Properties of positive feedback interconnected negative imaginary systems

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***Abstract*—This paper addresses the problem of finding a set of 'if and only if' criteria for a positive feedback interconnection of two Negative Imaginary (NI) systems to retain the NI property. First, we have derived a state-space characterisation for NI systems that does not impose the minimality restriction. Then, two more essential pre-requisite technical lemmas are provided to prove the main results of this paper, which deal with interconnected system properties. Under some technical assumptions, the paper shows that a positive feedback interconnection of two NI systems with possible pole(s) at $s = 0$ preserves the NI property. Subsequently, the result is specialised when the NI systems do not have any pole(s) at $s = 0$ and then to SNI systems. The proposed results have potential applications in decentralised control of large vehicle platoons. Furthermore, the results may be utilised in simplifying complex physical networks and in physical network analysis and synthesis.**

***Index Terms*—Negative imaginary systems, positive feedback, DC-gain, interconnected systems, internal stability.**

I. Introduction

A promising new development in the broad area of robust control and dissipative systems theory [1] is Negative Imaginary (NI) systems theory, first proposed in [2], which was originally motivated by the principle of position control of inertial systems via positive feedback [3]. These systems often arise from the models having colocated position sensors (or acceleration sensors) and force actuators [2], [4]. NI dynamics is highly correlated to the concept of Counter-clockwise Input-Output (I/O) systems [5], I/O Hamiltonian systems [6] and a new type of dissipative systems defined with respect to the input (u) and time derivative of the system's output ($\dot{y}$) [7]–[10]. Some potential applications of the NI toolkit can be found in multi-disciplinary engineering systems, such as large space structures [3], active electrical filters [11], robotic manipulators [12], large vehicle platoons [13], multi-agent networked systems [14], [15], Nano-positioning systems [16], [17]. Recently, NI property has been defined for improper (e.g. $-s$, $-s^2$, etc.) and non-rational (e.g. time-delay) systems [18], [19]. In addition, discrete-time version of NI theory is also proposed in the literature [20], [21]. Lately, NI arena is further broadened to capture *non-square* LTI SISO/MIMO systems with polytopic uncertainty [22].

This work was supported by the Engineering and Physical Sciences Research Council (EPSRC) [grant number EP/R008876/1]. All research data supporting this publication are directly available within this publication.

NI theory has become attractive primarily because of its closed-loop stability criterion that relies on the systems' gains only at $\omega = 0$. [2] first established that a positive feedback loop containing a stable NI and a strictly NI (abbreviated as SNI) systems remains asymptotically stable if the product of the systems' gains at $\omega = 0$ is smaller than one (celebrated as the *DC loop gain* criterion). This is the most fundamental result of the NI theory. Moreover, it is also an 'if and only if' criterion to ensure robust stability of an NI-SNI interconnection. Later, [23] showed that this stability criterion equally applies to the scenario where one of the systems may contain poles on the $j\omega$-axis except at $s = 0$. Further generalisation allows NI systems to have a maximum of two poles at $s = 0$ [12], [18], [19], [24]. Note that the stability results in [19] not only generalised the earlier stability results but also relaxed the restrictive assumptions imposed in the prior literature.

This brief establishes a set of 'if and only if' criteria for a positive feedback NI interconnection (as shown in Fig. 1) considering $j\omega$-axis poles, even at $s = 0$, to retain its NI property from input to output. Note that we particular focus on the cases where both the systems in an interconnection can have $j\omega$-axis poles. This study expands the developments of [25] to a wider set of NI systems allowing pole(s) at $s = 0$. In contrast to another earlier result proposed in [19], developed in a similar spirit, we herein remove the restrictive closed-loop internal stability assumption imposed in [19, Theorem 6]. Apart from the main results, this paper also derives a state-space characterisation for the whole NI class. It gives an LMI condition which is convenient for testing the NI property of a given system. We can now briefly outline the primary contributions of this work: (i) a set of 'if and only if' criteria to retain the NI property of an interconnection, comprised of two NI systems that may contain $j\omega$-axis poles including $s = 0$, in a closed-loop setting; (ii) a set of convenient and easy-to-test suppositions for which the celebrated *DC loop gain* criterion becomes 'if and only if'; and (iii) an LMI condition for checking the NI systems (allowing $j\omega$-axis poles including $s = 0$) property without imposing minimality assumption.

Notations

The sets of real and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$ respectively. The set of all $(m \times n)$ real matrices are denoted by $\mathbb{R}^{m\times n}$. A^* denotes the complex conjugate transpose of a matrix A while $A^\top$ indicates the transpose of a matrix. The symbol $\left(A^{-1}\right)^*$ is used as a shorthand for A^{-*} when exists. $(\cdot)'$ represents the transpose of a vector. For a matrix A having only real eigenvalues, $\lambda_{\max}(A)$ stands for its maximum eigenvalue. $R(j\omega)^* = R(-j\omega)^\top$ is noted for a transfer function matrix $R(s)$. $R(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ signifies a minimal state-space description of a proper, real, rational transfer function matrix $R(s)$.

