ONI (or TD-ONI) systems, which have been defined in the time domain w.r.t. an abstract energy supply rate function. This definition encompasses the existing strict/non-strict NI subsets, including those having imaginary-axis poles. This paper introduces the idea of a "shifted $(Q_\sigma(\omega), S_\sigma(\omega), R_\sigma(\omega))$-dissipativity", as an alternative to the conventional $(Q(\omega), S(\omega), R(\omega))$-dissipativity, to capture the TD-ONI systems, particularly the ones having imaginary-axis poles. The shifted $(Q_\sigma(\omega), S_\sigma(\omega), R_\sigma(\omega))$-dissipativity is defined w.r.t. a shifted imaginary axis $(\sigma + j\omega,\ \sigma > 0)$ and thereby, it overcomes the limitation of earlier frequency-domain dissipative frameworks to capture systems with imaginary-axis poles. The paper has also established the relationship between the time-domain and frequency-domain dissipativity of TD-ONI systems. Finally, a closed-loop stability theorem is also given for a positive feedback interconnection of two TD-ONI systems.**

I. Introduction

Negative Imaginary (NI) framework offers an energy-based control technique like passivity [1], dissipativity [2] and counter-clockwise dynamics [3]. NI theory came into the robust control literature in 2007–08 [4] and reinforced the notion of positive position feedback control of lightly damped systems. In the SISO setting, a system is said to have the NI property if its phase angle $\phi(\omega) \in [-\pi, 0]$ for all $\omega \geq 0$. The majority of the Euler-Lagrange systems with collocated force input and position output belong to the NI class. Similarly, a cantilever beam with a collocated force input and position output can be approximately modelled as $\Sigma(s) = \sum_{i=1}^{M} \frac{\omega_{ni}^2}{s^2+2\xi_i\omega_{ni}+\omega_{ni}^2}$, which exhibits the NI property [4], [5]. Most NI literature, since its inception, has concentrated on frequency-domain properties and behaviour. There are very few ones, for instance, [6]–[12] and [13], that explored and investigated the dissipative property of NI systems and redefined the class of Output (and/or Input) Negative Imaginary [ONI/IONI] systems.

In the NI literature, [14] took the first step to define the concept of Input and/or Output Negative Imaginary systems (IONI/ONI/INI), taking inspiration from [15]. The ideas of [14] were later utilised in [16], [17] to define IONI systems having 'mixed' properties (i.e. a combination of IONI, finite-gain and passive properties) in different frequency intervals. However, the supply rates defined in [14], [16], [17] had two significant disadvantages – they were applicable to only asymptotically stable IONI/ONI systems and they couldn't capture bi-proper Output Strictly NI (or OSNI) systems. Urged by these limitations, [8]–[10], [13] have introduced new supply rates for defining ONI and IONI classes, including bi-proper systems as well. They exploited an auxiliary output ($z \triangleq v - Df$) of the system instead of the physical output (v) to define a new appropriate supply rate $\mathbb{W}(f, \dot{z})$, which resolved the earlier difficulty in capturing bi-proper OSNI systems. Among these articles, [8] and [13] defined *asymptotically stable* ONI and IONI systems w.r.t. a frequency-domain dissipative [known as $(Q(\omega), S(\omega), R(\omega))$-dissipativity] supply rate, whereas [9] and [10] proposed a new time-domain dissipative supply rate for defining ONI/IONI systems allowing $j\omega$-axis poles. However, none of the NI literature published so far has addressed the problem of developing a frequency-domain dissipative framework for the entire class of ONI/IONI systems that includes poles on the $j\omega$ axis, even at the origin.

Being inquisitive about this gap in NI research, we will bring in the concept of a "shifted $(Q_\sigma(\omega), S_\sigma(\omega), R_\sigma(\omega))$-dissipativity" by exploiting the idea of the Fourier transform w.r.t. a shifted $j\omega$-axis [i.e. the Fourier integral is now evaluated on the $(\sigma + j\omega)$-axis for a specific $\sigma > 0$]. The "shifted $(Q_\sigma(\omega), S_\sigma(\omega), R_\sigma(\omega))$-dissipative" supply rate complies with the time-domain supply rate $\mathbb{W}(f, \dot{z})$ used to define the full class of ONI systems with possible poles on the $j\omega$-axis. Moreover, in the case of asymptotically stable systems, the proposed frequency-domain dissipative characterisation boils down to the conventional frequency-domain definitions of NI [4], [18] and OSNI systems [7], [8]. Finally, an asymptotic stability theorem is derived for a positive feedback ONI systems (without any poles at $s = 0$) interconnection relying on the Lyapunov stability approach.

II. Preliminaries and background

Section II caters some crucial pre-requisites that help to build the core results of this paper.

A. Mathematical preliminaries

For a matrix A (or a vector X), transpose and Hermitian transpose are denoted by $A^\top$ (or $X^\top$) and A^* (or X^*). $A^{-*} \triangleq (A^{-1})^*$ and $A^{-\top} \triangleq (A^{-1})^\top$. $\mathbb{L}_{2e}^m = \{\eta : \mathbb{R} \rightarrow$

