2015 IEEE 54th Annual Conference on Decision and Control (CDC)
December 15-18, 2015. Osaka, Japan

Descriptor Systems State-space Conditions to Guarantee Negative Imaginary Properties without Minimality Restrictions

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Abstract— **This paper is concerned with negative imaginary lemmas for descriptor systems. Without the minimal state-space realization assumption, sufficient conditions are developed for a negative imaginary lemma and a strictly negative imaginary lemma, respectively. As corollaries, sufficient conditions are derived to ensure the systems are both admissible and negative imaginary. Also, new sufficient negative imaginary lemmas are found for standard linear systems as special cases. The developed negative imaginary lemmas are applicable to descriptor systems with impulse modes. Two examples are used to illustrate the theory.**

I. Introduction

Negative imaginary properties can be found in many practical systems. For example, by choosing appropriate system inputs and outputs, the resultant transfer functions can exhibit such properties in vibration control systems [1]–[3] and circuit networks [4], [5]. Many valuable theoretical results have been developed for negative imaginary systems. For instance, the concept of negative imaginary systems has been refined in [6]–[9]; the negative imaginary lemmas for standard linear systems have been developed in [6], [7], [10]; applications of negative imaginary theory can be found in [3], [5], [11]. On the other hand, descriptor systems provide a suitable model for mechanical systems and circuit systems [12], [13]. The study of descriptor systems has attracted much attention. For instance, positive real theory has been studied in [14], [15]; model reduction techniques that can preserve the passivity have been investigated in [16], [17]. Descriptor systems are currently still an active research topic; see [18], [19].

Negative imaginary lemmas are an important class of the results developed for standard linear systems [6], [7], [10]. These lemmas give necessary and sufficient conditions to test the negative imaginary property of the systems according to their state-space realizations, and these conditions can be solved numerically efficiently. For a system with a descriptor state-space description, several versions of negative imaginary lemmas have been reported in [20]–[22]. In [20], the negative imaginary lemma is based on the Weierstrass form of the system. However, the Weierstrass form is usually difficult to obtain. In [21], a criteria to test the negative imaginary property is given in terms of a Kronecker canonical decomposition of a matrix pencil. However, the proposed conditions are numerically difficult to check. In [22], the authors established a negative imaginary lemma, a strictly negative imaginary lemma, and a lossless negative imaginary lemma. Necessary and sufficient conditions were derived based on the minimal realization assumption of the descriptor state-space model [22]. However, because of the minimal realization assumption and the properness requirement for transfer functions, the results in [22] are only applicable to systems that are impulse-free.

This paper follows the development in [22], but the minimal state-space realization assumption is removed. As a consequence, the developed negative imaginary lemmas are applicable to descriptor systems with impulse modes. Compared to the results in [21], our results can be tested numerically efficiently. When compared to the results in [20], the results here are not dependent on the Weierstrass form. The organization of the paper is as follows. Section II recalls basic concepts from descriptor systems and negative imaginary transfer functions. The main results are presented in Section III. Under an assumption on the fast subsystem, sufficient negative imaginary lemmas are established based on a decomposition of transfer functions. In addition, sufficient conditions are established to ensure descriptor systems are both admissible and negative imaginary. When descriptor systems reduce to standard linear systems, new negative imaginary lemmas for standard linear systems are deduced as special cases. Section IV gives two examples to illustrate the developed negative imaginary theory. Conclusions are drawn in Section V.

Notation: Let $\mathbb{R}^{m\times n}$ and $\mathscr{R}^{m\times n}$ denote the set of $m\times n$ real matrices and real-rational proper transfer function matrices, respectively. A^T and A^* denote the transpose and the complex conjugate transpose of a complex matrix A, respectively. $R^{\sim}(s)$ represents the adjoint of transfer function matrix $R(s)$ and is given by $R^T(-s)$. $\Re[\cdot]$ is the real part of a complex number. The notation $X>0$ or $X\geq 0$, where X is a real symmetric matrix, means that the matrix X is positive definite or positive semidefinite.

