Eigenvalue-Based Algorithms for Testing Negative Imaginariness of Scalar Transfer Functions

Mei Liu, Kai Feng, and Alexander Lanzon, *Senior Member, IEEE*

***Abstract*—This article proposes several algorithms based on eigenvalue computations for testing negative imaginariness of scalar transfer functions of single-input single-output time-invariant linear systems. The input of the proposed eigenvalue-based algorithms can be any state-space realization, not necessarily minimal. The algorithms can test negative imaginariness of descriptor systems whose transfer functions may be nonproper, and only involve eigenvalues computation of matrices or matrix pencils with fewer restrictions. Finally, illustrative examples are provided to show how the proposed eigenvalue-based algorithms are applied for negative imaginariness test.**

***Index Terms*—Eigenvalue computations, negative imaginariness, nonproper transfer functions, state-space realization.**

I. Introduction

POSITIVE realness, as a fundamental concept in system and control theory [1], [2], has been widely applied in the study of circuit analysis [1], robust control [3], adaptive control [4], and absolute stability [5], [6]. An important research topic of the study of positive real systems theory is how to test positive real properties of a given transfer function in terms of the state-space representation. Methods for checking positive real properties of a given transfer function based on the frequency conditions, positive real lemma, Riccati equation, and Hamiltonian matrix have been developed in [1], [5], [7], [8], and [9]. However, for large-dimensional systems or high-order systems, the computational cost of these methods may be expensive. Several eigenvalue-based algorithms in [10] and [11], which have less computational cost, provide an efficient method to test positive realness of transfer functions of single-input single-output (SISO) systems.

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Mei Liu and Kai Feng are with the Department of Automation, School of Electrical and Information Engineering, Tianjin University, Tianjin 300072, China (e-mail: liumeimei@tju.edu.cn; fengkai_1001@tju.edu.cn).

Alexander Lanzon is with the Department of Electrical and Electronic Engineering, School of Engineering, University of Manchester, M13 9PL Manchester, U.K. (e-mail: Alexander.Lanzon@manchester.ac.uk).

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Although the research on positive real systems theory has achieved fruitful results, one limitation of positive real systems theory is that the relative degree of positive real transfer functions must be unity or zero [2]. Such restriction has inspired researchers to study negative imaginary (NI) systems whose relative degree is between zero and two [12], [13], [14], [15], [16], [17], [18], [19]. The concept of NI systems was first established in [12], where NI systems were required to be stable. Furthermore, the definition of NI systems has been extended to allow poles on the imaginary axis [13], [14], [15], [16], [17]. Particularly, the authors in [16] and [17], respectively, studied the nonproper irrational symmetric NI systems and the nonproper rational NI systems by allowing poles at infinity. In addition, the study of NI systems theory was extended to descriptor systems [18], networked NI systems [20], [21], [22], [23], fractional-order systems [24], and nonlinear systems [25], [26].

A key result of NI theory is the stability result developed in [12], [13], [27], [28], [29], and [30]. Lanzon and Petersen [12] first presented the internal stability result, by characterizing the gain condition at zero and infinite frequencies of positive feedback interconnections, between an NI system and a strictly negative imaginary (SNI) system. Then, the internal stability result was extended in [13] and [27] when NI systems have poles on the imaginary axis. The internal stability results of NI systems have important application in the study of robust control. The authors in [28] and [29] derived converse NI theorems and established necessary and sufficient conditions for robust feedback stability against diverse NI uncertainties. Moreover, the robust stability of feedback interconnections via frequency-dependent constraints has been studied in [30], which can also be applicable to NI systems. Several application examples of the NI stability results could be found in the control of trajectory tracking of a quadcopter uncrewed aerial vehicle [31], the robust cooperative control of multiagent systems [20], and the positioning problem of atomic force microscopes [15], [32]. Moreover, the research on designing feedback controllers for non-NI systems such that the resulting closed-loop systems are NI, referred to as NI synthesis problem, has been addressed in [15], [33], [34], and [35].

Along this line of research, another main issue on NI systems theory is how to test NI properties of a given transfer function. The definition of NI systems was first characterized by frequency response conditions [12], [13]. Under the assumption of minimal state-space realizations, the authors in [12], [13], and [15] established NI lemmas based on the linear matrix inequalities (LMIs) to test negative imaginariness. By removing the assumption of minimal realizations, NI lemmas in [12] and [13] were proved to be a sufficient condition for NI systems [36]. A method based on spectral conditions was studied in [32] to test NI or SNI properties of a given transfer function, by computing the spectrum of Hamiltonian matrices. The authors in [15] and [37] developed NI or SNI test conditions in terms of Riccati equations, and Salcan-Reyes and Lanzon [35] further established several necessary and sufficient conditions to test SNI or strongly SNI properties based on Riccati equations. A time-domain characterization of NI systems was proposed in [38].

The objective of this article is to provide a novel method based on eigenvalue computations of matrix or matrix pencil to determine negative imaginariness of SISO scalar transfer functions. The results of this article are also applicable to descriptor systems whose transfer functions may be nonproper. Motivations for studying the NI test method based on eigenvalue computations are threefold.

1) Many practical systems with large dimension or high order can be modeled as NI systems, such as *RLC* networks [39] and lightly damped flexible structures with a collocated force actuator and position sensor [13]. When determining NI properties of large-scale systems, the checking of frequency conditions and solving the LMIs are computationally expensive. Moreover, if solutions of the LMIs exist at the boundary of the convex sets, it is hard to compute the values of these solutions [32]. Motivated by the method of eigenvalue-based algorithms in [10] and [11], this article proposes several eigenvalue-based algorithms to test NI properties of a given transfer function.
2) The existing methods of judging negative imaginariness need to satisfy technical assumptions. For example, a minimal state-space realization is needed in [12], [13], [15], and [32] and the product of input matrix and output matrix is required to be positive in [15] and [32]. Removing those assumptions, the eigenvalue-based methods provided in this article are more extensive than the existing methods for testing NI properties of a given transfer function.
3) Descriptor systems have been applied in many engineering fields, such as electric power engineering [40] and integrated circuit design [41]. Recently, the study of NI properties of descriptor systems has been addressed in [18]. However, the results based on LMIs in [18] were only applicable to proper descriptor systems, while some engineering systems in practice may be modeled as descriptor systems with nonproper transfer functions [42]. Inspired by this, the eigenvalue-based algorithms proposed in this article provide a new negative imaginariness test approach, which is also applicable to descriptor systems with nonproper transfer functions.