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$\mathbb{R}^m : \eta(t) = 0$ when $t < 0, \int_0^\Gamma \eta(t)^\top \eta(t)\,\mathrm{d}t < \infty\ \forall \Gamma \in [0,\infty)\}$ indicates the space of all finite-energy signals. An energy supply rate $\mathbb{W}(f,\upsilon)$ follows the relationship $\int_0^\Gamma \mathbb{W}(f,\upsilon)\,\mathrm{d}t < \infty$ for all $\Gamma \in [0,\infty)$ and all admissible input-output pairs $(f,\upsilon) \in \mathbb{U} \times \mathbb{Y}$ where $\mathbb{U} \in \mathbb{L}_{2e}^m$ and $\mathbb{Y} \in \mathbb{L}_{2e}^p$. If $(f,\upsilon) \in \mathbb{L}_2^m \times \mathbb{L}_2^p$, then $\int_0^\infty \mathbb{W}(f,\upsilon)\,\mathrm{d}t < \infty$. A storage function $\mathscr{V}(\chi)$ belongs to the C^1 category if it is continuously differentiable in $\chi \in \mathbb{R}^n$. A frequency-domain Lebesgue space $\mathscr{L}_2^m(j\mathbb{R})$ can be defined w.r.t. the inner product $\langle \eta_1, \eta_2 \rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} \eta_1(j\omega)^* \eta_2(j\omega)\,\mathrm{d}\omega < \infty$ when $\eta_1, \eta_2 \in \mathscr{L}_2^m(j\mathbb{R})$ [14], [19]. The $\mathscr{L}_2$ norm of a signal $\eta \in \mathscr{L}_2^m(j\mathbb{R})$ is defined as $\|\eta\| = \sqrt{\langle \eta, \eta \rangle} = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \eta(j\omega)^* \eta(j\omega)\,\mathrm{d}\omega} < \infty$. Following the notion of $\mathscr{L}_2^m(j\mathbb{R})$ space, another frequency-domain space $\mathscr{L}_2^m(\sigma + j\mathbb{R})$ for a given $\sigma > 0$ can be defined w.r.t. the inner product $\langle \eta_1, \eta_2 \rangle_\sigma = \frac{1}{2\pi}\int_{-\infty}^{\infty} \eta_1(\sigma + j\omega)^* \eta_2(\sigma + j\omega)\,\mathrm{d}\omega < \infty$ for the signals that are not bounded on the $j\omega$-axis but are bounded on the $(\sigma + j\omega)$-axis.

B. NI and SNI properties

This subsection includes the frequency-domain definitions of NI and SNI systems[1].

Definition 1: (**NI System**) [4], [18] A system $\varSigma(s) \in \mathscr{R}^{m\times m}$ with no open-RHP poles is NI if $j[\varSigma(j\omega) - \varSigma(j\omega)^*] \geq 0\ \forall \omega \in (0,\infty)$ except those $\omega_0 \in \omega$ where $s = j\omega_0$ is a pole of $\varSigma(s)$. If $\omega_0 \in (0,\infty)$, the multiplicity of the pole $(s^2 + \omega_0^2)$ cannot be more than one and the residue matrix $\Delta|_{s=j\omega_0} \triangleq \lim_{s\to j\omega_0}(s - j\omega_0)j\varSigma(s) = \Delta|_{s=j\omega_0}^* \geq 0$. If $s = 0$ is a pole of $\varSigma(s)$, then $\lim_{s\to 0} s^k \varSigma(s) = 0\ \forall k \geq 3$ and the residue matrix $\Delta|_{s=0} \triangleq \lim_{s\to 0} s^2 \varSigma(s) \Delta|_{s=0}^* \geq 0$.

Definition 2: (**SNI System**) [4], [18] A system $\varSigma(s) \in \mathscr{RH}_\infty^{m\times m}$ is SNI if $j[\varSigma(j\omega) - \varSigma(j\omega)^*] > 0\ \forall \omega \in (0,\infty)$.

C. Basics of dissipativity theory

This subsection recalls the classical concepts of dissipativity in both time and frequency domains. It also proposes a new idea of a "shifted $(Q_\sigma(\omega), S_\sigma(\omega), R_\sigma(\omega))$-dissipativity" to capture the systems with $j\omega$-axis poles.

Definition 3: (**Dissipative systems**) [2] Consider a dynamical system $\varSigma$ governed by $\dot{\chi} = A\chi + Bf$, $\chi_0 = \chi(0)$, and $\upsilon = C\chi + Df$ where $\chi \in \mathbb{R}^n$, $f \in \mathbb{R}^m$ and $\upsilon \in \mathbb{R}^m$. Suppose there exists a positive semidefinite storage function $\mathscr{V}(\chi)$. Then, $\varSigma$ is called dissipative w.r.t. an energy supply rate $\mathbb{W}(f,\upsilon)$ if

$$\mathscr{V}(\chi(0)) + \int_0^\Gamma \mathbb{W}(f,\upsilon)\,\mathrm{d}t \geq \mathscr{V}(\chi(\Gamma)) \tag{1}$$

for any final time $\Gamma \in [0,\infty)$, any admissible input $f(\cdot)$ and any $\chi(0) \in \mathbb{R}^n$.
The relationship (1) is distinguished as the 'dissipation inequality', originally proposed by Prof J. C. Willems. If the storage function $\mathscr{V}(\chi)$ is differentiable, then we get the *differential form* of (1) as mentioned below:

$$\mathbb{W}(f,\upsilon) \geq \dot{\mathscr{V}}(\chi). \tag{2}$$

After the time-domain dissipativity, we will now recap the notion of classical frequency-domain dissipativity, termed as $(Q(\omega), S(\omega), R(\omega))$**-dissipativity**[2].

Definition 4: ($(Q(\omega), S(\omega), R(\omega))$**-dissipativity**) [14], [15] Consider a system $\varSigma$ having the transfer function mapping $\Upsilon(s) = \varSigma(s)F(s)$ where $\varSigma(s) \in \mathscr{RH}_\infty^{p\times m}$. Then, $\varSigma$ is a $(Q(\omega), S(\omega), R(\omega))$-dissipative system if there exists a set of frequency-dependent operators (or matrices) $Q(\omega) = Q(\omega)^\top \in \mathbb{R}^{p\times p}$, $S(\omega) \in \mathbb{C}^{p\times m}$ and $R(\omega) = R(\omega)^\top \in \mathbb{R}^{m\times m}$ for all $\omega \in \mathbb{R}$ such that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \big[\Upsilon(j\omega)^* Q(\omega)\Upsilon(j\omega) + \Upsilon(j\omega)^* S(\omega) F(j\omega) + F(j\omega)^* S(\omega)^* \Upsilon(j\omega) + F(j\omega)^* R(\omega) F(j\omega)\big]\mathrm{d}\omega \geq 0 \tag{3}$$

for all admissible inputs $F \in \mathscr{L}_2^m(j\mathbb{R})$.
Note that for marginally-stable systems, the frequency-domain integral in (3) does not remain bounded on the $j\omega$-axis and hence, $(Q(\omega), S(\omega), R(\omega))$-dissipative supply rate cannot be defined. To overcome this limitation, in this paper, we introduce the notion of a "shifted $(Q_\sigma(\omega), S_\sigma(\omega), R_\sigma(\omega))$-dissipativity" that relies on the Fourier transformation evaluated w.r.t. a shifted $j\omega$-axis, denoted as the $(\sigma + j\omega)$-axis, where the parameter σ is chosen to be a positive constant such that $\sigma \geq 0$ and $\sigma > \Re[\lambda_i(A)]\ \forall i$. In such cases, inequality (3) takes the form (4), as introduced via the following definition.