II. Problem Formulation

This section reviews some basic concepts from descriptor systems and negative imaginary transfer function matrices.

This work was financially supported by the NSFC (61374026), the EPSRC (EP/F06022X/1), the Royal Society, and the ARC.

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Consider a class of dynamical systems described by

$$\begin{cases} E\dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t), \end{cases} \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the system state, $u(t) \in \mathbb{R}^m$ is the control input and $y(t) \in \mathbb{R}^m$ is the measurement output. The matrices $E \in \mathbb{R}^{n\times n}$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$ and $D \in \mathbb{R}^{m\times m}$ are constant matrices. The system in (1) is called a descriptor linear system when E is a singular matrix. The pair (E,A) is called regular if $\det(sE-A) \neq 0$ for some $s \in \mathbb{C}$. When (E,A) is regular, the descriptor system (1) has a transfer function

$$R(s) = C(sE-A)^{-1}B + D. \quad (2)$$

The regularity of (E,A) is a necessary and sufficient condition for the existence and uniqueness of the solution to descriptor system (1), and hence a common assumption in almost every study of descriptor systems. Moreover, the regularity of (E,A) ensures that there exist non-singular matrices $Q \in \mathbb{R}^{n\times n}$, $P \in \mathbb{R}^{n\times n}$ such that

$$QEP = \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \quad QAP = \begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix}, \quad (3a)$$

$$QB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \qquad CP = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \quad (3b)$$

where $N \in \mathbb{R}^{n_2\times n_2}$ is a nilpotent matrix, $A_1 \in \mathbb{R}^{n_1\times n_1}$, $B_1 \in \mathbb{R}^{n_1\times m}$, $B_2 \in \mathbb{R}^{n_2\times m}$, $C_1 \in \mathbb{R}^{m\times n_1}$, $C_2 \in \mathbb{R}^{m\times n_2}$ and $n_1+n_2 = n$. The matrices on the right sides of equations (3) are called the Weierstrass form of the descriptor system (1).

Next, we recall some definitions of negative imaginary transfer function matrices.

Definition 1: [7] A transfer function matrix $R(s) \in \mathscr{R}^{m\times m}$ is *negative imaginary* if

1) $R(s)$ has no poles at the origin and in $\Re[s] > 0$;
2) $j[R(j\omega) - R^*(j\omega)] \geq 0$ for all $\omega \in (0,\infty)$ except values of ω where $j\omega$ is a pole of $R(s)$;
3) If $j\omega_0$, $\omega_0 \in (0,\infty)$, is a pole of $R(s)$, it is at most a simple pole, and the residue matrix $K_0 \triangleq \lim_{s\to j\omega_0}(s - j\omega_0)jR(s)$ is positive semidefinite Hermitian.

Remark 1: In the above definition, the transfer function matrix $R(s)$ is required to have no poles at the origin. This requirement can be removed by a careful modification of the definition; see for example [23]. In [9], the authors studied symmetric transfer function matrices, and generalized the negative imaginary concept to the case of non-proper symmetric transfer function matrices.

Definition 2: [6] A transfer function matrix $R(s) \in \mathscr{R}^{m\times m}$ is *strictly negative imaginary* if

1) $R(s)$ has no poles in $\Re[s] \geq 0$;
2) $j[R(j\omega) - R^*(j\omega)] > 0$ for $\omega \in (0,\infty)$.

Remark 2: It deserves mentioning that the strict inequality in Condition 2) of Definition 2 does not hold at both zero and infinite frequencies for any real-rational proper transfer functions. Actually, we always have $j[R(j\omega) - R^*(j\omega)] = 0$ at those two frequencies. This is different from the definition of strictly positive real transfer functions. Also, because of this difference, an equivalence relationship between strictly negative imaginary transfer functions and strictly positive real transfer functions cannot be established like those in Lemma 3 of [7] for negative imaginary transfer functions and Lemma 1 of [24] for lossless negative imaginary transfer functions. This fact has complicated the strictly negative imaginary control or synthesis problem considerably.