II. Technical Background

This section will define the NI and SNI systems and provide some essential technical results.

Definition 1: (**NI System**) [12], [24] Let R be a square, dynamical, LTI system with a proper, real, rational transfer function matrix $R(s)$. Then, $R(s)$ is called an NI system if

- $j[R(j\omega) - R(j\omega)^*] \geq 0 \ \forall \omega \in (0,\infty)$ except the values of ω where $s = j\omega$ is a pole of $R(s)$;
- If $s = j\omega_0$ with $\omega_0 \in (0,\infty)$ is a pole of $R(s)$, then it is at most a simple pole and the residue matrix $\lim_{s\to j\omega_0}(s - j\omega_0)jR(s)$ is Hermitian and positive semidefinite;
- If $s = 0$ is a pole of $R(s)$, then $\lim_{s\to 0} s^k R(s) = 0$ for all $k \geq 3$ and $\lim_{s\to 0} s^2 R(s)$ is Hermitian and positive semidefinite.

Definition 2: (**SNI System**) [2] Let R be a square, dynamical, asymptotically stable, LTI system with a proper, real, rational transfer function matrix $R(s) \in \mathcal{RH}_\infty^{m\times m}$. Then, $R(s)$ is called an SNI system if $j[R(j\omega) - R(j\omega)^*] > 0$ $\forall \omega \in (0,\infty)$.

We will now present a state-space characterisation for NI systems with possible poles on the $j\omega$ axis including the origin. Lemma 1 can be considered as an unified version of [26, Lemma 2] and [27, Lemma 2] since this result also applies to NI systems with possible poles at $s = 0$ and does not require a minimal state-space description.

Lemma 1: Let R be a square, LTI, dynamical system with a proper, real, rational transfer function matrix $R(s)$ and a state-space description $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$. Then,

1) R is NI if $D = D^\top$ and $\exists$ a $P = P^\top$ such that
$$\begin{bmatrix} PA + A^\top P & PB - A^\top C^\top \\ B^\top P - CA & -CB - B^\top C^\top \end{bmatrix} \leq 0; \qquad (1)$$
2) $\exists$ a $P = P^\top \geq 0$ satisfying (1) and $D = D^\top$ if R is NI and $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ does not have any observable but uncontrollable modes.

Proof: We first define $F(s) = s[R(s) - D]$, which has a state-space description $\left[\begin{array}{c|c} A & B \\ \hline CA & CB \end{array}\right]$.

Part 1: Suppose there exists $P = P^\top \geq 0$ satisfying LMI (1). Then, there also exist $L \in \mathbb{R}^{m\times n}$ and $W \in \mathbb{R}^{m\times m}$ such that $\begin{bmatrix} PA + A^\top P & PB - A^\top C^\top \\ B^\top P - CA & -(CB + B^\top C^\top) \end{bmatrix} = \begin{bmatrix} -L^\top L & -L^\top W \\ -W^\top L & -W^\top W \end{bmatrix} \leq 0$. Hence, $F(s)$ is Positive Real (PR) via Corollaries 1 and 3 of [28]. We can then conclude that $R(s)$ is NI by following the same arguments given in the sufficiency part of the proof of [27, Lemma 2] (see the proof from the third sentence to the end of the sufficiency proof of [27, Lemma 2]).

Part 2: Since $R(s)$ is NI, $F(s)$ is PR via the necessity part of the proof of [27, Lemma 2] (see the proof from the first sentence to the second last sentence of the necessity proof of [27, Lemma 2]). As the realisation $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ does not have any uncontrollable but observable modes, $F(s) = \left[\begin{array}{c|c} A & B \\ \hline CA & CB \end{array}\right]$ also does not have any such modes. This happens due to the fact that the controllable/uncontrollable/unobservable modes remain intact during the transformation from NI to PR domain. Note also that as $F(s)$ is PR, Corollaries 2 and 3 of [28] can be appropriately used to guarantee the existence of a matrix $P = P^\top \geq 0$ satisfying (1). ■

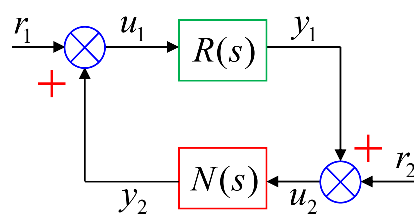

Fig. 1. Two NI systems $R(s)$ and $N(s)$ are connected via positive feedback.

We will now present the closed-loop stability theorem for an NI-SNI interconnection without imposing the earlier assumptions on the systems' gains at $\omega = \infty$ (which in turn imposed strict properness of the loop).

Theorem 1: (**NI-SNI stability theorem**) [24] Consider an NI system $R(s)$ without poles at $s = 0$ and an SNI system $N(s)$. Then the positive feedback closed-loop system comprised of $R(s)$ and $N(s)$, shown in Fig. 1, is asymptotically stable if and only if
$$\begin{cases} \det[I - R(\infty)N(\infty)] \neq 0, \\ \lambda_{\max}\left[(I - R(\infty)N(\infty))^{-1}(R(\infty)N(0) - I)\right] < 0, \\ \lambda_{\max}\left[(I - N(0)R(\infty))^{-1}(N(0)R(0) - I)\right] < 0. \end{cases} \qquad (2)$$
The readers are referred to [24] for results when $R(s)$ contains poles at $s = 0$.