Definition 5: (**Shifted** $(Q_\sigma(\omega), S_\sigma(\omega), R_\sigma(\omega))$**-dissipativity**) Consider a system $\varSigma$ having the transfer function mapping $\Upsilon(s) = \varSigma(s)F(s)$ where $\varSigma(s) \in \mathscr{R}^{p\times m}$. Then, $\varSigma$ is defined as a "shifted $(Q_\sigma(\omega), S_\sigma(\omega), R_\sigma(\omega))$-dissipative" system if there exists a set of frequency-dependent operators (or matrices) $Q_\sigma(\omega) = Q_\sigma(\omega)^\top \in \mathbb{R}^{p\times p}$, $S_\sigma(\omega) \in \mathbb{C}^{p\times m}$ and $R_\sigma(\omega) = R_\sigma(\omega)^\top \in \mathbb{R}^{m\times m}$ for all $\omega \in \mathbb{R}$ such that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \big[\Upsilon(\sigma + j\omega)^* Q_\sigma(\omega)\Upsilon(\sigma + j\omega) + \Upsilon(\sigma + j\omega)^* S_\sigma(\omega) F(\sigma + j\omega) + F(\sigma + j\omega)^* S_\sigma(\omega)^* \Upsilon(\sigma + j\omega) + F(\sigma + j\omega)^* R_\sigma(\omega) F(\sigma + j\omega)\big]\mathrm{d}\omega \geq 0 \tag{4}$$

for all admissible inputs $F(\sigma + j\omega) \in \mathscr{L}_2^m(j\mathbb{R})$, where $F(\sigma + j\omega) = \mathscr{F}[e^{-\sigma t} f(t)]$ and $\Upsilon(\sigma + j\omega) = \mathscr{F}[e^{-\sigma t}\upsilon(t)]$ on noting that $f_\sigma = e^{-\sigma t} f(t) \in \mathbb{L}_2^m$ and $\upsilon_\sigma = e^{-\sigma t}\upsilon(t) \in \mathbb{L}_2^m$ for all $t \geq 0$, subject to an appropriate choice of $\sigma > 0$ and restricting the time-domain input signals $f \in \mathbb{L}_2^m$.

III. Time-domain output negative imaginary systems

We will now define the class of time-domain Output Negative Imaginary (TD-ONI) systems using a time-domain dissipative approach w.r.t. a new supply rate $\mathbb{W}(f, \dot{z})$ where $z \triangleq \upsilon - Df$ is an auxiliary output of the system. This

[1]In this paper, NI, SNI, ONI and IONI all properties are defined for finite-dimensional, causal and square systems having proper, real and rational transfer function matrices in the space $\mathscr{R}^{m\times m}$.

[2]$(Q(\omega), S(\omega), R(\omega))$-dissipativity is the frequency-domain analogue of the time-domain (Q, S, R)-dissipativity proposed by Hill and Moylan [1].

definition does not impose any *a priori* conditions (such as stability, minimality, full normal rank constraint, etc. – commonly used in the NI literature) on the system to be defined and thereby, pose no difficulty in acquiring systems containing poles on the imaginary axis, even at the origin.

Definition 6: (**TD-ONI systems**) Consider a dynamical system Σ : $\begin{cases} \dot{\chi} = A\chi + Bf, & \chi_0 = 0; \\ \upsilon = C\chi + Df, \end{cases}$ where $\chi \in \mathbb{R}^n$, $f \in \mathbb{R}^m$, $\upsilon \in \mathbb{R}^m$ and (A, B, C, D) is minimal. Let $z \triangleq \upsilon - Df$ be an auxiliary output of the system. Then, Σ is said to be a time-domain Output Negative Imaginary (TD-ONI) system if there exists a parameter $\delta \geq 0$ such that

$$\int_0^\Gamma \left(2\dot{z}^\top f - \delta\, \dot{z}^\top \dot{z}\right) \mathrm{d}t \geq 0 \tag{5}$$

for all final time $\Gamma \in [0, \infty)$ and all admissible $f \in \mathbb{L}_{2e}^m$.

Inequality (5) is regarded as the "TD-ONI inequality". This definition captures all the existing variants of TD-ONI and TD-OSNI systems [7]–[9], [13], [14], [16] and opens the door to accept marginally system ONI systems (i.e. those contain imaginary-axis poles including the origin). Surprisingly, according to this definition, a subset of the *marginally-stable* LTI systems, especially the ones having *a single pole at* $s = 0$ (e.g. $\frac{1}{s}$, $\frac{s+4}{s(s+2)}$, etc.) satisfies the TD-OSNI property (i.e. $\delta > 0$). This in contrast to the existing notion that TD-OSNI property used to be defined only for asymptotically stable systems.

Classification of TD-ONI systems:

Under the TD-ONI class, there exists several strict and non-strict subsets, which can be classified depending on the output negative imaginary index $\delta \geq 0$ and type of stability of the system. Fig. 1 illustrates the relationship through a comprehensive Venn diagram.

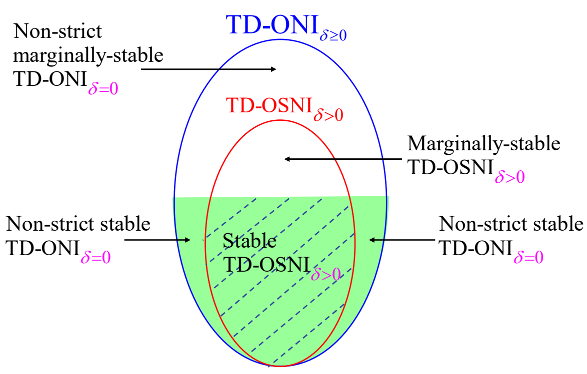

Fig. 1: The classification of asymptotically stable and marginally-stable subsets of TD-ONI and TD-OSNI systems.