The rest of this article is organized as follows. In Section II, the definitions of NI transfer functions of SISO systems are presented. In Section III, the eigenvalue-based characterization of negative imaginariness is provided. In Section IV, an algorithm based on eigenvalue computations is proposed for testing the NI properties of scalar transfer functions. In Section V, the NI test algorithm is extended to be applicable to descriptor systems. In Section VI, three examples are provided to verify the proposed eigenvalue-based algorithms. Finally, Section VII concludes this article.

Notations: $\mathbb{R}^{m\times n}$ denotes the set of $m \times n$ real matrices. $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. $\mathrm{Re}[\cdot]$ and $\mathrm{Im}[\cdot]$ are the real and imaginary part of complex numbers, respectively. A^T presents the transpose of a matrix A. $A > 0$ or $A \geq 0$ denotes a real symmetric positive definite or real symmetric positive semidefinite matrix. $\mathbf{0}$ denotes the zero vector with appropriate dimensions.

II. Preliminaries

Considering the following class of SISO linear time-invariant system:

$$\begin{cases} E\dot{x}(t) = Ax(t) + bu(t) \\ y(t) = c^T x(t) + du(t) \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}$ is the control input, $y(t) \in \mathbb{R}$ is the system output, $E, A \in \mathbb{R}^{n\times n}$, $b, c \in \mathbb{R}^n$, $d \in \mathbb{R}$. If E is singular, the system (1) is called a singular system or a descriptor system [43]. The matrix pencil $sE - A$ is called regular if $\det(sE - A) \neq 0$ for some $s \in \mathbb{C}$. The transfer function of system (1) is given by

$$h(s) = c^T (sE - A)^{-1} b + d. \tag{2}$$

To make the results meaningful, assume that $h(s) \not\equiv 0$ and the matrix pencil $sE - A$ is regular throughout of rest of this article. One important case of system (1) is that $E = I$. In this case, the form in (1) is a normal SISO system with the transfer function

$$h(s) = c^T (sI - A)^{-1} b + d.$$

The definition of NI transfer functions is introduced in the following to determine whether a SISO real-rational transfer function is NI.

Definition 1 (See [17]): A real-rational transfer function $h(s)$ is NI if the following conditions hold.

1) $h(s)$ has no poles in $\mathrm{Re}[s] > 0$.
2) For all $\omega > 0$ such that $j\omega$ is not a pole of $h(s)$, $\mathrm{Im}[h(j\omega)] \leq 0$.
3) If $s = j\omega_0$, $\omega_0 > 0$ is a pole of $h(s)$, then it is a simple pole, and the residue $K_0 = \lim_{s\to j\omega_0}(s - j\omega_0)jh(s) \geq 0$.
4) If $s = 0$ is a pole of $h(s)$, then $\lim_{s\to 0} s^k h(s) = 0$ for all $k \geq 3$ and $\lim_{s\to 0} s^2 h(s) \geq 0$.
5) If $s = j\infty$ is a pole of $h(s)$, then $\lim_{\omega\to\infty} \frac{h(j\omega)}{(j\omega)^k} = 0$ for all $k \geq 3$ and $\lim_{\omega\to\infty} \frac{h(j\omega)}{(j\omega)^2} \leq 0$.

Remark 1: Liu and Xiong [17] extended the definition of NI systems to the case where transfer functions may be nonproper and have poles at $j\infty$. The nonproper transfer function in this article means that the order of numerator polynomial is higher than the order of denominator polynomial. If only one proper transfer function $h(s)$ is considered, conditions 1–4 of Definition 1 are used to determine negative imaginariness of $h(s)$ [15]. The condition $\text{Im}[h(j\omega)] \leq 0$ is equivalent to $j[h(j\omega) - h(-j\omega)] \geq 0$.

The definitions of the lossless NI transfer function and SNI transfer function are introduced in the following. To avoid the ambiguity in determining SNI properties of nonproper transfer functions in the case of poles at infinity, we only consider proper SNI transfer function in this article.

Definition 2 (See [17] and [44]): A real-rational transfer function $h(s)$ is lossless NI if the following conditions hold.

1) $h(s)$ is NI.
2) $\text{Im}[h(j\omega)] = 0$ for all $\omega > 0$ except values of ω where $j\omega$ is a pole of $h(s)$.

Definition 3 (See [13]): A real-rational proper transfer function $h(s)$ is SNI if the following conditions hold.

1) $h(s)$ has no poles in $\text{Re}[s] \geq 0$.
2) $\text{Im}[h(j\omega)] < 0$ for all $\omega > 0$.

III. Eigenvalue-Based Characterization

In this section, characterizations of negative imaginariness based on eigenvalue computations are developed. In Section III-A, the transfer function $h(s)$ is firstly transformed into a new form of (3). In Section III-B, the transfer function $h(s)$ is further transformed into the form of Lemma 1, which is characterized in terms of determinants. The poles of $h(s)$ are transformed into eigenvalue computations of matrix pencils in Lemma 2. Moreover, Lemma 3 is established to test the condition $\text{Im}[h(j\omega)] \leq 0$ in NI definitions. Combining the results of Lemmas 1–3, eigenvalue-based characterizations of negative imaginariness are developed in Theorems 1 and 2.

A. Transformation of Transfer Functions

In this section, a transformation is introduced such that the transfer function $h(s)$ in (2) can be transformed into a simplified form (3) with the unit vector.

If $h(s) = c^T(sE - A)^{-1}b + d \equiv d$, then it follows that either $b = 0$ or $c = 0$. Then, $h(s) = d$ is lossless NI according to Definition 2.

For $h(s) \not\equiv d$, which implies that $b \neq 0$ and $c \neq 0$, two cases of $E = I$ and $E \neq I$ are studied to transform the transfer function $h(s) = c^T(sE - A)^{-1}b + d$ into the form

$$h(s) = \gamma e^T(sT - S)^{-1}b_1 + d \tag{3}$$

where $TS = ST, T, S \in \mathbb{R}^{n\times n}$, $b_1 \in \mathbb{R}^n$, $\gamma > 0$, $\gamma \in \mathbb{R}$, and e is the first unit vector of length n (i.e., $e = [1\ 0 \cdots 0]^T_{1\times n}$).