The descriptor system in (1) is called a negative imaginary descriptor system if its transfer function matrix $R(s)$ in (2) is a negative imaginary transfer function.

The objective of the paper is to develop negative imaginary lemmas for descriptor systems without minimal realization assumptions.

III. SUFFICIENT NEGATIVE IMAGINARY LEMMAS

In this section, under an assumption on the fast subsystem, sufficient conditions are developed for a negative imaginary lemma and a strictly negative imaginary lemma, respectively. Also, new versions of the negative imaginary lemma for standard linear systems are obtained as special cases.

Assumption 1: The fast subsystem in (3) is observable.

Assumption 2: $NB_2 = 0$.

Assumption 1 is equivalent to the requirement that the pair (N,C_2) is observable. A sufficient but not necessary condition for Assumption 2 being true is that the system (1) is impulse free.

The following lemma plays a critical role to obtain the sufficient conditions in the negative imaginary lemmas for descriptor systems.

Lemma 1: Consider a state-space realization (E,A,B,C,D) of the transfer function $R(s) \in \mathscr{R}^{m\times m}$. Suppose either Assumption 1 or Assumption 2 is satisfied and that the following two conditions hold:

1) $\det(A) \neq 0$, $R(\infty) = R^T(\infty)$;
2) there exist matrices $X \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{n\times m}$ and $L \in \mathbb{R}^{n\times n}$ such that

$$A^TX + X^TA + L^TL = 0 \quad (4)$$

$$C^T + X^TEA^{-1}B = A^TY \quad (5)$$

$$E^TX = X^TE \quad (6)$$

$$E^TY = 0. \quad (7)$$

Then

$$R(s) - R^{\sim}(s) = -sM^{\sim}(s)M(s) \quad (8)$$

for all s with s not a pole of $R(s)$, where

$$M(s) = LA^{-1}E(sE-A)^{-1}B. \quad (9)$$

Proof: Firstly, the nonsingularity of A implies that the transfer function $R(s)$ in (2) has the Weierstrass form given in (3). Therefore, one has that

$$R(s) = C_1(sI-A_1)^{-1}B_1 + D - C_2B_2 - \sum_{i=1}^{h-1} s^iC_2N^iB_2,$$

where h is the smallest integer such that $N^h = 0$. Note that $h \leq n_2$. Furthermore, the properness of $R(s)$ leads to

$C_2N^iB_2 = 0$ for $i = 1,2,\ldots,h-1$; that is

$$\begin{bmatrix} C_2 \\ C_2N \\ \vdots \\ C_2N^{n_2-1} \end{bmatrix} NB_2 = 0.$$

Because the fast subsystem is observable, it follows that the observability matrix in the above equation has full column rank. Hence, we have $NB_2 = 0$. (that is, Assumption 1 implies Assumption 2.) Note that N is not necessarily zero under either Assumption 1 or Assumption 2. Hence the system need not to be impulse-free.

Let

$$Q^{-T}XP = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix},\ Q^{-T}Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix},\ LP = \begin{bmatrix} L_1 & L_2 \end{bmatrix}.$$

Then

$$\begin{aligned}(6) &\iff (QEP)^TQ^{-T}XP = (Q^{-T}XP)^TQEP \\ &\iff \begin{bmatrix} I & 0 \\ 0 & N^T \end{bmatrix}\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} = \begin{bmatrix} X_1^T & X_3^T \\ X_2^T & X_4^T \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} \\ &\iff \begin{bmatrix} X_1 & X_2 \\ N^TX_3 & N^TX_4 \end{bmatrix} = \begin{bmatrix} X_1^T & X_3^TN \\ X_2^T & X_4^TN \end{bmatrix} \\ &\implies X_1 = X_1^T,\ X_2 = X_3^TN.\end{aligned}$$