For better understanding, a few numerical examples are given here: $\Sigma_1(s) = \frac{1}{s+1}$ and $\Sigma_2(s) = \frac{s^2+12.5}{s^4+s^3+42.5s^2+12.5s+150}$ are TD-OSNI systems with $\delta_1, \delta_2 > 0$; $\Sigma_3(s) = \frac{2s^2+s+1}{(s+1)(2s+1)(s^2+2s+5)}$ is a non-strict asymptotically stable TD-ONI system with $\delta_3 = 0$; $\Sigma_4(s) = \frac{1}{s}$ and $\Sigma_5(s) = \frac{(s+4)}{s(s+1)}$ are marginally-stable TD-OSNI systems with $\delta_4, \delta_5 > 0$; and $\Sigma_6(s) = \frac{1}{s^2}$ and $\Sigma_7(s) = \frac{1}{s^2+1}$ are marginally-stable TD-ONI systems with $\delta_6, \delta_7 = 0$.

To this end, before going to frequency-domain dissipativity of TD-ONI systems, we can shed some light on time-domain dissipativity of such systems. It can be proved in the spirit of [13] and [2] that TD-ONI systems are equivalent to a class of minimally-realised dissipative systems characterised by the time-domain supply rate $\mathbb{W}(f, \dot{z}) = 2\dot{z}^\top f - \delta \dot{z}^\top \dot{z}$. That is, for such systems, there always exists a positive semidefinite storage function $\mathscr{V}(\chi)$ such that

$$\mathscr{V}(\chi(0)) + \int_0^\Gamma (2\dot{z}^\top f - \delta \dot{z}^\top \dot{z})\, \mathrm{d}t \geq \mathscr{V}(\chi(\Gamma)) \tag{6}$$

for all final time $\Gamma \in [0, \infty)$ and admissible inputs $f \in \mathbb{L}_{2e}^m$.

IV. Frequency-domain dissipative characterisation for TD-ONI systems

We will now put forward the main development of this research study. It develops a new frequency-domain dissipative characterisation for the full set of the TD-ONI systems in contrast to the related earlier results reported in [14], [16], [17] and [8] where $(Q(\omega), S(\omega), R(\omega))$-dissipativity was used only for asymptotically stable TD-ONI systems because $(Q(\omega), S(\omega), R(\omega))$-dissipativity cannot capture the marginally-stable systems. To overcome this limitation, we leverage on the idea of "shifted $(Q_\sigma(\omega), S_\sigma(\omega), R_\sigma(\omega))$-dissipativity" (discussed at the end of Subsection II-C) instead of the conventional $(Q(\omega), S(\omega), R(\omega))$-dissipativity.

Theorem 1: Consider an initially relaxed TD-ONI system Σ with a minimal (A, B, C, D) where $D = \Sigma(\infty) = D^\top$. Choose $\sigma > 0$ when $\max_i \Re\{\lambda_i[A]\} = 0$, or else $\sigma = 0$. Then, Σ satisfies the "shifted $(Q_\sigma(\omega), S_\sigma(\omega), R_\sigma(\omega))$-dissipativity" property w.r.t. $Q_\sigma(\omega) = -\delta\,(\sigma^2 + \omega^2) I_m$, $S_\sigma(\omega) = (\sigma - j\omega) I_m + \delta\,(\sigma^2 + \omega^2) D$ and $R_\sigma(\omega) = -\sigma(D + D^\top) - \delta\,(\sigma^2 + \omega^2) D^\top D$ for all $\omega \in \mathbb{R}$.

Proof. We begin the proof on noting that $\upsilon = \mathscr{L}^{-1}[\Sigma(s)] \star f$, $z \triangleq \upsilon - Df$ where $D = \Sigma(\infty) = D^\top$.

Part I. Let $\Sigma(s)$ contain pole(s) on the $j\omega$-axis for $\omega \in \mathbb{R}$. In this part of the proof, $\Upsilon(\sigma + j\omega)$, $Z(\sigma + j\omega)$, $F(\sigma + j\omega)$ are the Fourier Transformations of the signals $e^{-\sigma t}\upsilon(t)$, $e^{-\sigma t} z(t)$, $e^{-\sigma t} f(t)$ for all $t \geq 0$ and for a specific $\sigma > 0$.

$$\begin{aligned}
&\Sigma \text{ is TD-ONI with } \delta \geq 0 \\
\Leftrightarrow\ & \text{there exists } \mathscr{V} : \mathbb{R}^n \to \mathbb{R}_{\geq 0} \text{ with } \mathscr{V}(0) = 0 \text{ such that} \\
&\quad \int_0^\Gamma \left(2\dot{z}^\top f - \delta\, \dot{z}^\top \dot{z}\right) \mathrm{d}t \geq \mathscr{V}(\chi(\Gamma)) - \mathscr{V}(\chi(0)) \\
&\quad \text{for all } \Gamma \in [0, \infty) \text{ and all admissible inputs } f \in \mathbb{L}_2^m \\
\Rightarrow\ & \int_0^\Gamma \left(2\dot{z}^\top f - \delta\, \dot{z}^\top \dot{z}\right) \mathrm{d}t \geq 0 \quad \text{for all } \Gamma \in [0, \infty) \\
&\quad \text{and all admissible inputs } f \in \mathbb{L}_2^m \\
&\quad [\text{since } \mathscr{V}(\chi(\Gamma)) \geq 0 \text{ for all } \Gamma \in [0, \infty) \text{ and } \mathscr{V}(0) = 0] \\
\Rightarrow\ & \int_0^\infty e^{-2\sigma t} \left(2\dot{z}^\top f - \delta\, \dot{z}^\top \dot{z}\right) \mathrm{d}t \geq 0 \\
&\quad \text{for all admissible inputs } f \in \mathbb{L}_2^m \quad [\text{choosing} \\
&\quad \sigma > 0 \text{ when } \max_i \Re\{\lambda_i[A]\} = 0; \text{ or else } \sigma = 0]
\end{aligned}$$