We now recall the following lemma which gives a criterion for a system to become NI with poles on the $j\omega$ axis excluding $s = 0$. This lemma will be invoked in Section III-B to derive Theorem 2, which is the main contribution of this paper.

Lemma 2: [24] Suppose $R(s)$ is NI. Choose a negative definite matrix Ψ that satisfies $\lambda_{\max}[R(\infty)\Psi] < 1$. Then, $R_1(s) = R(s)[I - \Psi R(s)]^{-1}$ is NI without any pole at $s = 0$.

Lemma 2 states that, under some technical assumptions, the closed-loop system interconnection comprised of $R(s)$ and Ψ, via positive feedback, exhibits NI property.

III. Main results: Conditions for a positive feedback closed-loop interconnection comprised of NI systems to retain the NI property

Article [13] introduced a set of sufficient conditions to ensure stability of a string of coupled SNI subsystems. In contrast to [13] which requires SISO stable systems, we provide if and only if conditions for MIMO interconnected systems, as shown in Fig. 1, to be NI. Using this result, together with the stability results reported earlier, we can conclude the stability for arbitrarily many coupled MIMO NI systems with possible poles on the $j\omega$ axis, even at $s=0$. It has also been shown that, under some assumptions, the DC loop gain condition works as a simple testing criterion for an interconnected system to check whether it retains the NI property. Before presenting the main results (Theorems 2, 3, 4 and 5), we provide two technical lemmas in Subsection III-A as essential pre-requisites to prove the main theorems.

A. Pre-requisite technical lemmas

With reference to Lemma 2, the next lemma shows existence of a positive definite matrix $\bar{P}$ that can be described in terms of $R(s)$ and Ψ. In essence, Lemma 3 extends Lemma 2.

Lemma 3: Suppose $R(s)$ is an NI system with a minimal state-space description $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$. Choose a negative definite matrix Ψ that satisfies the condition $\lambda_{\max}[R(\infty)\Psi] < 1$. Then $\bar{P} = P - C^\top \Psi (I - D\Psi)^{-1} C > 0$ where $P = P^\top \geq 0$ satisfies (1).

Proof: Due to the fact that $R(s)$ is NI with $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ being minimal, there exist $P = P^\top \geq 0$, $L \in \mathbb{R}^{m\times n}$ and $W \in \mathbb{R}^{m\times m}$ such that $PA + A^\top P = -L^\top L$, $PB - A^\top C^\top = -L^\top W$ and $CB + B^\top C^\top = W^\top W$. We readily have $\bar{P} = P - C^\top \Psi (I - D\Psi)^{-1} C = P + C^\top (D - \Psi^{-1})^{-1} C = \begin{bmatrix} P^{1/2} \\ (D-\Psi^{-1})^{-1/2} C \end{bmatrix}^\top \begin{bmatrix} P^{1/2} \\ (D-\Psi^{-1})^{-1/2} C \end{bmatrix} \geq 0$. To prove $\bar{P}$ is nonsingular, we need to show that $\begin{bmatrix} P^{1/2} \\ C \end{bmatrix}$ has full column rank since $(D - \Psi^{-1})^{-1/2} > 0$. Suppose via contradiction that $\begin{bmatrix} P^{1/2} \\ C \end{bmatrix}$ lacks full column rank. Then, there exists $x \in \mathbb{R}^n$, $x \neq 0$, such that $\begin{bmatrix} P^{1/2} \\ C \end{bmatrix} x = 0$. Pre and post multiplying the expression $PA + A^\top P = -L^\top L$ with $x^\top$ and x, we get $x^\top (PA + A^\top P) x = -x^\top (L^\top L) x$, which implies $Lx = 0$. It then follows from $PB - A^\top C^\top = -L^\top W$ that $x^\top (PB - A^\top C^\top) = -x^\top L^\top W$, which implies $CAx = 0$ due to the results $Lx = 0$ and $P^{1/2} x = 0$. Continuing the same procedure, we can readily show that $\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} x = 0$. This contradicts the observability assumption. Therefore, we can ensure that $\begin{bmatrix} P^{1/2} \\ C \end{bmatrix}$ has full column rank. Hence, $\bar{P} = P - C^\top \Psi (I - D\Psi)^{-1} C > 0$. ■

Then, in Lemma 4, we show that the positive definite matrix $\bar{P}$ is a solution of the LMI conditions given in (3).

Lemma 4: Suppose the assumptions of Lemma 3 hold and furthermore, let $R_1(s) = R(s)[I - \Psi R(s)]^{-1}$ have a minimal state-space description $\left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right]$ with $\bar{A} = A + B\Psi(I - D\Psi)^{-1}C$, $\bar{B} = B(I - \Psi D)^{-1}$, $\bar{C} = (I - D\Psi)^{-1}C$ and $\bar{D} = D(I - \Psi D)^{-1}$. Then, $\bar{D} = \bar{D}^\top$, $\det(\bar{A}) \neq 0$ and $\bar{P} = P - C^\top \Psi (I - D\Psi)^{-1} C > 0$ satisfies

$$\bar{A}\bar{P}^{-1} + \bar{P}^{-1}\bar{A}^\top \leq 0 \quad \text{and} \quad \bar{B} + \bar{A}\bar{P}^{-1}\bar{C}^\top = 0 \qquad (3)$$

where $P = P^\top \geq 0$ is a solution of (1).