Note that we are not interested in the definiteness of X_1. Similarly, it follows from (4) that

$$\begin{aligned}& A^TX + X^TA + L^TL = 0 \\ &\iff (QAP)^T(Q^{-T}XP) + (Q^{-T}XP)^T(QAP) + (LP)^TLP = 0 \\ &\iff \begin{bmatrix} A_1^TX_1 & A_1^TX_2 \\ X_3 & X_4 \end{bmatrix} + \begin{bmatrix} X_1^TA_1 & X_3^T \\ X_2^TA_1 & X_4^T \end{bmatrix} + \begin{bmatrix} L_1^TL_1 & L_1^TL_2 \\ L_2^TL_1 & L_2^TL_2 \end{bmatrix} = 0 \\ &\implies A_1^TX_1 + X_1A_1 + L_1^TL_1 = 0.\end{aligned}$$

Also, it can be verified that

$$\begin{aligned}(7) &\iff (QEP)^T(Q^{-T}Y) = 0 \\ &\iff \begin{bmatrix} Y_1 \\ N^TY_2 \end{bmatrix} = 0 \\ &\implies Y_1 = 0.\end{aligned}$$

Moreover, from (5), one has

$$\begin{aligned}& C^T + X^TEA^{-1}B = A^TY \\ &\iff (CP)^T + (Q^{-T}XP)^TQEP(QAP)^{-1}QB \\ &\qquad\qquad = (QAP)^TQ^{-T}Y \\ &\iff \begin{bmatrix} C_1^T \\ C_2^T \end{bmatrix} + \begin{bmatrix} X_1A_1^{-1}B_1 + X_3^TNB_2 \\ N^TX_3A_1^{-1}B_1 + X_4^TNB_2 \end{bmatrix} = \begin{bmatrix} 0 \\ Y_2 \end{bmatrix} \\ &\iff \begin{bmatrix} C_1^T + X_1A_1^{-1}B_1 \\ C_2^T + N^TX_3A_1^{-1}B_1 \end{bmatrix} = \begin{bmatrix} 0 \\ Y_2 \end{bmatrix}.\end{aligned}$$

The last equivalence holds because $NB_2 = 0$.

Let $R_1(s) = C_1(sI - A_1)^{-1}B_1$. Then $R(s) = R_1(s) + R(\infty)$. The symmetric structure of $R(\infty)$ implies that

$$\begin{aligned}R(s) - R^{\sim}(s) &= [R_1(s) + R(\infty)] - [R_1^{\sim}(s) + R^T(\infty)] \\ &= R_1(s) - R_1^{\sim}(s). \qquad (10)\end{aligned}$$

Let

$$\begin{aligned}M_1(s) &= L_1A_1^{-1}(sI - A_1)^{-1}B_1, \\ F_1(s) &= sR_1(s) = C_1A_1(sI - A_1)^{-1}B_1 + C_1B_1, \\ W_1(s) &= sM_1(s) = L_1(sI - A_1)^{-1}B_1 + L_1A_1^{-1}B_1.\end{aligned}$$

We prove that $F_1(s) + F_1^{\sim}(s) = W_1^{\sim}(s)W_1(s)$ as shown in (11) at the top of the next page, where the first equality follows from the definition of $W_1^{\sim}(s)$; that is, $W_1^{\sim}(s) = B_1^T(sI + A_1^T)^{-1}(-L_1^T) + B_1^TA_1^{-T}L_1^T$, the second from the formula for the product of two transfer function matrices; see Chapter 3 of [25], the third from replacing $L_1^TL_1$ with $-A_1^TX_1 - X_1A_1$, the fourth from a similar transformation, the fifth from replacing $X_1A_1^{-1}B_1$ with $-C_1^T$, the sixth from the formula for the addition of two transfer function matrices; see Chapter 3 of [25], the last from the definition of $F_1^{\sim}(s)$; that is, $F_1^{\sim}(s) = B_1^T(sI + A_1^T)^{-1}(-A_1^TC_1^T) + B_1^TC_1^T$.