$$\Leftrightarrow \int_0^{\infty} \left[2(e^{-\sigma t}\dot{z})^\top (e^{-\sigma t} f) - \delta\,(e^{-\sigma t}\dot{z})^\top (e^{-\sigma t}\dot{z})\right] \mathrm{d}t \geq 0$$

for all admissible inputs $f \in \mathbb{L}_2^m$

$$\Leftrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} (\sigma - j\omega) Z(\sigma + j\omega)^* F(\sigma + j\omega)\, \mathrm{d}\omega + \frac{1}{2\pi}\int_{-\infty}^{\infty} (\sigma + j\omega) F(\sigma + j\omega)^* Z(\sigma + j\omega)\, \mathrm{d}\omega - \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta\,(\sigma^2 + \omega^2) Z(\sigma + j\omega)^* Z(\sigma + j\omega)\, \mathrm{d}\omega \geq 0$$

for all admissible inputs $F \in \mathscr{L}_2^m(\sigma + j\mathbb{R})$

[using the result $(\sigma + j\omega)\Upsilon(\sigma + j\omega) = \mathscr{F}[e^{-\sigma t}\dot{v}(t)]$ for a specific $\sigma > 0$ such that $\Upsilon(\sigma + j\omega)$ exists $\forall \omega \in \mathbb{R}$ [20]]

$$\Leftrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} (\sigma - j\omega) \Big[\big[\Upsilon(\sigma + j\omega)^* - F(\sigma + j\omega)^* D^\top\big] \times F(\sigma + j\omega)\Big] \mathrm{d}\omega + \frac{1}{2\pi}\int_{-\infty}^{\infty} (\sigma + j\omega) \Big[F(\sigma + j\omega)^* [\Upsilon(\sigma + j\omega) - DF(\sigma + j\omega)]\Big] \mathrm{d}\omega - \frac{1}{2\pi}\int_{-\infty}^{\infty} \Big[\delta\,(\sigma^2 + \omega^2)\,\{\Upsilon(\sigma + j\omega)^* - F(\sigma + j\omega)^* D^\top\}\{\Upsilon(\sigma + j\omega) - DF(\sigma + j\omega)\}\Big] \mathrm{d}\omega \geq 0$$

for all admissible inputs $F \in \mathscr{L}_2^m(\sigma + j\mathbb{R})$

$$\Leftrightarrow \int_{-\infty}^{\infty} \Big[(\sigma - j\omega)\Upsilon(\sigma + j\omega)^* F(\sigma + j\omega) + (\sigma - j\omega) F(\sigma + j\omega)^* \Upsilon(\sigma + j\omega) - \sigma F(\sigma + j\omega)^* (D + D^\top) F(\sigma + j\omega) - \delta\,(\sigma^2 + \omega^2)\Upsilon(\sigma + j\omega)^* \Upsilon(\sigma + j\omega) + \delta\,(\sigma^2 + \omega^2) F(\sigma + j\omega)^* D^\top \Upsilon(\sigma + j\omega) + \delta\,(\sigma^2 + \omega^2) \Upsilon(\sigma + j\omega)^* D F(\sigma + j\omega) - \delta\,(\sigma^2 + \omega^2) F(\sigma + j\omega)^* D^\top D F(\sigma + j\omega)\Big] \mathrm{d}\omega \geq 0$$

for all admissible inputs $F \in \mathscr{L}_2^m(\sigma + j\mathbb{R})$

$$\Leftrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} \begin{bmatrix} \Upsilon(\sigma + j\omega)^* & F(\sigma + j\omega)^* \end{bmatrix} \begin{bmatrix} -\delta\,(\sigma^2 + \omega^2) I_m & (\sigma - j\omega) I_m + \delta\,(\sigma^2 + \omega^2) D \\ (\sigma + j\omega) I_m + \delta\,(\sigma^2 + \omega^2) D^\top & -\sigma(D + D^\top) - \delta\,(\sigma^2 + \omega^2) D^\top D \end{bmatrix} \times \begin{bmatrix} \Upsilon(\sigma + j\omega) \\ F(\sigma + j\omega) \end{bmatrix} \mathrm{d}\omega \geq 0 \text{ for all } F \in \mathscr{L}_2^m(\sigma + j\mathbb{R})$$

$$\Leftrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} \begin{bmatrix} \Upsilon(\sigma + j\omega)^* & F(\sigma + j\omega)^* \end{bmatrix} \begin{bmatrix} Q_\sigma(\omega) & S_\sigma(\omega) \\ S_\sigma(\omega)^* & R_\sigma(\omega) \end{bmatrix} \begin{bmatrix} \Upsilon(\sigma + j\omega) \\ F(\sigma + j\omega) \end{bmatrix} \mathrm{d}\omega \geq 0$$

for all admissible inputs $F \in \mathscr{L}_2^m(\sigma + j\mathbb{R})$ denoting $Q_\sigma(j\omega) = -\delta\,(\sigma^2 + \omega^2) I_m,\ S_\sigma(j\omega) = (\sigma - j\omega) I_m + {} + \delta\,(\sigma^2 + \omega^2) D$ and $R_\sigma(j\omega) = -\sigma(D + D^\top) - \delta\,(\sigma^2 + \omega^2) D^\top D$.

Part II. Let $\Sigma(s) \in \mathscr{RH}_\infty^{m \times m}$. Then,

Σ is asymptotically stable TD-ONI with $\delta \geq 0$

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} \begin{bmatrix} \Upsilon(j\omega)^* & F(j\omega)^* \end{bmatrix} \begin{bmatrix} Q(\omega) & S(\omega) \\ S(\omega)^* & R(\omega) \end{bmatrix} \begin{bmatrix} \Upsilon(j\omega) \\ F(j\omega) \end{bmatrix} \mathrm{d}\omega \geq 0$$

for all admissible inputs $F \in \mathscr{L}_2^m(j\mathbb{R})$ and denoting $Q(\omega) = -\delta\omega^2 I_m,\ S(\omega) = -j\omega I_m + \delta\,\omega^2 D$ and $R(\omega) = -\delta\,\omega^2 D^\top D \quad \forall \omega \in \mathbb{R}$

[follows directly from Part I on setting $\sigma = 0$].