Proof. According to Lemma 2, $R_1(s) = R(s)[I - \Psi R(s)]^{-1}$ is NI with no poles at $s = 0$. It is easy to show that $\left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right]$ is a minimal state-space description of $R_1(s)$ where $\det(\bar{A}) \neq 0$ because $R_1(s)$ has no poles at the origin. Now, $\bar{D} = D(I - \Psi D)^{-1} = (I - D^\top \Psi^\top)^{-1} D^\top = \bar{D}^\top$ since $D = D^\top$ and $\Psi = \Psi^\top < 0$.

Moreover, as $R(s)$ is NI, there exist $P = P^\top \geq 0$, $L \in \mathbb{R}^{m\times n}$ and $W \in \mathbb{R}^{m\times m}$ via Lemma 1 such that

$$\begin{cases} PA + A^\top P = -L^\top L, \\ PB - A^\top C^\top = -L^\top W, \\ CB + B^\top C^\top = W^\top W. \end{cases}$$

We then define $\bar{P} = P - C^\top \Psi (I - D\Psi)^{-1} C$, $\bar{L} = L + W\Psi(I - D\Psi)^{-1} C$ and $\bar{W} = W(I - \Psi D)^{-1}$. Lemma 3 ensures that $\bar{P} = \bar{P}^\top > 0$. Via some algebraic manipulation, it can be readily shown that

$$\begin{cases} \bar{P}\bar{A} + \bar{A}^\top \bar{P} = -\bar{L}^\top \bar{L}, \\ \bar{P}\bar{B} - \bar{A}^\top \bar{C}^\top = -\bar{L}^\top \bar{W}, \\ \bar{C}\bar{B} + \bar{B}^\top \bar{C}^\top = \bar{W}^\top \bar{W}, \end{cases}$$

which in turn implies $\bar{A}\bar{P}^{-1} + \bar{P}^{-1}\bar{A}^\top \leq 0$ and $\bar{B} + \bar{A}\bar{P}^{-1}\bar{C}^\top = 0$ via [2] and [23]. ■

B. Properties of a positive feedback NI interconnection allowing poles at $s = 0$

This subsection caters the main contributions of the paper. Theorem 2 is a fundamental result that proposes a set of 'if and only if' criteria required for an NI interconnection, constructed via positive feedback, allowing poles on the $j\omega$ axis including $s = 0$ to preserve the NI property.

Theorem 2: Suppose $R(s)$ and $N(s)$ are NI systems. Choose $\Psi_1 < 0$ and $\Psi_2 < 0$ such that $\lambda_{\max}[N(\infty)\Psi_1] < 1$ and $\lambda_{\max}[\Psi_2(R(\infty) - \Psi_1)] < 1$. Then, the input-output transfer function mapping from $\begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$ to $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, designated by $\Sigma(s)$, is NI and has no poles at $s = 0$ (resp. $\det\left(\lim_{s\to 0}\left[[R(\infty)N(s) - I][I - \Psi_1 N(s)]^{-1}\right]\right) \neq 0$) if and only if

$\det[I - R(\infty)N(\infty)] \neq 0$,

$\lambda_{\max}[\lim_{s\to 0}[[I - \Psi_1 N(\infty)][I - R(\infty)N(\infty)]^{-1}[R(\infty)N(s) - I][I - \Psi_1 N(s)]^{-1}]] < 0$ and

$\lambda_{\max}[\lim_{s\to 0}[[I - \Psi_2(R(\infty) - \Psi_1)][I - N(s)R(\infty)]^{-1}[N(s)R(s) - I][I - \Psi_2(R(s) - \Psi_1)]^{-1}]] < 0$ (resp. ≤ 0).

Proof: The proofs for the cases where one or both of the systems are static are omitted since they can be obtained via similar (yet appropriately modified) arguments to those given below. Hence, we only consider the case where both $R(s)$ and $N(s)$ are dynamical systems.