On the other hand, we have

$$\begin{aligned}W_1^{\sim}(s)W_1(s) &= -s^2M_1^{\sim}(s)M_1(s), \\ F_1(s) + F_1^{\sim}(s) &= sR_1(s) - sR_1^{\sim}(s).\end{aligned}$$

Therefore,

$$\begin{aligned}s[R_1(s) - R_1^{\sim}(s)] &= F_1(s) + F_1^{\sim}(s) \\ &= W_1^{\sim}(s)W_1(s) \\ &= -s^2M_1^{\sim}(s)M_1(s).\end{aligned}$$

When $s \neq 0$, one has that $R_1(s) - R_1^{\sim}(s) = -sM_1^{\sim}(s)M_1(s)$. When $s = 0$, using $R_1(0) = -C_1A_1^{-1}B_1 = B_1^TA_1^{-T}X_1A_1^{-1}B_1 = R_1^T(0)$, one has that $R_1(0) - R_1^T(0) = 0$. So we have

$$R_1(s) - R_1^{\sim}(s) = -sM_1^{\sim}(s)M_1(s) \qquad (12)$$

for all s with s not a pole of $R(s)$.

On the other hand, one has that

$$\begin{aligned}M(s) &= LA^{-1}E(sE - A)^{-1}B \\ &= LP(QAP)^{-1}(QEP)(sQEP - QAP)^{-1}QB \\ &= L_1A_1^{-1}(sI - A_1)^{-1}B_1 + L_2N(sN - I)^{-1}B_2 \\ &= M_1(s) + L_2(sN - I)^{-1}NB_2 \\ &= M_1(s). \qquad (13)\end{aligned}$$

By noting (10), (12) and (13), we have that (8) holds for all s with s not a pole of $R(s)$. This completes the proof. ■

Remark 3: On one hand, Lemma 1 is applicable to descriptor systems that may have impulse modes. On the other hand, when the descriptor systems are impulse-free or reduced to standard linear systems, neither Assumption 1 nor Assumption 2 is needed for Lemma 1 to be true. This remark is also applicable to the following results where Assumption 1 is required.

Theorem 1: Consider a state-space realization (E,A,B,C,D) of the transfer function $R(s) \in \mathscr{R}^{m\times m}$. Suppose either Assumption 1 or Assumption 2 is satisfied and that the following conditions hold:

1) (E,A) is stable, $R(\infty) = R^T(\infty)$;

$$
\begin{aligned}
W_1^{\sim}(s)W_1(s) &= \left[\begin{array}{c|c} -A_1^T & -L_1^T \\ \hline B_1^T & B_1^T A_1^{-T} L_1^T \end{array}\right] \left[\begin{array}{c|c} A_1 & B_1 \\ \hline L_1 & L_1 A_1^{-1} B_1 \end{array}\right] \\
&= \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ -L_1^T L_1 & -A_1^T & -L_1^T L_1 A_1^{-1} B_1 \\ \hline B_1^T A_1^{-T} L_1^T L_1 & B_1^T & B_1^T A_1^{-T} L_1^T L_1 A_1^{-1} B_1 \end{array}\right] \\
&= \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ A_1^T X_1 + X_1 A_1 & -A_1^T & A_1^T X_1 A_1^{-1} B_1 + X_1 B_1 \\ \hline -B_1^T A_1^{-T} X_1 A_1 - B_1^T X_1 & B_1^T & -B_1^T X_1 A_1^{-1} B_1 - B_1^T A_1^{-T} X_1 B_1 \end{array}\right] \\
&= \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & -A_1^T & A_1^T X_1 A_1^{-1} B_1 \\ \hline -B_1^T A_1^{-T} X_1 A_1 & B_1^T & -B_1^T X_1 A_1^{-1} B_1 - B_1^T A_1^{-T} X_1 B_1 \end{array}\right] \\
&= \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & -A_1^T & -A_1^T C_1^T \\ \hline C_1 A_1 & B_1^T & C_1 B_1 + B_1^T C_1^T \end{array}\right] \\
&= \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 A_1 & C_1 B_1 \end{array}\right] + \left[\begin{array}{c|c} -A_1^T & -A_1^T C_1^T \\ \hline B_1^T & B_1^T C_1^T \end{array}\right] \\
&= F_1(s) + F_1^{\sim}(s).
\end{aligned} \tag{11}
$$