The above two parts together accomplish the proof. ■

A. Frequency-domain dissipativity of stable TD-ONI systems: A necessary and sufficient result

For asymptotically stable systems, the proposed idea of "shifted $(Q_\sigma(\omega), S_\sigma(\omega), R_\sigma(\omega))$-dissipativity" boils down to the conventional $(Q(\omega), S(\omega), R(\omega))$-dissipativity. We will show that for asymptotically stable TD-ONI systems, the notion of "shifted $(Q_\sigma(\omega), S_\sigma(\omega), R_\sigma(\omega))$-dissipativity" is not only a sufficient-type result, implied by the time-domain dissipative supply rate $\mathbb{W}(f, \dot{z}) = 2\dot{z}^\top f - \delta \dot{z}^\top \dot{z}$, but it is also a necessary and sufficient property. To prove this claim, we will specialise the time-domain dissipation inequality (6) for asymptotically stable TD-ONI systems as follows: there exists a positive semidefinite storage function $\mathscr{V}(\chi)$ such that

$$\mathscr{V}(\chi(0)) + \int_0^{\infty} \left(2\dot{z}^\top f - \delta\,\dot{z}^\top \dot{z}\right) \mathrm{d}t \geq \mathscr{V}(\chi(\infty)) \qquad (7)$$

for all admissible inputs $f \in \mathbb{L}_2^m$, since $\dot{z} = C\dot{\chi} = CA\chi + CBf$ now belongs to the space $\mathbb{L}_2^m$ as $\chi \in \mathbb{L}_2^n$ subject to $f \in \mathbb{L}_2^m$. Note also that $\chi(\infty)$ exists and is finite, since $\Sigma(s)$ belongs to $\mathscr{RH}_\infty$. Therefore, $\mathscr{V}(\chi(\infty))$ does also exists and is a finite quantity.

Theorem 2: Consider an initially relaxed LTI system Σ with a minimal (A, B, C, D) where $D = \Sigma(\infty) = D^\top$ and A is Hurwitz. Then, Σ is an asymptotically stable TD-ONI system $\Leftrightarrow$ Σ is $(Q(\omega), S(\omega), R(\omega))$-dissipative with $Q(\omega) = -\delta\omega^2 I_m$, $S(\omega) = -j\omega I_m + \delta\omega^2 D$ and $R(\omega) = -\delta\omega^2 D^\top D$ for all $\omega \in \mathbb{R}$.

Proof. First note that $\Sigma(s)$ belongs to the asymptotically stable TD-ONI class implies that $\Sigma(s)$ is asymptotically stable NI according to Definition 1.

$\Sigma(s)$ is asymptotically stable TD-ONI with $\delta \geq 0$

$$\Leftrightarrow \int_0^{\infty} \left(2\dot{z}^\top f - \delta\,\dot{z}^\top \dot{z}\right) \mathrm{d}t \geq 0 \quad \forall f \in \mathbb{L}_2^m$$

[via (7) and since $\mathscr{V}(\chi(\infty)) = \mathscr{V}(0)$ as $\chi(\infty) = 0$]

$$\Leftrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} \Big[(j\omega Z(j\omega))^* F(j\omega) + F(j\omega)^* (j\omega Z(j\omega)) - \delta\,(j\omega Z(j\omega))^* (j\omega Z(j\omega))\Big] \mathrm{d}\omega \geq 0 \quad \forall F \in \mathscr{L}_2^m(j\mathbb{R})$$

[via Parseval's Theorem [20]]

$$\Leftrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} \Big[\Upsilon(j\omega)^* (-\delta\omega^2 I_m) \Upsilon(j\omega) + \Upsilon(j\omega)^* (-j\omega I_m +$$

$$
\begin{aligned}
&\delta\omega^2 D)F(j\omega) + F(j\omega)^*(j\omega I_m + \delta\omega^2 D^\top)\Upsilon(j\omega) + \\
&F(j\omega)^*\left(-\delta\omega^2 D^\top D\right)F(j\omega)\Big]\, \mathrm{d}\omega \geq 0 \quad \forall F \in \mathscr{L}_2^m(j\mathbb{R}) \\
&\quad [\text{substituting } Z(j\omega) = \Upsilon(j\omega) - DF(j\omega)] \\
\Leftrightarrow &\frac{1}{2\pi}\int_{-\infty}^{\infty}\Big[\Upsilon(j\omega)^* Q(\omega)\Upsilon(j\omega) + \Upsilon(j\omega)^* S(\omega)F(j\omega) + \\
&\quad F(j\omega)^* S(\omega)^*\Upsilon(j\omega) + F(j\omega)^* R(\omega)F(j\omega)\Big]\mathrm{d}\omega \geq 0 \\
&\quad \forall F \in \mathscr{L}_2^m(j\mathbb{R}) \\
\Leftrightarrow &\ \Sigma \text{ is } (Q(\omega), S(\omega), R(\omega))\text{-dissipative.}
\end{aligned}
$$

This completes the proof. ■

V. Closed-loop stability analysis for a TD-ONI systems interconnection

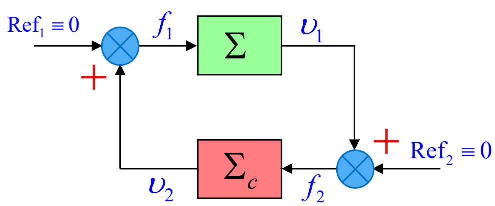

Fig. 2: A closed-loop interconnection of TD-ONI systems.

The class of TD-ONI systems with no poles at $s = 0$ offers an appealing and useful asymptotic stability result. Theorem 3 derives a set of sufficient conditions to guarantee the asymptotic stability (in the sense of Lyapunov) of a TD-ONI system with no poles at $s = 0$ and an asymptotically stable TD-OSNI system connected via positive feedback as shown in Fig. 2). The following lemma is a crucial technical result for proving Theorem 3. Lemma 1 provides the criteria to be imposed on the TD-ONI systems under consideration to ensure that the feedback interconnection of $\Sigma(s)$ and $\Sigma_c(s)$ in Fig. 2 does not contain any complex pole-pair on the imaginary axis.