First, consider the case $E = I$. In view of the Householder transformation [45], since $c \neq 0$, there exists an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ satisfying $Q = Q^T = Q^{-1}$ such that

$$Qc = \|c\|_2 e \tag{4}$$

where $\|c\|_2$ stands for 2-norm of the vector c. In view of (4), $h(s) = c^T(sI - A)^{-1}b + d$ is rewritten as the form $h(s) = \gamma e^T(sT - S)^{-1}b_1 + d$ in (3), where

$$\gamma = \|c\|_2, T = I, S = Q^{-T}AQ^T \text{and } b_1 = Q^{-T}b.$$

Second, consider the case $E \neq I$. Select any $s_0 \in \mathbb{R}$ such that $\det(s_0E - A) \neq 0$. Then, a factorization of $s_0E - A$ is given as

$$s_0E - A = LU \tag{5}$$

where $L, U \in \mathbb{R}^{n\times n}$. The factorization in (5) can be any formal factorization, such as, a LU factorization with a lower triangular matrix L and a upper triangular matrix U, or a commonly used factorization with $L = s_0E - A$ and $U = I$. Obviously, matrices L and U are nonsingular and $L^{-1}(s_0E - A)U^{-1} = I$ holds. In view of (5) and $L^{-1}(s_0E - A)U^{-1} = I$, the transfer function $h(s) = c^T(sE - A)^{-1}b + d$ can be rewritten as

$$\begin{aligned} h(s) &= c^TU^{-1}(sL^{-1}EU^{-1} - L^{-1}AU^{-1})^{-1}L^{-1}b + d \\ &= c^TU^{-1}(sG - s_0G + I)^{-1}L^{-1}b + d \end{aligned} \tag{6}$$

where

$$G = L^{-1}EU^{-1}.$$

Moreover, it follows from $c \neq 0$ that $U^{-T}c \neq 0$. Thus, there exists a Householder transformation $\bar{Q} \in \mathbb{R}^{n\times n}$ such that

$$\bar{Q}U^{-T}c = \|U^{-T}c\|_2 e. \tag{7}$$

According to (6) and (7), the transfer function $h(s) = c^T(sI - A)^{-1}b + d$ has the form $h(s) = \gamma e^T(sT - S)^{-1}b_1 + d$ in (3), where

$$\gamma = \|U^{-T}c\|_2, T = \bar{Q}^{-T}G\bar{Q}^T$$
$$S = s_0T - I, b_1 = \bar{Q}^{-T}L^{-1}b.$$

B. Characterization of Negative Imaginariness

In this section, the eigenvalue-based characterization is introduced for testing NI properties of scalar transfer functions.

According to Section III-A, the transfer function $h(s)$ of system (1) can be written as the form $h(s) = \gamma e^T(sT - S)^{-1}b_1 + d$ in (3). Then, $h(s)$ is further transformed into a form with determinant in the following lemma.

Lemma 1: The transfer function $h(s) = \gamma e^T(sT - S)^{-1}b_1 + d$ in (3) can be transformed into the form

$$h(s) = \gamma \frac{\det(s\tilde{T} - \tilde{S})}{\det(sT - S)} + d \tag{8}$$

where

$$\tilde{T} = T(I - ee^T),\ \tilde{S} = S(I - ee^T) - b_1e^T. \tag{9}$$

Proof: Let $\mathbf{adj}_{(sT-S)}$ be the adjoint matrix of $sT - S$. In terms of the algebraic cofactor in linear algebra, the following condition holds:

$$e^T(sT - S)^{-1}b_1 = \frac{e^T\mathbf{adj}_{(sT-S)}b_1}{\det(sT - S)} = \frac{\det(s\tilde{T} - \tilde{S})}{\det(sT - S)} \tag{10}$$

where $s\tilde{T} - \tilde{S}$ is the matrix pencil that replaces the first column of matrix pencil $sT - S$ by the vector b_1. Furthermore, $\tilde{T}$ and $\tilde{S}$ can be given by

$$\tilde{T} = T(I - ee^T), \quad \tilde{S} = S(I - ee^T) - b_1e^T.$$

Thus, the transfer function $h(s)$ in (3) is rewritten as the form in (8). □

In view of (8), the poles of transfer function $h(s)$ are characterized in Lemma 2, which indicates that the poles of $h(s)$ can be obtained by computing eigenvalues of matrix pencils $sT - S$ and $s\tilde{T} - \tilde{S}$.

Lemma 2: Considering $h(s) = \gamma e^T(sT - S)^{-1}b_1 + d$ in (3). The poles of $h(s)$ are the finite eigenvalues of the matrix pencil $sT - S$, which are not the finite eigenvalues of the matrix pencil $s\tilde{T} - \tilde{S}$, where $\tilde{T}$ and $\tilde{S}$ are given by (9).

Proof: According to the form (8) in Lemma 1, the proof is trivial. □

To test negative imaginariness of the transfer function $h(s)$, we also need to check the condition $\text{Im}[h(j\omega)] \le 0$ in Definition 1. The following lemma shows that the checking condition $\text{Im}[h(j\omega)] \le 0$ can be transformed into a problem of eigenvalue computations.

Lemma 3: Considering $h(s) = \gamma e^T(sT - S)^{-1}b_1 + d$ in (3), $\text{Im}[h(j\omega)] \le 0$ holds for all $\omega > 0$ except values of ω_0, where $j\omega_0$ is a pole of $h(s)$, if and only if

$$\det(\lambda M + N) \ge 0 \tag{11}$$

for all $\lambda > 0$, where

$$M = T^2(I - ee^T),\ N = S^2(I - ee^T) + Tb_1e^T. \tag{12}$$

Proof: According to $h(s) = \gamma e^T(sT - S)^{-1}b_1 + d$ with $TS = ST$, it follows that:

$$\begin{aligned}\text{Im}[h(j\omega)] &= -\frac{1}{2}j[h(j\omega) - h(-j\omega)] \\ &= -\frac{1}{2}j\gamma e^T[(j\omega T - S)^{-1} - (-j\omega T - S)^{-1}]b_1.\end{aligned} \tag{13}$$

Due to $TS = ST$, it follows that:

$$\begin{aligned}\omega^2T^2 + S^2 &= -(j\omega T - S)(j\omega T + S) \\ &= -(j\omega T + S)(j\omega T - S).\end{aligned} \tag{14}$$

It follows from (14) that

$$(j\omega T - S)^{-1} - (-j\omega T - S)^{-1} = -2j\omega[\omega^2T^2 + S^2]^{-1}T. \tag{15}$$

Thus

$$\text{Im}[h(j\omega)] = -\omega\gamma e^T(\omega^2T^2 + S^2)^{-1}Tb_1. \tag{16}$$