2) there exist matrices $X \in \mathbb{R}^{n\times n}$ and $Y \in \mathbb{R}^{n\times m}$ such that

$$A^T X + X^T A \le 0 \tag{14}$$

$$C^T + X^T E A^{-1} B = A^T Y \tag{15}$$

$$E^T X = X^T E \tag{16}$$

$$E^T Y = 0. \tag{17}$$

Then $R(s)$ is negative imaginary.

Proof: The proof is completed by verifying the conditions in Definition 1. First, because the pair (E,A) is stable, the first and third conditions in Definition 1 hold. Next, in view of Lemma 1, one has $R(j\omega) - R^*(j\omega) = -j\omega M^*(j\omega)M(j\omega)$. Therefore, $j[R(j\omega) - R^*(j\omega)] = \omega M^*(j\omega)M(j\omega) \ge 0$ for all $\omega \in (0,\infty)$; that is, the second condition in Definition 1 is true. According to Definition 1, we conclude that $R(s)$ is negative imaginary. ■

Remark 4: The condition of (E,A) being stable may be further relaxed since it is only used to show that $R(s)$ is analytic in $\Re[s] \ge 0$. Once this condition is relaxed to allow purely imaginary poles, it may be possible to derive new versions of lossless negative imaginary lemmas.

Similarly, a sufficient strict negative imaginary lemma can be obtained as follows.

Theorem 2: Consider a state-space realization (E,A,B,C,D) of the transfer function $R(s) \in \mathscr{R}^{m\times m}$. Suppose either Assumption 1 or Assumption 2 is satisfied and that the following conditions hold:

1) (E,A) is stable, $R(\infty) = R^T(\infty)$;
2) there exist matrices $X \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{n\times m}$, such that (14)–(17) hold.
3) $\mathrm{rank}(M(j\omega)) = m$ for all $\omega \in (0,\infty)$, where $M(s) = LA^{-1}E(sE-A)^{-1}B$ is defined in (9).

Then $R(s)$ is strictly negative imaginary.

Proof: Firstly, the condition that (E,A) is stable means that $R(s)$ has no poles in $\Re[s] \ge 0$. Secondly, in view of Theorem 1, the first and the second conditions in the theorem imply that $R(s)$ is negative imaginary. Furthermore, according to Lemma 1 we have

$$R(j\omega) - R^*(j\omega) = -j\omega M^*(j\omega)M(j\omega), \quad \forall \omega \in (0,\infty).$$

Multiplying both sides of the above equation by j leads to

$$j[R(j\omega) - R^*(j\omega)] = \omega M^*(j\omega)M(j\omega), \quad \forall \omega \in (0,\infty).$$

The third condition implies that $M^*(j\omega)M(j\omega) > 0$. Therefore $j[R(j\omega) - R^*(j\omega)] > 0$ for all $\omega \in (0,\infty)$. According to Definition 2, $R(s)$ is strictly negative imaginary. ■

The following corollary gives a sufficient condition to check if the system is both admissible and negative imaginary.

Corollary 1: Consider a state-space realization (E,A,B,C,D) of the transfer function $R(s) \in \mathscr{R}^{m\times m}$. Suppose that the following conditions hold:

1) $R(\infty) = R(\infty)^T$;
2) there exist matrices $X \in \mathbb{R}^{n\times n}$ and $Y \in \mathbb{R}^{n\times m}$ such that

$$
\begin{aligned}
A^T X + X^T A &< 0 \\
C^T + X^T E A^{-1} B &= A^T Y \\
E^T X = X^T E &\ge 0 \\
E^T Y &= 0.
\end{aligned}
$$

Then $R(s)$ is both admissible and negative imaginary.