(**Sufficiency**) Assume $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ and $\left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right]$ are two minimal state-space descriptions of $R(s)$ and $N(s)$ respectively. Then, via assumptions and by applying Lemma 1, $D = D^\top$, $\bar{D} = \bar{D}^\top$ and there exist the matrices $P = P^\top \geq 0$, $\bar{P} = \bar{P}^\top \geq 0$, W, L, $\bar{W}$ and $\bar{L}$ such that

$$\begin{cases} PA + A^\top P = -L^\top L, \\ PB - A^\top C^\top = -L^\top W, \\ CB + B^\top C^\top = W^\top W; \end{cases} \qquad \begin{cases} \bar{P}\bar{A} + \bar{A}^\top \bar{P} = -\bar{L}^\top \bar{L}, \\ \bar{P}\bar{B} - \bar{A}^\top \bar{C}^\top = -\bar{L}^\top \bar{W}, \\ \bar{C}\bar{B} + \bar{B}^\top \bar{C}^\top = \bar{W}^\top \bar{W}. \end{cases}$$

We define $U = (I - D\bar{D})$ and $V = (I - \bar{D}D)$. Now, the closed-loop system $\Sigma(s)$ has a state-space description $\left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right]$ where $\hat{A} = \begin{bmatrix} A + BV^{-1}\bar{D}C & BV^{-1}\bar{C} \\ \bar{B}U^{-1}C & \bar{A} + \bar{B}DV^{-1}\bar{C} \end{bmatrix}$, $\hat{B} = \begin{bmatrix} BV^{-1} & BV^{-1}\bar{D} \\ \bar{B}U^{-1}D & \bar{B}U^{-1} \end{bmatrix}$, $\hat{C} = \begin{bmatrix} U^{-1}C & DV^{-1}\bar{C} \\ V^{-1}\bar{D}C & V^{-1}\bar{C} \end{bmatrix}$ and $\hat{D} = \begin{bmatrix} U^{-1}D & DV^{-1}\bar{D} \\ V^{-1}\bar{D}D & V^{-1}\bar{D} \end{bmatrix}$.

Define $R_1(s) = [R(s) - \Psi_1]$ with a minimal state-space description $\left[\begin{array}{c|c} A & B \\ \hline C & D_1 \end{array}\right]$ where $D_1 = (D - \Psi_1)$ and $N_1(s) = [I - N(s)\Psi_1]^{-1} N(s)$ with a minimal state-space description $\left[\begin{array}{c|c} \bar{A}_1 & \bar{B}_1 \\ \hline \bar{C}_1 & \bar{D}_1 \end{array}\right]$ where $\bar{A}_1 = \bar{A} + \bar{B}\Psi_1(I - \bar{D}\Psi_1)^{-1}\bar{C}$, $\bar{B}_1 = \bar{B}(I - \Psi_1\bar{D})^{-1}$, $\bar{C}_1 = (I - \bar{D}\Psi_1)^{-1}\bar{C}$ and $\bar{D}_1 = \bar{D}(I - \Psi_1\bar{D})^{-1}$. Also, define $R_2(s) = [I - R_1(s)\Psi_2]^{-1} R_1(s)$ with a minimal state-space description $\left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right]$ where $A_2 = A + B\Psi_2(I - D_1\Psi_2)^{-1}C$, $B_2 = B(I - \Psi_2 D_1)^{-1}$, $C_2 = (I - D_1\Psi_2)^{-1}C$ and $D_2 = D_1(I - \Psi_2 D_1)^{-1}$ and $N_2(s) = [N_1(s) - \Psi_2]$ with a minimal state-space description $\left[\begin{array}{c|c} \bar{A}_1 & \bar{B}_1 \\ \hline \bar{C}_1 & \bar{D}_2 \end{array}\right]$ where $\bar{D}_2 = (\bar{D}_1 - \Psi_2)$. Note that $R_1(s)$ is NI because $R(s)$ is NI and Lemma 2 implies $N_1(s)$ is NI with no poles at $s = 0$. Then, via Lemma 3 and Lemma 4, there exist the matrices $P = P^\top \geq 0$, $L \in \mathbb{R}^{m \times n}$, $W \in \mathbb{R}^{m \times m}$, $\bar{P}_1 = \bar{P}_1^\top = \bar{P} - \bar{C}^\top\Psi_1(I - \bar{D}\Psi_1)^{-1}\bar{C} > 0$, $\bar{L}_1 = \bar{L} + \bar{W}\Psi_1(I - \bar{D}\Psi_1)^{-1}\bar{C}$ and $\bar{W}_1 = \bar{W}(I - \Psi_1\bar{D})^{-1}$ such that

$$\begin{cases} PA + A^\top P = -L^\top L, \\ PB - A^\top C^\top = -L^\top W, \\ CB + B^\top C^\top = W^\top W; \end{cases} \qquad \begin{cases} \bar{P}_1\bar{A}_1 + \bar{A}_1^\top \bar{P}_1 = -\bar{L}_1^\top \bar{L}_1, \\ \bar{P}_1\bar{B}_1 - \bar{A}_1^\top \bar{C}_1^\top = -\bar{L}_1^\top \bar{W}_1, \\ \bar{C}_1\bar{B}_1 + \bar{B}_1^\top \bar{C}_1^\top = \bar{W}_1^\top \bar{W}_1. \end{cases}$$