Similar to Lemma 1, let $\mathbf{adj}_{(\omega^2T^2+S^2)}$ be the adjoint matrix of $\omega^2T^2 + S^2$. In terms of the algebraic cofactor in linear algebra, it follows that:

$$\begin{aligned}e^T(\omega^2T^2 + S^2)^{-1}Tb_1 &= \frac{e^T\mathbf{adj}_{(\omega^2T^2+S^2)}Tb_1}{\det(\omega^2T^2 + S^2)} \\ &= \frac{\det(\omega^2M + N)}{\det(\omega^2T^2 + S^2)}\end{aligned} \tag{17}$$

where $\omega^2M + N$ is the matrix pencil that replaces the first column of matrix pencil $\omega^2T^2 + S^2$ by the vector Tb_1. Thus, M and N are given by

$$M = T^2(I - ee^T), N = S^2(I - ee^T) + Tb_1e^T.$$

Then, $\text{Im}[h(j\omega)]$ is written as

$$\text{Im}[h(j\omega)] = -\omega\gamma\frac{\det(\omega^2M + N)}{\det(\omega^2T^2 + S^2)}. \tag{18}$$

In view of [10], since T and S are real matrices and (14) holds, one has that

$$\det(\omega^2T^2 + S^2) = |\det(S + j\omega T)|^2. \tag{19}$$

Then, (18) is rewritten as

$$\text{Im}[h(j\omega)] = -\omega\gamma\frac{\det(\omega^2M + N)}{|\det(S + j\omega T)|^2}. \tag{20}$$

Since $\gamma > 0$ and $|\det(S + j\omega T)|^2$ is nonnegative for all $\omega > 0$, then $\text{Im}[h(j\omega)] \le 0$ holds for all $\omega > 0$ except values of ω_0, where $j\omega_0$ is a pole of $h(s)$, if and only if

$$\det(\lambda M + N) \ge 0$$

for all $\lambda = \omega^2 > 0$ except values of $\lambda = \omega_0^2$, where $\omega_0 > 0$ and $j\omega_0$ is a pole of $h(s)$. Obviously, $\det(\lambda M + N)$ is a real polynomial with degree at most $n - 1$. By the continuity of $\det(\lambda M + N)$ with respect to λ, the condition $\det(\lambda M + N) \ge 0$ also holds for $\lambda = \omega_0^2$. Thus, the condition of $\text{Im}[h(j\omega)] \le 0$ in Definition 1 for judging negative imaginariness of $h(s)$ is equivalent to judging $\det(\lambda M + N) \ge 0$ for all $\lambda > 0$. □

Remark 2: If a real λ is a root of $\det(\lambda M + N) = 0$ with odd algebraic multiplicity, the polynomial $\det(\lambda M + N)$ changes its sign at the real λ. Thus, to ensure $\det(\lambda M + N) \ge 0$ for all $\lambda > 0$, any possible real positive root λ of $\det(\lambda M + N) = 0$ has even algebraic multiplicity and there exists at least one point $\lambda_0 > 0$ such that $\det(\lambda_0 M + N) > 0$.

According to the transformations in Sections III-A and III-B, an eigenvalue-based characterization of negative imaginariness is introduced in the following Theorem 1. The idea of Theorem 1 is motivated by the eigenvalue-based characterization of positive realness in [10]. To simplify Theorem 1, let Ω denote the set of the finite eigenvalues of the matrix pencil $sT - S$, which are not the finite eigenvalues of the matrix pencil $s\tilde{T} - \tilde{S}$. In view of Lemma 2, Ω is the set of poles of $h(s)$ after possible zero-pole cancellation.

Theorem 1: Consider the transfer function $h(s)$ in (3), the matrices $\tilde{T}$ and $\tilde{S}$ in (9), M and N in (12). Then, $h(s)$ is NI if and only if the following holds.

1) All values in Ω have no positive real part.
2) For any purely imaginary values in Ω, conditions 3–5 of Definition 1 must hold.
3) The matrix pencil $\lambda M + N$ is either singular, or the following two conditions hold.
   i) Any real positive eigenvalue, $\lambda > 0$, of the matrix pencil $\lambda M + N$ has even algebraic multiplicity.
   ii) There exists at least one point $\lambda_0 > 0$ such that $\det(\lambda_0 M + N) > 0$.

Furthermore, if $\lambda M + N$ is singular, then $\text{Im}[h(j\omega)] = 0$ holds, and hence, the transfer function $h(s)$ is lossless NI.

Proof: In view of Lemma 2, conditions 1) and 2) are equivalent to conditions 1, 3, 4, and 5 of Definition 1. According to Lemma 3, condition 2 of Definition 1 holds if and only if

$\det(\lambda M + N) \geq 0$ for all $\lambda > 0$, which is equivalent to condition 3).

Furthermore, according to Definition 2, $h(s)$ is lossless NI if $\lambda M + N$ is singular. □

Remark 3: Theorem 1 is applicable to a nonproper transfer function $h(s)$. If $h(s)$ is proper, then condition 5 of Definition 1 is not considered.

In view of (20), $\mathrm{Im}[h(j\omega)] < 0$ for all $\omega > 0$ if and only if $\det(\lambda M + N) > 0$ for all $\lambda > 0$. Thus, the next theorem provides an eigenvalue-based criterion to check whether $h(s)$ is SNI.

Theorem 2: Consider the proper transfer function $h(s)$ in (3), the matrices $\tilde{T}$ and $\tilde{S}$ in (9), and M and N in (12). Then, the proper transfer function $h(s)$ is SNI if and only if the following conditions hold.

1) All values in Ω have negative real part.
2) All the finite eigenvalues of the matrix pencil $\lambda M + N$ are not real positive eigenvalues.
3) There exists at least one point $\lambda_0 > 0$ such that $\det(\lambda_0 M + N) > 0$.

Proof: In view of Lemma 2, condition 1) in this theorem is equivalent to condition 1 of Definition 3. In view of (20), condition 2 of Definition 3 holds if and only if $\det(\lambda M + N) > 0$ for all $\lambda > 0$, which is equivalent to the conditions 2) and 3). □

IV. Eigenvalue-Based Algorithms for $E = I$

In this section, an algorithm based on eigenvalue computations is developed for testing negative imaginariness of normal SISO systems with $E = I$.

According to Theorems 1 and 2, the eigenvalue-based test of negative imaginariness involves computations of eigenvalues of matrix pencils $sT - S$, $s\tilde{T} - \tilde{S}$, and $\lambda M + N$. In view of [10], [11], and [45], the eigenvalue computations of matrix pencils cost less than the eigenvalue computations of matrices. Thus, in this section, the eigenvalue-based algorithm only involving eigenvalue computations of matrices is proposed to check NI properties of system (1) with $E = I$.