Proof: In view of Lemma 2 of [26], the two conditions $E^T X = X^T E \ge 0$ and $A^T X + X^T A < 0$ guarantee that the pair (E,A) is admissible. Moreover, according to Theorem 1, $R(s)$ is negative imaginary. ■

Corollary 2: Consider a state-space realization (E,A,B,C,D) of the transfer function $R(s) \in \mathscr{R}^{m\times m}$. Suppose that the following conditions hold:

1) $R(\infty) = R^T(\infty)$;
2) there exist matrices $X \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{n\times m}$, such that

$$A^TX + X^TA < 0$$
$$C^T + X^TEA^{-1}B = A^TY$$
$$E^TX = X^TE \geq 0$$
$$E^TY = 0;$$

3) $\operatorname{rank}(M(j\omega)) = m$ for all $\omega \in (0,\infty)$, where $M(s) = LA^{-1}E(sE-A)^{-1}B$ is defined in (9).

Then $R(s)$ is both admissible and strictly negative imaginary.

When $E = I$, we have the following negative imaginary lemmas for standard linear systems without minimal state-space realization assumptions.

Corollary 3: Suppose that the following conditions hold:

1) A is Hurwitz, $D = D^T$;
2) there exists a matrix $X = X^T \in \mathbb{R}^{n\times n}$, such that

$$A^TX + XA \leq 0$$
$$C^T + XA^{-1}B = 0.$$

Then $R(s) = C(sI-A)^{-1}B + D$ is negative imaginary.

Proof: When $E = I$, the conditions in Theorem 1 reduce to the conditions in this corollary. Therefore, according to Theorem 1, $R(s) = C(sI-A)^{-1}B + D$ is negative imaginary. ■

Remark 5: Other versions of negative imaginary lemmas for standard linear systems without minimality assumptions have been reported in [10]. Corollary 3 may be considered as a dual of Lemma 2 in [10]. The main difference between Corollary 3 and Lemma 2 of [10] is that X is not required to be positive definite in Corollary 3. The reason is that we require the matrix A is Hurwitz.

Corollary 4: Suppose that the following conditions hold:

1) A is Hurwitz, $D = D^T$;
2) there exists a matrix $X = X^T \in \mathbb{R}^{n\times n}$, such that

$$A^TX + XA \leq 0$$
$$C^T + XA^{-1}B = 0;$$

3) $\operatorname{rank}(M(j\omega)) = m$ for all $\omega \in (0,\infty)$, where $M(s) = LA^{-1}(sI-A)^{-1}B$, and L is defined to satisfy $A^TX + XA + L^TL = 0$.

Then $R(s) = C(sI-A)^{-1}B + D$ is strictly negative imaginary.

IV. Illustrative Examples

Two examples are presented in this section. The first illustrates that the negative imaginary lemma developed in this paper is applicable to descriptor state-space realizations with impulse modes. The other illustrates the application of the negative imaginary theory to circuit networks.

Example 1: Consider the transfer function $R(s) = \frac{b}{(s+1)^2+1}$ with $b > 0$. On one hand, it follows from Definition 1 that $R(s)$ is negative imaginary. The Nyquist plot of $R(s)$ with $b = 10$ is shown in Fig. 1.

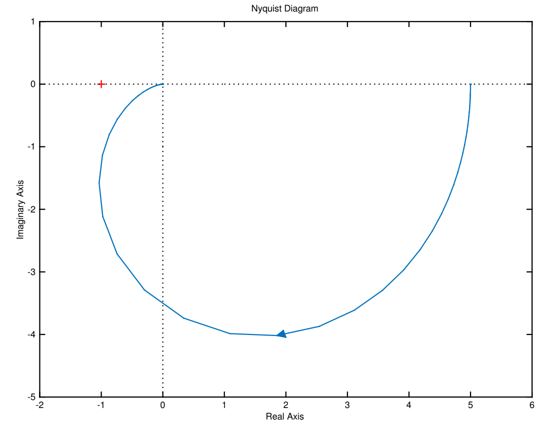

Fig. 1. Nyquist plot of closed-loop system $R(s) = \frac{10}{s^2+2s+2}$.