Lemma 1: [13] Consider a positive feedback loop (Fig. 2) containing a TD-ONI system $\Sigma(s) \in \mathscr{R}^{m\times m}$ and a TD-OSNI system $\Sigma_c(s) \in \mathscr{R}\mathscr{H}_\infty^{m\times m}$. $\Sigma(s)$ may be marginally-stable but does not contain any pole at $s = 0$. Suppose the matrix $[I - \Sigma(s)\Sigma_c(s)]$ has full normal rank. Define a finite set of frequencies, Ψ, consisting of $\omega_i \in (0,\infty)$ where $i \in \{1,2,\ldots\}$ such that $\Sigma(s)$ has no complex pole-pair at $s = \pm j\omega_i$ and

$$
j\left[\Sigma_c(j\omega_0) - \Sigma_c(j\omega_0)^*\right] > 0 \qquad \forall \omega_0 \in (0,\infty)\backslash\Psi. \quad (8)
$$

Also suppose $\nexists$ any $\omega \in \Psi$ such that $\det[\Sigma(j\omega) - \Sigma(j\omega)^*] = 0$ and $\det[\Sigma_c(j\omega) - \Sigma_c(j\omega)^*] = 0$ together. Then, $\det[I - \Sigma(j\omega)\Sigma_c(j\omega)] \neq 0$ at any $\omega \in (0,\infty)$.

Theorem 3: Consider a positive feedback loop (shown in Fig. 2) containing a TD-ONI system $\Sigma(s) \in \mathscr{R}^{m\times m}$ and a TD-OSNI system $\Sigma_c(s) \in \mathscr{R}\mathscr{H}_\infty^{m\times m}$. $\Sigma(s)$ may be marginally-stable but does not contain any pole at $s = 0$. Define a finite set of frequencies, Ψ, consisting of $\omega_i \in (0,\infty)$ where $i \in \{1,2,\ldots\}$ such that $\Sigma(s)$ has no complex pole-pair at $s = \pm j\omega_i$ and $j\left[\Sigma_c(j\omega_0) - \Sigma_c(j\omega_0)^*\right] > 0$ $\forall \omega_0 \in (0,\infty)\backslash\Psi$. Also suppose $\nexists$ any $\omega \in \Psi$ such that $\det[\Sigma(j\omega) - \Sigma(j\omega)^*] = 0$ and $\det[\Sigma_c(j\omega) - \Sigma_c(j\omega)^*] = 0$. Suppose further $\Sigma(\infty)\Sigma_c(\infty) = 0$, $\Sigma_c(\infty) \geq 0$, and

$$
\lambda_{\max}\left[\Sigma_c(0)\Sigma(0)\right] < 1. \quad (9)
$$

Then, the origin $\underline{0}$ of the unforced (i.e. $\mathrm{Ref}_1 \equiv 0$ and $\mathrm{Ref}_2 \equiv 0$) closed-loop control system shown in Fig. 2 is asymptotically stable in the sense of Lyapunov.

Proof. Here, we provide the proof in nutshell. The detailed proof could not be accommodated due to a lack of space. Suppose there exist two storage functions $\mathscr{V}_1 = \chi_1^\top \mathscr{P}_1 \chi_1$ with $\mathscr{P}_1 = \mathscr{P}_1^\top > 0$ and $\mathscr{V}_2 = \chi_2^\top \mathscr{P}_2 \chi_2$ with $\mathscr{P}_2 = \mathscr{P}_2^\top > 0$ such that Σ satisfies $2\dot{z}_1^\top f_1 - \delta_1 \dot{z}_1^\top \dot{z}_1 \geq \dot{\mathscr{V}}_1(\chi_1)$ with $\delta_1 \geq 0$ and Σ_c satisfies $2\dot{z}_2^\top f_2 - \delta_2 \dot{z}_2^\top \dot{z}_2 \geq \dot{\mathscr{V}}_2(\chi_2)$ with $\delta_2 > 0$. $z_1 \triangleq \upsilon_1 - Df_1$ and $z_2 \triangleq \upsilon_2 - D_c f_2$. Assume Σ and Σ_c have minimal state-space realisations (A, B, C, D) and (A_c, B_c, C_c, D_c), respectively, where $D = D^\top$, $D_c = D_c^\top$, $\det[A] \neq 0$ and A_c has Hurwitz property.

Part I – Let $\Sigma(s)$ contain one or more simple pole-pair(s) on the imaginary axis (but not at $s = 0$) and $\Sigma_c(s)$ is asymptotically stable TD-OSNI. It can be readily established under the suppositions of this theorem that the loop transfer function $\Sigma(s)\Sigma_c(s)$ does not give rise to any pole-zero cancellation in $\{s : \Re[s] \geq 0\}$.

We now define a specific storage function $\mathscr{V}_{cl}(\chi_1,\chi_2) \triangleq \mathscr{V}_1(\chi_1) + \mathscr{V}_2(\chi_2) - \upsilon_1^\top \upsilon_2 - z_1^\top z_2$ for the closed-loop system shown in Fig. 2 and can be expressed in the form:

$$
\begin{bmatrix} \chi_1^\top & \chi_2^\top \end{bmatrix}
\begin{bmatrix} \mathscr{P}_1 - C^\top D_c C & -C^\top C_c \\ -C_c^\top C & \mathscr{P}_2 - C_c^\top D C_c \end{bmatrix}
\begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix},
$$

which is positive definite due to the conditions $\mathscr{P}_1 > 0$, $\mathscr{P}_2 > 0$ and $\lambda_{\max}[\Sigma_c(0)\Sigma(0)] < 1$. Note that $\mathscr{V}_{cl}$ is a C^1 function of $\chi_{cl} = \begin{bmatrix} \chi_1^\top & \chi_2^\top \end{bmatrix}^\top$. Hence, $\mathscr{V}_{cl}$ qualifies to be a Lyapunov function candidate for the closed-loop system. We will now calculate the time-derivative of $\mathscr{V}_{cl}$ as:

$$
\begin{aligned}
&\dot{\mathscr{V}}_{cl}(\chi_1,\chi_2) \\
=&\dot{\mathscr{V}}_1(\chi_1) + \dot{\mathscr{V}}_2(\chi_2) - \dot{\upsilon}_1^\top \upsilon_2 - \upsilon_1^\top \dot{\upsilon}_2 - \dot{z}_1^\top z_2 - z_1^\top \dot{z}_2 \\
=&\ \dot{\mathscr{V}}_1(\chi_1) + \dot{\mathscr{V}}_2(\chi_2) - 2\dot{z}_1^\top f_1 - 2\dot{z}_2^\top f_2 \\
=&\left(\dot{\mathscr{V}}_1(\chi_1) - 2\dot{z}_1^\top f_1\right) + \left(\dot{\mathscr{V}}_2(\chi_2) - 2\dot{z}_2^\top f_2\right) \\
\leq&\ -\delta_1 \dot{z}_1^\top \dot{z}_1 - \delta_2 \dot{z}_2^\top \dot{z}_2.
\end{aligned} \quad (10)
$$