In the case of $E = I$, due to

$$T = I \quad \text{and} \quad S = Q^{-T} A Q^{T}$$

it can be verified that

$$\begin{aligned}\det(sT - S) &= \det(Q^{-T})\det(sI - A)\det(Q^{T}) \\ &= \det(sI - A)\end{aligned}$$

holds. Thus, the eigenvalues of the matrix A and the matrix pencil $sT - S$ are equivalent. Then, the following Lemma 4, which only involves the eigenvalue computations of matrices, provides a method to address the eigenvalues of the matrix pencil $s\tilde{T} - \tilde{S}$.

Lemma 4: Let $\tilde{T} = T(I - ee^T)$ and $\tilde{S} = S(I - ee^T) - b_1 e^T$ as in (9). Partition $\tilde{S}$ as $\tilde{S} = \left[\begin{smallmatrix} s_{11} & s_{12}^T \\ s_{21} & S_1 \end{smallmatrix}\right] \in \mathbb{R}^{n\times n}$, where $s_{11} \in \mathbb{R}$, $S_1 \in \mathbb{R}^{(n-1)\times(n-1)}$, and $s_{12}, s_{21} \in \mathbb{R}^{n-1}$. Then, the following three cases are considered to compute the eigenvalues of the matrix pencil $s\tilde{T} - \tilde{S}$.

1) If $s_{11} \neq 0$, then the eigenvalues of the matrix pencil $s\tilde{T} - \tilde{S}$ are the eigenvalues of the matrix $\hat{S} = S_1 - (1/s_{11}) s_{21} s_{12}^T$.
2) If $s_{11} = 0$ and $s_{21} = \mathbf{0}$, then $\det(s\tilde{T} - \tilde{S}) = 0$ holds for all values of $s \in \mathbb{C}$.
3) If $s_{11} = 0$ and $s_{21} \neq \mathbf{0}$, then compute a Householder transformation matrix Q_1 such that $Q_1 s_{21} = \|s_{21}\|_2 e$. Let $\tilde{S} = (I - ee^T) Q_1 S_1 Q_1^{-1} + e s_{12}^T Q_1^{-1}$ and repartition $\tilde{S}$ as $\left[\begin{smallmatrix} s_{11} & s_{12}^T \\ s_{21} & S_1 \end{smallmatrix}\right]$. Then, Cases 1–3 are considered again to compute the eigenvalues of the matrix pencil $s\tilde{T} - \tilde{S}$.

Moreover, a special case is that $\tilde{S} \in \mathbb{R}^{2\times 2}$ and $s_{11} = 0$. In this case, the above Cases 2 and 3 are not applicable, and $\det(s\tilde{T} - \tilde{S})$ is a constant for all values of $s \in \mathbb{C}$.

Proof: In view of $\tilde{T}$, $\tilde{S}$ in (9), and $T = I$, let

$$\det(s\tilde{T} - \tilde{S}) = \det\left(\begin{bmatrix} -s_{11} & -s_{12}^T \\ -s_{21} & sI - S_1 \end{bmatrix}\right) \tag{21}$$

where $s_{11} \in \mathbb{R}$, $s_{12}, s_{21} \in \mathbb{R}^{n-1}$, and $S_1 \in \mathbb{R}^{(n-1)\times(n-1)}$.

If $s_{11} \neq 0$, it follows that:

$$\det(s\tilde{T} - \tilde{S}) = -s_{11}\det(sI - S_1 + (1/s_{11}) s_{21} s_{12}^T).$$

Thus, the eigenvalues of the matrix pencil $s\tilde{T} - \tilde{S}$ are the eigenvalues of the matrix $S_1 - (1/s_{11}) s_{21} s_{12}^T$.

Then, consider the case of $s_{11} = 0$. If $s_{21} = \mathbf{0}$, then one has that $\det(s\tilde{T} - \tilde{S}) = 0$ and $h(s)$ is lossless NI. If $s_{21} \neq \mathbf{0}$, then there exists a Householder transformation $Q_1 \in \mathbb{R}^{(n-1)\times(n-1)}$ such that

$$Q_1 s_{21} = \|s_{21}\|_2 e \tag{22}$$

where e is the first unit vector of length $n-1$. Let

$$\hat{Q} = \begin{bmatrix} 1 & 0 \\ 0 & Q_1 \end{bmatrix}.$$

Then, it follows that:

$$\begin{aligned}&\det(s\tilde{T} - \tilde{S}) \\ &= \det(\hat{Q})\det(s\tilde{T} - \tilde{S})\det(\hat{Q}^{-1}) \\ &= \det\left(\begin{bmatrix} 0 & -s_{12}^T Q_1^{-1} \\ -\|s_{21}\|_2 e & sI - Q_1 S_1 Q_1^{-1} \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} 0 & -s_{22} & -s_{13}^T \\ -\|s_{21}\|_2 & s - s_1 & -s_2 \\ \mathbf{0} & -s_{31} & sI - S_2 \end{bmatrix}\right) \\ &= \|s_{21}\|_2 \det\left(\begin{bmatrix} -s_{22} & -s_{13}^T \\ -s_{31} & sI - S_2 \end{bmatrix}\right),\end{aligned} \tag{23}$$

where

$$s_{12}^T Q_1^{-1} = [s_{22}\ s_{13}^T],$$

$$Q_1 S_1 Q_1^{-1} = \begin{bmatrix} s_1 & s_2 \\ s_{31} & S_2 \end{bmatrix}$$

and s_1, $s_{22} \in \mathbb{R}$, s_2^T, $s_{13}, s_{31} \in \mathbb{R}^{n-2}$, and $S_2 \in \mathbb{R}^{(n-2)\times(n-2)}$. It follows that:

$$\begin{bmatrix} s_{22} & s_{13}^T \\ s_{31} & S_2 \end{bmatrix} = (I - ee^T) Q_1 S_1 Q_1^{-1} + e s_{12}^T Q_1^{-1}.$$

Thus, in the Case 3 of Lemma 4, we set

$$\tilde{S} = (I - ee^T)Q_1 S_1 Q_1^{-1} + es_{12}^T Q_1^{-1}$$

and repartition $\tilde{S}$ as

$$\tilde{S} = \begin{bmatrix} s_{11} & s_{12}^T \\ s_{21} & S_1 \end{bmatrix} \in \mathbb{R}^{(n-1)\times(n-1)}$$

where $s_{11} = s_{22} \in \mathbb{R}$, $s_{12} = s_{13}, s_{21} = s_{31} \in \mathbb{R}^{n-2}$, $S_1 = S_2 \in \mathbb{R}^{(n-2)\times(n-2)}$.