On the other hand, the underlying dynamical system might be a descriptor system that is not impulse free. For instance, a state-space realization is given by

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} b \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix}, \quad D = 0.$$

This realization is of the Weierstrass form. So we have

$$N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Because $N \neq 0$, the system is not impulse-free. Because $NB_2 = 0$, Theorem 1 holds and can be used to test the negative imaginary property of the system. A set of solutions to equations (14)–(17) of Theorem 1 is given by

$$X = \begin{bmatrix} \frac{1}{b} & -\frac{1}{b} & 0 & 0 \\ -\frac{1}{b} & \frac{3}{b} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore, $R(s)$ is negative imaginary for any $b > 0$ according to Theorem 1.

Example 2: Consider the single-loop circuit depicted in Fig. 2, which was used in [12], [13] as a motivation example for the study of descriptor systems.

The input of the circuit is the output of the controlled voltage source $V_s(t)$, the output of the circuit is the charge on the capacitor $Q_c(t)$. Let $I(t)$ be the current through the circuit, $V_R(t)$, $V_L(t)$ and $V_C(t)$ the voltages across the resistor, the inductor and the capacitor, respectively. A descriptor state-space system model (1) can be established, where

$$x(t) = \begin{bmatrix} I(t) \\ V_L(t) \\ V_C(t) \\ C_R(t) \end{bmatrix}, \quad u = V_s(t), \quad y(t) = Q_c(t),$$

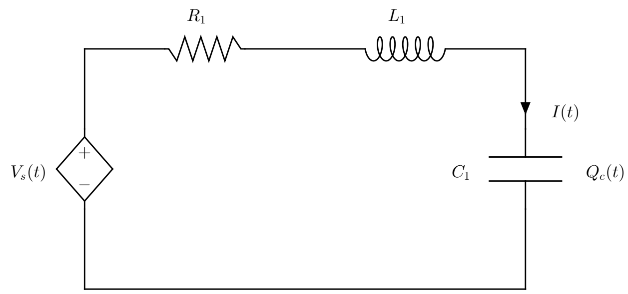

Fig. 2. A single-loop circuit network.

$$E=\begin{bmatrix}L_1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix},\quad A=\begin{bmatrix}0 & 1 & 0 & 0\\ \frac{1}{C_1} & 0 & 0 & 0\\ -R_1 & 0 & 0 & 1\\ 1 & 1 & 1 & 1\end{bmatrix},$$

$$B=\begin{bmatrix}0\\ 0\\ 0\\ -1\end{bmatrix},\quad C=\begin{bmatrix}0 & 0 & C_1 & 0\end{bmatrix},\quad D=0.$$

Because the equality $\deg\det(sE-A)=2=\operatorname{rank}(E)$ holds, the system is impulse free in view of Theorem 7.1 of [13]. As a result, Theorem 1 is applicable to check the negative imaginary property of the system. A set of solutions to equations (14)–(17) was found to be

$$X=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & C_1 & 0\\ 1 & 0 & 0 & 0\\ -1 & 0 & 0 & 0\end{bmatrix},\quad Y=\begin{bmatrix}0\\ 0\\ 0\\ 0\end{bmatrix}.$$

We can conclude that the system is negative imaginary according to Theorem 1.

V. Conclusions

This paper studied negative imaginary lemmas for descriptor linear systems. The state-space realization is not needed to be a minimal realization, and impulse modes may exist. Sufficient negative imaginary lemmas have been developed in terms of the state-space realization of the system. As corollaries, sufficient conditions have been derived to guarantee the system is both admissible and negative imaginary. Moreover, when descriptor systems reduce to standard linear systems, new negative imaginary lemmas have been found. Finally, two examples have been used to illustrate the developed theory.

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