Similarly, $R_2(s)$ and $N_2(s)$ are both NI systems with no poles at $s = 0$ via Lemma 2. Then, via Lemma 3 and Lemma 4, there exist the matrices $P_2 = P_2^\top = P - C^\top\Psi_2(I - D_1\Psi_2)^{-1}C > 0$, $L_2 = L + W\Psi_2(I - D_1\Psi_2)^{-1}C$, $W_2 = W(I - \Psi_2 D_1)^{-1}$ along with $\bar{P}_1$, $\bar{L}_1$ and $\bar{W}_1$ such that

$$\begin{cases} P_2A_2 + A_2^\top P_2 = -L_2^\top L_2, \\ P_2B_2 - A_2^\top C_2^\top = -L_2^\top W_2, \\ C_2B_2 + B_2^\top C_2^\top = W_2^\top W_2; \end{cases} \qquad \begin{cases} \bar{P}_1\bar{A}_1 + \bar{A}_1^\top \bar{P}_1 = -\bar{L}_1^\top \bar{L}_1, \\ \bar{P}_1\bar{B}_1 - \bar{A}_1^\top \bar{C}_1^\top = -\bar{L}_1^\top \bar{W}_1, \\ \bar{C}_1\bar{B}_1 + \bar{B}_1^\top \bar{C}_1^\top = \bar{W}_1^\top \bar{W}_1. \end{cases}$$

Now define $\hat{P} = \begin{bmatrix} P - C^\top\bar{D}U^{-1}C & -C^\top V^{-1}\bar{C} \\ -\bar{C}^\top U^{-1}C & \bar{P} - \bar{C}^\top U^{-1}D\bar{C} \end{bmatrix}$.

$\hat{P} > 0$ (resp. ≥ 0) due to satisfying the set of three conditions given in Theorem 2. The proof follows from [24, Theorems 9 and 24]. Also, it can be shown via straightforward algebraic manipulations that

$$\begin{cases} \hat{P}\hat{A} + \hat{A}^\top \hat{P} = -\hat{L}^\top \hat{L}, \\ \hat{P}\hat{B} - \hat{A}^\top \hat{C}^\top = -\hat{L}^\top \hat{W}, \\ \hat{C}\hat{B} + \hat{B}^\top \hat{C}^\top = \hat{W}^\top \hat{W}, \end{cases} \tag{4}$$

where $\hat{L} = \begin{bmatrix} L + WV^{-1}\bar{D}C & WV^{-1}\bar{C} \\ \bar{W}U^{-1}C & \bar{L} + \bar{W}U^{-1}D\bar{C} \end{bmatrix}$ and $\hat{W} = \begin{bmatrix} WV^{-1} & WV^{-1}\bar{D} \\ \bar{W}U^{-1}D & \bar{W}U^{-1} \end{bmatrix}$. When $\hat{P} \geq 0$, the set of matrix equations (4) confirms that $\Sigma(s)$ is NI via Lemma 1 and $\lim_{s\to 0}\left[[R(\infty)N(s) - I][I - \Psi_1 N(s)]^{-1}\right]$ is nonsingular on noting that $\hat{D} = \hat{D}^\top$ and $\det[I - R(\infty)N(\infty)] \neq 0$. When $\hat{P} > 0$, it follows from [24, Theorem 9] that $\hat{A} = \begin{bmatrix} A_2P_2^{-1} & 0 \\ 0 & \bar{A}_1\bar{P}_1^{-1} \end{bmatrix}\hat{P}$ is nonsingular, $[I - R(\infty)N(\infty)]$ is nonsingular and $\left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right]$ is minimal. We can then conclude that in this case (i.e. when $\hat{P} > 0$), the input-output transfer function mapping $\Sigma(s)$ is NI with no poles at $s = 0$.

(**Necessity**) It follows through the following arguments.

$\Sigma(s)$ is NI and has no poles at $s = 0$
(resp. $\det(\lim_{s\to 0}[[R(\infty)N(s) - I][I - \Psi_1 N(s)]^{-1}]) \neq 0$)
and $\left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right]$ is minimal

$\Rightarrow \hat{P} = \begin{bmatrix} P - C^\top\bar{D}U^{-1}C & -C^\top V^{-1}\bar{C} \\ -\bar{C}^\top U^{-1}C & \bar{P} - \bar{C}^\top U^{-1}D\bar{C} \end{bmatrix} > 0$
(resp. ≥ 0) satisfies (4) with
$\hat{L} = \begin{bmatrix} L + WV^{-1}\bar{D}C & WV^{-1}\bar{C} \\ \bar{W}U^{-1}C & \bar{L} + \bar{W}U^{-1}D\bar{C} \end{bmatrix}$,
$\hat{W} = \begin{bmatrix} WV^{-1} & WV^{-1}\bar{D} \\ \bar{W}U^{-1}D & \bar{W}U^{-1} \end{bmatrix}$ and $\det[I - R(\infty)N(\infty)] \neq 0$ (resp. $\det[I - R(\infty)N(\infty)] \neq 0$ and $\det[\lim_{s\to 0}[[R(\infty)N(s) - I][I - \Psi_1 N(s)]^{-1}]] \neq 0$)

$\Leftrightarrow \det[I - R(\infty)N(\infty)] \neq 0$,
$\lambda_{\max}[\lim_{s\to 0}[[I - \Psi_1 N(\infty)][I - R(\infty)N(\infty)]^{-1} \times [R(\infty)N(s) - I][I - \Psi_1 N(s)]^{-1}]] < 0$ and
$\lambda_{\max}[\lim_{s\to 0}[[I - \Psi_2(R(\infty) - \Psi_1)][I - N(s)R(\infty)]^{-1}[N(s)R(s) - I][I - \Psi_2(R(s) - \Psi_1)]^{-1}]] < 0$ (resp. ≤ 0).