This implies $\dot{\mathscr{V}}_{cl}(\chi_1,\chi_2) \leq 0$ as $\delta_1 \geq 0$ and $\delta_2 > 0$. Therefore, we can assert that the states $\chi_1(t)$ and $\chi_2(t)$ of the unforced closed-loop system remain bounded for all $t \geq 0$. But we also need to show the asymptotic convergence of $\chi_1(t)$ and $\chi_2(t)$ towards the origin, which is guaranteed if the closed-loop system matrix

$$
A_{cl} = \begin{bmatrix} A & BC_c \\ 0 & A_c \end{bmatrix} + \begin{bmatrix} BD_c \\ B_c \end{bmatrix}\begin{bmatrix} C & DC_c \end{bmatrix}
$$

is Hurwitz, which is equivalent to the criterion that $[I - \Sigma(j\omega)\Sigma_c(j\omega)]^{-1}$ does not have any complex pole-pair on the $j\omega$-axis for any $\omega \in (0,\infty)$. In the present situation, the said condition is fulfilled by Lemma 1. Subsequently, the condition $\det[\Sigma_c(0)\Sigma(0) - I] \neq 0$, implied from (9), ensures that $[I - \Sigma(s)\Sigma_c(s)]^{-1}$ does not have any pole at $s = 0$ via [21, Lemma 3.38] on noting that $[I - \Sigma(s)\Sigma_c(s)]$ does not have any pole at $s = 0$. Combining all the preceding arguments, it can be concluded that the unforced closed-loop system is asymptotically stable in the sense of Lyapunov.

Part II – Let $\Sigma(s) \in \mathscr{RH}_\infty^{m\times m}$ be TD-ONI and $\Sigma_c(s) \in \mathscr{RH}_\infty^{m\times m}$ be TD-OSNI. This part can be easily proved from Part I and it re-establishes the sufficiency parts of proofs of [7, Theorem 1] and [18, Theorem 5] respectively.

The above two parts together constitutes the complete proof like Theorem 3. ∎

Example 1: Consider a marginally-stable TD-ONI system $\Sigma(s) = \frac{9}{s^2+9}$ and an asymptotically stable TD-OSNI system $\Sigma_c(s) = \frac{s^2+12.5}{s^4+s^3+42.5s^2+12.5s+150}$ connected via a positive feedback. This interconnection satisfies all the suppositions of Theorem 3 and also the DC-gain condition: $\Sigma_c(0)\Sigma(0) = \frac{12.5}{150} < 1$. Hence, the closed-loop system is asymptotically stable in the sense of Lyapunov via Theorem 3. We also verify the closed-loop pole locations $\{-0.3742 \pm j6.2044, -0.0018 \pm j3.0315, -0.1240 \pm j1.8628\}$, which confirms our claim.

Example 2: Consider an interconnection of $\Sigma(s) = \dfrac{s^2+4s+12}{5(s+2)(s^2+4)}\begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix}$ being a TD-ONI system without any pole at $s = 0$ and $\Sigma_c(s) = \dfrac{0.25s^2+3.75}{s^4+0.25s^3+25s^2+3.75s+83.33}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ being an asymptotically stable TD-OSNI system. It is checked that both $[\Sigma(s) - \Sigma(s)^*]$ and $[\Sigma_c(s) - \Sigma_c(s)^*]$ have full normal rank, $\Sigma(s)\Sigma_c(s)$ does not have any pole-zero cancellation in the entire closed right-half plane, $j\,[\Sigma_c(j\omega) - \Sigma_c(j\omega)^*] > 0$ at $\omega = 2$ rad/s, $j\,[\Sigma(j\omega) - \Sigma(j\omega)^*] > 0$ at $\omega = \sqrt{15}$ rad/s and there does not exist any $\omega_z \in (0, \infty)$ such that $\det[\Sigma(\omega_z) - \Sigma(\omega_z)^*] = 0$ and $\det[\Sigma_c(\omega_z) - \Sigma_c(\omega_z)^*] = 0$. Thus, $\Sigma(s)$ and $\Sigma_c(s)$ satisfy all the assumptions of Theorem 3. Finally, the DC loop gain condition $\lambda_{\max}[\Sigma_c(0)\Sigma(0)] = \lambda_{\max}\begin{bmatrix} 0.2295 & -0.1485 \\ -0.1485 & 0.2295 \end{bmatrix} = 0.3780 < 1$ concludes the closed-loop asymptotic stability.

VI. Conclusions

The concept of time-domain Output Negative Imaginary (TD-ONI) systems has been brought in, which unifies the strict and non-strict subsets of the existing ONI and NI systems, including those with $j\omega$-axis poles [7]–[9], [13]. TD-ONI systems have been defined w.r.t. an unconventional supply rate $\mathbb{W}(f, \dot{z})$ instead of relying on the classical frequency-domain definitions, where $z \triangleq v - Df$ is an auxiliary output of the system. A new type of frequency-domain dissipativity, termed as the "shifted $(Q_\sigma(\omega), S_\sigma(\omega), R_\sigma(\omega))$-dissipativity", is proposed here to describe the whole class of the TD-ONI systems (i.e. allowing $j\omega$-axis poles). This idea is in contrast to the conventional $(Q(\omega), S(\omega), R(\omega))$-dissipativity, which applies to only asymptotically stable ONI/IONI/NI systems [8], [14], [16]. The paper finally derives an asymptotic stability theorem for a positive feedback TD-ONI systems interconnection in which at least one must be asymptotically stable TD-OSNI.

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