Now, according to (23), to compute the eigenvalues of the matrix pencil $s\tilde{T} - \tilde{S}$, the roots of

$$\det(s\tilde{T} - \tilde{S}) = \|s_{21}\|_2 \det\left(\begin{bmatrix} -s_{11} & -s_{12}^T \\ -s_{21} & sI - S_1 \end{bmatrix}\right) = 0$$

need to be computed. $\|s_{21}\|_2$ is a nonzero constant, which does not change the eigenvalues of the matrix pencil $s\tilde{T} - \tilde{S}$. Thus, we only compute the roots of

$$\det\left(\begin{bmatrix} -s_{11} & -s_{12}^T \\ -s_{21} & sI - S_1 \end{bmatrix}\right) = 0$$

which has the same block structure as (21). Thus, by repeating the above analysis process, the eigenvalues of the matrix pencil $s\tilde{T} - \tilde{S}$ or $\det(s\tilde{T} - \tilde{S}) = 0$ are obtained. In addition, considering a special case that $\tilde{S} \in \mathbb{R}^{2\times 2}$ and $s_{11} = 0$, it follows from (21) that

$$\begin{aligned} \det(s\tilde{T} - \tilde{S}) &= \det\left(\begin{bmatrix} 0 & -s_{12}^T \\ -s_{21} & sI - S_1 \end{bmatrix}\right) \\ &= -s_{21}s_{12}^T \end{aligned} \tag{24}$$

which is a constant for all values of $s \in \mathbb{C}$. □

Example 1: A transfer function is provided in this example to show how Lemma 4 is applied. Consider the following transfer function:

$$h(s) = c^T(sI - A)^{-1}b + d = \frac{4s + 3}{s(s + 1)(s + 3)} \tag{25}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$c = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}, d = 0. \tag{26}$$

Thus, $h(s)$ is transformed into the form $h(s) = \gamma e^T(sT - S)^{-1}b_1 + d$, where $\gamma = 5$, $T = I$, and

$$S = Q^{-T}AQ^T = \begin{bmatrix} 0.48 & -0.36 & 0.8 \\ 0.64 & -0.48 & -0.6 \\ -2.4 & 1.8 & -4 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0.6 & 0.8 & 0 \\ 0.8 & -0.6 & 0 \\ 0 & 0 & 1 \end{bmatrix}, b_1 = Q^{-T}b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

It follows from Lemma 4 that:

$$\tilde{S} = \begin{bmatrix} 0 & -0.36 & 0.8 \\ 0 & -0.48 & -0.6 \\ -1 & 1.8 & -4 \end{bmatrix}$$

with $s_{11} = 0$ and $s_{21} \neq \mathbf{0}$. Thus, according to Case 3 of Lemma 4, a new matrix

$$\tilde{S} = \begin{bmatrix} -0.8 & 0.36 \\ -0.6 & -0.48 \end{bmatrix}$$

is established with $s_{11} = -0.8 \neq 0$. Then, Case 1 of Lemma 4 is applied and the eigenvalue of $s\tilde{T} - \tilde{S}$ is -0.75.

The eigenvalues of the matrix pencil $\lambda M + N$ can be computed by using the similar method in Lemma 4. The specific analysis is as follows.

Partition N as

$$N = \begin{bmatrix} n_{11} & n_{12}^T \\ n_{21} & N_1 \end{bmatrix} \tag{27}$$

where $n_{11} \in \mathbb{R}$, $n_{12}, n_{21} \in \mathbb{R}^{n-1}$, and $N_1 \in \mathbb{R}^{(n-1)\times(n-1)}$. Then, in view of the matrix $M = T^2(I - ee^T)$ in (12), $T = I$, and the matrix N in (27), let

$$\det(\lambda M + N) = \det\left(\begin{bmatrix} n_{11} & n_{12}^T \\ n_{21} & \lambda I + N_1 \end{bmatrix}\right). \tag{28}$$

If $n_{11} < 0$, it follows that:

$$\det(\lambda M + N) = n_{11}\det(\lambda I + N_1 - (1/n_{11})n_{21}n_{12}^T).$$

Obviously, there exists $\lambda > 0$ such that $\det(\lambda M + N) < 0$. Thus, $h(s)$ is not NI.

If $n_{11} > 0$

$$\det(\lambda M + N) = n_{11}\det(\lambda I + N_1 - (1/n_{11})n_{21}n_{12}^T)$$

holds. Hence, the eigenvalues of the matrix pencil $\lambda M + N$ are the eigenvalues of the matrix $(1/n_{11})n_{21}n_{12}^T - N_1$.

If $n_{11} = 0$, then we assume that $n_{21} \neq \mathbf{0}$; otherwise, $h(s)$ is lossless NI. Due to $n_{21} \neq \mathbf{0}$, there exists a Householder transformation $Q_2 \in \mathbb{R}^{(n-1)\times(n-1)}$ such that

$$Q_2 n_{21} = -\|n_{21}\|_2 e \tag{29}$$

where e is the first unit vector of length $n - 1$. Let

$$\hat{Q} = \begin{bmatrix} 1 & 0 \\ 0 & Q_2 \end{bmatrix}.$$

Then, it follows that:

$$\begin{aligned} \det(\lambda M + N) &= \det(\hat{Q})\det(\lambda M + N)\det(\hat{Q}^{-1}) \\ &= \det\left(\begin{bmatrix} 0 & n_{12}^T Q_2^{-1} \\ -\|n_{21}\|_2 e & \lambda I + Q_2 N_1 Q_2^{-1} \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} 0 & n_{22} & n_{13}^T \\ -\|n_{21}\|_2 & \lambda + n_1 & n_2 \\ \mathbf{0} & n_{31} & \lambda I + N_2 \end{bmatrix}\right) \\ &= \|n_{21}\|_2 \det\left(\begin{bmatrix} n_{22} & n_{13}^T \\ n_{31} & \lambda I + N_2 \end{bmatrix}\right) \end{aligned} \tag{30}$$

where

$$n_{12}^T Q_2^{-1} = [n_{22}\ n_{13}^T]$$

$$Q_2 N_1 Q_2^{-1} = \begin{bmatrix} n_1 & n_2 \\ n_{31} & N_2 \end{bmatrix}$$

and $n_1, n_{22} \in \mathbb{R}$, $n_2^T, n_{13}, n_{31} \in \mathbb{R}^{n-2}$, and $N_2 \in \mathbb{R}^{(n-2)\times(n-2)}$. The last line of (30) has the same block structure as (28). Thus, by repeating the above analysis, the computation of the eigenvalues of the matrix pencil $\lambda M + N$ is transformed into a eigenvalue computation problem of a matrix $(1/n_{11})n_{21}n_{12}^T - N_1$. Similar to Lemma 4, the special case of $N \in \mathbb{R}^{2\times 2}$ and $n_{11} = 0$ also needs to be considered, which is noted in Remark 4.