Hence the proof is done. ■

Example 1 illustrates the usefulness of Theorem 2.

Example 1: Suppose $R(s) = \frac{1}{s^2}$ and $N(s) = \frac{1}{s+1} - 1$, being respectively an NI and SNI systems, are connected via positive feedback according to Fig. 1. We select $\Psi_1 = -0.5 < 0$ and $\Psi_2 = -1 < 0$ satisfying $\lambda_{\max}[N(\infty)\Psi_1] = 0.5 < 1$ and $\lambda_{\max}[\Psi_2(R(\infty) - \Psi_1)] = -0.5 < 1$. Therefore, the input-output mapping $\Sigma(s)$ from $\begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$ to $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ is NI via Theorem 2 because $[I - R(\infty)N(\infty)] = 1$ is nonsingular, $\lambda_{\max}[\lim_{s\to 0}[[I - \Psi_1 N(\infty)][I - R(\infty)N(\infty)]^{-1}[R(\infty)N(s) - I][I - \Psi_1 N(s)]^{-1}]] = -0.5 < 0$ and $\lambda_{\max}[\lim_{s\to 0}[[I - \Psi_2(R(\infty) - \Psi_1)][I - N(s)R(\infty)]^{-1} [N(s)R(s) - I][I - \Psi_2(R(s) - \Psi_1)]^{-1}]] = 0 \le 0$. To confirm this outcome, we compute $\Sigma(s) = \frac{1}{s^3+s^2+s}\begin{bmatrix} s+1 & -s \\ -s & -s^3 \end{bmatrix}$ which is NI directly via definition.

Note that there are large number of matrices $\Psi_1 < 0$ and $\Psi_2 < 0$ that satisfy $\lambda_{\max}[N(\infty)\Psi_1] < 1$ and $\lambda_{\max}[\Psi_2(R(\infty) - \Psi_1)] < 1$. One can arbitrarily select $\Psi_1 < 0$ and $\Psi_2 < 0$ within these sets. Hence, the theorems given in this paper can be easily applied. Theorem 3 offers another set of conditions for an NI interconnection to retain the NI property, that is different from Theorem 2.

Theorem 3: Suppose $R(s)$ and $N(s)$ are NI systems. Choose $\Psi_1 < 0$ and $\Psi_2 < 0$ such that $\lambda_{\max}[R(\infty)\Psi_1] < 1$ and $\lambda_{\max}[\Psi_2(N(\infty) - \Psi_1)] < 1$. Then, the input-output transfer function mapping from $\begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$ to $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, designated by $\Sigma(s)$, is NI and has no pole(s) at $s = 0$ [resp. $\det\left(\lim_{s\to 0}\left[[R(\infty)N(s) - I][I - \Psi_2(N(s) - \Psi_1)]^{-1}\right]\right) \neq 0$] if and only if

$\det[I - R(\infty)N(\infty)] \neq 0$,
$\lambda_{\max}[\lim_{s\to 0}[[I - \Psi_2(N(\infty) - \Psi_1)][I - R(\infty)N(\infty)]^{-1}[R(\infty) N(s) - I][I - \Psi_2(N(s) - \Psi_1)]^{-1}]] < 0$ and
$\lambda_{\max}[\lim_{s\to 0}[[I - \Psi_1 R(\infty)][I - N(s)R(\infty)]^{-1}[N(s)R(s) - I][I - \Psi_1 R(s)]^{-1}]] < 0$ (resp. ≤ 0).

Proof: This proof follows along the same lines of Theorem 2 except that $R_1(s) = [I - R(s)\Psi_1]^{-1}R(s)$, $N_1(s) = [N(s) - \Psi_1]$, $R_2(s) = [R_1(s) - \Psi_2]$ and $N_2(s) = [I - N_1(s)\Psi_2]^{-1}N_1(s)$ are used instead of the particular $R_1(s)$, $N_1(s)$, $R_2(s)$ and $N_2(s)$ considered in Theorem 2. ■

Remark 1: It is not difficult to show that if we assume $N(s)$ to be an NI system with no pole(s) at $s = 0$ and $R(s)$ to be strictly proper, we can obtain a convenient and easy-to-test condition $\lambda_{\max}\left[\lim_{s\to 0}\left[[N(0)R(s) - I][I - \Psi_1 R(s)]^{-1}\right]\right] < 0$ (resp. ≤ 0) from Theorem 3, to verify whether the interconnected system is NI.

Remark 2: [19, Thereom 6] shows that the closed-loop feedback interconnection of two (strictly) NI systems is (strictly) NI if it is internally asymptotically stable. However, internal stability is an excessively restrictive assumption. In contrast, this paper does not impose such a restriction and instead, derives a set of less conservative 'if and only if' criteria for the results to hold. In Example 2, Theorem 4 is used to guarantee that the NI interconnection, constructed via feedback, inherits the NI property. However, the result [19, Thereom 6] cannot be applied in Example 2 since the closed-loop transfer function is not internally stable.