In view of Theorem 1, Theorem 2, and the above analysis for eigenvalue computations of matrix pencils $sT - S$, $s\tilde{T} - \tilde{S}$, and $\lambda M + N$, the method only involving eigenvalue computations of matrices is summarized in Algorithm 1 for testing NI properties of scalar transfer functions with $E = I$.

Remark 4: When the matrix $N \in \mathbb{R}^{2\times 2}$ and $n_{11} = 0$ hold in Algorithm 1, $\det(\lambda M + N) = -n_{21}n_{12}^T$ is a constant in view of (28). In this case, Steps 7–9 in Algorithm 1 can be omitted. Furthermore, if $\det(\lambda M + N) = -n_{21}n_{12}^T = 0$, $h(s)$ is lossless NI. If $n_{21}n_{12}^T > 0$, then $h(s)$ is not NI. If $n_{21}n_{12}^T < 0$, whether $h(s)$ is NI or SNI depends on the judgment of the pole condition of $h(s)$ in Step 4.

Remark 5: In Algorithm 1, Steps 1–3 transform $h(s)$ into the form (3). Step 4 is used to test conditions 1 and 2 in Theorem 1. Condition 3 in Theorem 1 is tested in Steps 5–8. Step 9 identified SNI properties according to Theorem 2. The frameworks of Algorithms 2 and 3 in the next section are similar to Algorithm 1. The Householder transformation matrix Q is an orthogonal matrix satisfying $Q = Q^T = Q^{-1}$, which can reduce some unnecessary calculations in Algorithm 1. Moreover, [45, Ch. 5] provides a method to compute the matrix Q. In addition to obtaining the eigenvalues of the matrix pencil $s\tilde{T} - \tilde{S}$, Lemma 4 may also output that $\det(s\tilde{T} - \tilde{S}) = 0$ or $\det(s\tilde{T} - \tilde{S})$ is equal to a nonzero constant for all values of $s \in \mathbb{C}$. If $\det(s\tilde{T} - \tilde{S}) = 0$ holds for all values of $s \in \mathbb{C}$, then it follows that $h(s) = d$ is lossless NI. If $\det(s\tilde{T} - \tilde{S})$ is a nonzero constant, then the poles of $h(s)$ are equivalent to the eigenvalues of the matrix A and the discussion of zero-pole cancellation in Step 4 of Algorithm 1 is not required, in view of Lemma 2 and Theorem 1.

Algorithm 1: Test of Negative Imaginariness for $E = I$.

INPUT: The matrices $A \in \mathbb{R}^{n\times n}$, $b, c \in \mathbb{R}^n$ and $d \in \mathbb{R}$ of a transfer function $h(s) = c^T(sI - A)^{-1}b + d$.
OUTPUT: not NI, lossless NI, NI or SNI.

1: If $b = 0$ or $c = 0$, then $h(s)$ is lossless NI.
2: Compute $\gamma = \|c\|_2$, and construct a Householder transformation matrix Q such that $Qc = \gamma e$.
3: Set $S = Q^{-T}AQ^T$ and $b_1 = Q^{-T}b$.
4: Compute the eigenvalues of A. If all the eigenvalues of A have negative real part, go to Step 5. Otherwise, use Lemma 4 to compute the eigenvalues of the matrix pencil $s\tilde{T} - \tilde{S}$, which is the matrix pencil obtained by (9). If A has eigenvalues with positive real part, check if they are also the eigenvalues of the matrix pencil $s\tilde{T} - \tilde{S}$. If no, stop: $h(s)$ is not NI. If A has purely imaginary eigenvalues which are not the eigenvalues of the matrix pencil $s\tilde{T} - \tilde{S}$, check if they satisfy conditions 3 and 4 of Definition 1. If no, stop: $h(s)$ is not NI.
5: Set $N = S^2(I - ee^T) + b_1e^T$. Partition N as $\begin{bmatrix} n_{11} & n_{12}^T \\ n_{21} & N_1 \end{bmatrix}$.
6: If $n_{11} < 0$, stop: $h(s)$ is not NI. If $n_{11} = 0$, go to Step 7. If $n_{11} > 0$, set $\hat{N} = (1/n_{11})n_{21}n_{12}^T - N_1$ and go to Step 8.
7: If $n_{21} = \mathbf{0}$, stop: $h(s)$ is lossless NI. Otherwise, compute a Householder transformation matrix Q_2 such that $Q_2 n_{21} = -\|n_{21}\|_2 e$. Set $N = (I - ee^T)Q_2N_1Q_2^{-1} + en_{12}^TQ_2^{-1}$. Repartition N as $\begin{bmatrix} n_{11} & n_{12}^T \\ n_{21} & N_1 \end{bmatrix}$.Then, go to Step 6.
8: Compute the eigenvalues of $\hat{N}$. If there has any real positive eigenvalue with odd multiplicity, stop: $h(s)$ is not NI.
9: Select any $\lambda_0 > 0$ that is not an eigenvalue of $\hat{N}$. If $f(\lambda_0) = \det(\lambda_0 M + N) < 0$, then $h(s)$ is not NI. Otherwise, if all the eigenvalues of A which are not the eigenvalues of the matrix pencil $s\tilde{T} - \tilde{S}$ have negative real part, and if all the eigenvalues of $\hat{N}$ are not real positive eigenvalues, $h(s)$ is SNI. Otherwise, $h(s)$ is NI.

V. Eigenvalue-Based Algorithms for $E \neq I$

In this section, two eigenvalue-based algorithms are developed for testing negative imaginariness of scalar transfer functions with $E \neq I$.

A. Algorithms Based on Eigenvalue Computations of Matrix Pencils

According to Theorems 1 and 2, a method based on eigenvalue computations of matrix pencils $sT - S$, $s\tilde{T} - \tilde{S}$, and $\lambda M + N$ is summarized in Algorithm 2 to test the NI properties of scalar transfer functions with $E \neq I$, which may be nonproper transfer functions of descriptor systems. Moreover, since this article only studies proper SNI transfer functions, the SNI properties test is limited to proper case in Step 9 of Algorithm 2.