C. Properties of an NI interconnection with possible $j\omega$-axis poles excluding $s = 0$

We will now specialise the earlier result developed in Subsection III-B to the cases when neither of the NI systems in an interconnection contains pole(s) at $s = 0$. Theorem 4 gives a set of 'if and only if' criteria for a positive-feedback interconnection of two NI systems (see Fig. 1) allowing poles on the imaginary axis (excluding the origin) to maintain the NI property in closed-loop.

Theorem 4: Suppose $R(s)$ and $N(s)$ are NI systems with no poles at $s = 0$. Then, the input-output transfer function mapping from $\begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$ to $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, designated by $\Sigma(s)$, is NI and has no pole(s) at $s = 0$ $\big($resp. $\det(I - R(\infty)N(0)) \neq 0\big)$ if and only if

$$\begin{cases} \det[I - R(\infty)N(\infty)] \neq 0, \\ \lambda_{\max}\left[(I - R(\infty)N(\infty))^{-1}(R(\infty)N(0) - I)\right] < 0, \\ \lambda_{\max}\left[(I - N(0)R(\infty))^{-1}(N(0)R(0) - I)\right] < 0 \\ \qquad \text{(resp. } \le 0). \end{cases}$$

Proof: The proof can be readily done following that of Theorem 2, but with no use of the negative definite matrices Ψ_1 and Ψ_2. ■

Remark 3: When one of the systems, $R(s)$, is assumed to be strictly proper [i.e. $R(\infty) = 0$], the set of three criteria in Theorem 4 can be simplified to the celebrated *DC-gain* criterion [i.e. $\lambda_{\max}[N(0)R(0)] \le 1$], which is an 'if and only if' criterion for two interconnected systems (Fig. 1) to be NI.

Example 2 demonstrates the usefulness of Theorem 4.

Example 2: Let $R(s) = \frac{1}{s^2+1} - 2$ and $N(s) = \frac{1}{s^2+1} - 2$ be two NI systems with no poles at $s = 0$, connected via a positive feedback according to the configuration shown in Fig. 1. Hence, the input-output transfer function mapping $\Sigma(s)$ is an NI system via Theorem 4 because $[I - R(\infty)N(\infty)] = -3$ is nonsingular, $\lambda_{\max}[[I - R(\infty)N(\infty)]^{-1}(R(\infty)N(0) - I)] = -\frac{1}{3} < 0$ and $\lambda_{\max}[[I - N(0)R(\infty)]^{-1}(N(0)R(0) - I)] = 0 \le 0$. Direct calculation of $\Sigma(s)$ confirms this result.

D. Properties of an SNI interconnection

This subsection further specialises the results proposed in Subsection III-B and Subsection III-C (i.e. Theorems 2 and 4) to the situation where both the systems in the configuration given in Fig. 1 are SNI.

Theorem 5: Suppose $R(s)$ and $N(s)$ are SNI systems. Then, the input-output transfer function mapping from $\begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$ to

$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, designated by $\Sigma(s)$, is an SNI system if and only if

$$\begin{cases} \det[I - R(\infty)N(\infty)] \neq 0, \\ \lambda_{\max}\left[(I - R(\infty)N(\infty))^{-1}(R(\infty)N(0) - I)\right] < 0, \\ \lambda_{\max}\left[(I - N(0)R(\infty))^{-1}(N(0)R(0) - I)\right] < 0. \end{cases}$$

Proof: In the sufficiency part of the proof, the stability of $\Sigma(s)$ is ensured via [24]. As both $R(s)$ and $N(s)$ have SNI dynamics and $\Sigma(s) \in \mathcal{RH}_\infty$, $\Sigma(s)$ can be easily classified as an SNI system invoking [19, Theorem 6]. The necessity part of the proof is trivial due to [24] and hence is omitted. ■

To this end, we want to highlight that the stability problem considered in [13] could be resolved by utilising the aforementioned theorem and the stability results proposed in [24]. Unlike [13], which mainly focuses on the SISO cases, Theorem 5 applies to MIMO situation as well.

IV. Conclusion

In this brief, we derive a set of 'if and only if' criteria for the NI property to be preserved in a closed-loop when considering NI systems connected via positive feedback. It also demonstrates that the conditions required to maintain the NI dynamics of the closed-loop transfer function of an NI interconnection with/without any pole at $s = 0$ are similar. They differ only in a non-strict inequality, which replaces a strict inequality in one of the conditions. In contrast to earlier works in the literature, we propose 'if and only if' criteria without imposing asymptotic stability constraint for the overall closed-loop system and generalise the prior works to include NI systems allowing pole(s) at $s = 0$. The proposed theorems are useful for determining stability of a broader class of interconnected NI systems (e.g. stability analysis of a string of coupled NI/SNI subsystems with possible pole(s) at $s = 0$) and find potential applications in decentralised control of a platoon of car-like vehicles (as shown in [25]).

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