Remark 6: If $\det(E) \neq 0$, then $h(s)$ can be transformed into the form $h(s) = c^T(sI - E^{-1}A)^{-1}E^{-1}b + d$. Therefore, Algorithm 1 is also applicable to the case of $\det(E) \neq 0$. Compared with Algorithm 2, the method in Algorithm 1 only involves the eigenvalue computations of matrices, and hence, Algorithm 1 is preferred to test negative imaginariness of transfer functions when $\det(E) \neq 0$. Compared with Algorithm 1, an advantage of Algorithm 2 is that it is applicable to descriptor systems, which may be nonproper.

Algorithm 2: Test of Negative Imaginariness for $E \neq I$.

INPUT: The matrices $E, A \in \mathbb{R}^{n\times n}$, $b, c \in \mathbb{R}^n$ and $d \in \mathbb{R}$ of a transfer function $h(s) = c^T(sE - A)^{-1}b + d$.

OUTPUT: not NI, lossless NI, NI or SNI.

1: If $b = 0$ or $c = 0$, then $h(s)$ is lossless NI.

2: Choose a scalar $s_0 \in \mathbb{R}$ such that $\det(s_0E - A) \neq 0$. Compute a formal factorization $s_0E - A = LU$. Compute $G = L^{-1}EU^{-1}$. Then, compute $\gamma = \|U^{-T}c\|_2$, and construct a Householder transformation matrix $\bar{Q} \in \mathbb{R}^{n\times n}$ such that $\bar{Q}U^{-T}c = \gamma e$.

3: Set $T = \bar{Q}^{-T}G\bar{Q}^T$, $S = s_0T - I$ and $b_1 = \bar{Q}^{-T}L^{-1}b$.

4: Compute the finite eigenvalues of the matrix pencil $sT - S$. If all the finite eigenvalues of $sT - S$ have negative real part, go to Step 5. Otherwise, compute the finite eigenvalues of the matrix pencil $s\tilde{T} - \tilde{S}$, which is the matrix pencil obtained by (9). If $sT - S$ has eigenvalues with positive real part, check if they are also the eigenvalues of the matrix pencil $s\tilde{T} - \tilde{S}$. If no, stop: $h(s)$ is not NI. If $sT - S$ has purely imaginary eigenvalues which are not the eigenvalues of the matrix pencil $s\tilde{T} - \tilde{S}$, check if they satisfy the conditions 3-5 of Definition 1. If no, stop: $h(s)$ is not NI.

5: Set $M = T^2(I - ee^T)$ and $N = S^2(I - ee^T) + Tb_1e^T$.

6: If the matrix pencil $\lambda M + N$ is singular, then stop: $h(s)$ is lossless NI.

7: Compute the finite eigenvalues of the matrix pencil $\lambda M + N$. If there has any real positive eigenvalue with odd multiplicity, stop: $h(s)$ is not NI.

8: Select any $\lambda_0 > 0$ that is not an eigenvalue of $\lambda M + N$. If $f(\lambda_0) = \det(\lambda_0 M + N) < 0$, then $h(s)$ is not NI. Otherwise, if the order of the numerator polynomial of $h(s)$ is higher than the order of the denominator polynomial of $h(s)$, $h(s)$ is NI. Otherwise, go to Step 9.

9: If all the eigenvalues of the matrix pencil $sT - S$ which are not the eigenvalues of the matrix pencil $s\tilde{T} - \tilde{S}$ have negative real part, and if all the finite eigenvalues of the matrix pencil $\lambda M + N$ are not real positive eigenvalues, $h(s)$ is SNI. Otherwise, $h(s)$ is NI.

B. Algorithms Based on Eigenvalue Computations of Matrices

Algorithm 2 in Section V-A calculates the eigenvalues of matrix pencils, which is more expensive than calculating the eigenvalues of matrices. Thus, under the assumption of $\det(A) \neq 0$, an algorithm based on eigenvalue computations of matrices is provided in this section to check the NI properties of scalar transfer functions with $E \neq I$.

Since $\det(A) \neq 0$, the transfer function $h(s) = c^T(sE - A)^{-1}b + d$ is rewritten as

$$h(s) = c^T(sA^{-1}E - I)^{-1}A^{-1}b + d. \tag{31}$$

Similar to Section III, $h(s)$ in (31) can also be transformed into the form $h(s) = \gamma e^T(sT - S)^{-1}b_1 + d$ in (3) and further into the form

$$h(s) = \gamma\frac{\det(s\tilde{T} - \tilde{S})}{\det(sT - S)} + d$$

in Lemma 1, where

$$\gamma = \|c\|_2, T = Q^{-T}A^{-1}EQ^T$$
$$S = I, b_1 = Q^{-T}A^{-1}b$$
$$\tilde{T} = T(I - ee^T), \tilde{S} = S(I - ee^T) - b_1e^T.$$

Using the same analysis in Lemmas 1–3, Theorems 1 and 2 also hold. Since $\det(A) \neq 0$, $h(s)$ has no poles at the origin, and hence condition 4 of Definition 1 is not considered in Theorem 1. Then, the eigenvalue computations of matrix pencils $sT - S$ and $s\tilde{T} - \tilde{S}$ can be transformed into the eigenvalue computations of matrices, respectively. Since

$$\det(sT - S) = \det(sT - I)$$

holds, it is verified that $sT - S$ has no eigenvalues at zero. Furthermore, for $s \neq 0$

$$\det(sT - S) = \det(sT - I) = (-s)^n \det(s^{-1}I - T). \tag{32}$$

It is obvious that the eigenvalues of the matrix pencil $sT - S$ are the inverse of the nonzero eigenvalues of T.

Since $sT - S$ has no eigenvalues at zero, only the nonzero eigenvalues of the matrix pencil $s\tilde{T} - \tilde{S}$ need to be computed. For $s \neq 0$

$$\det(s\tilde{T} - \tilde{S}) = (-s)^{n-1} \det(s^{-1}\bar{T} - \bar{S}) \tag{33}$$

where

$$\bar{T} = I - ee^T,\ \bar{S} = T(I - ee^T) - b_1e^T. \tag{34}$$

Thus, the nonzero eigenvalues of the matrix pencil $s\tilde{T} - \tilde{S}$ are the inverse of the nonzero eigenvalues of the matrix pencil $s\bar{T} - \bar{S}$.

Since $\bar{T}$ in (34) is equivalent to $\tilde{T}$ in Lemma 4, the similar method of computing the eigenvalues of the matrix pencil $s\tilde{T} - \tilde{S}$ in Lemma 4 can be used to compute the nonzero eigenvalues of the matrix pencil $s\bar{T} - \bar{S}$ by partitioning $\bar{S}$ as

$$\bar{S} = \begin{bmatrix} s_{11} & s_{12}^T \\ s_{21} & S_1 \end{bmatrix}. \tag{35}$$

Thus, the following Lemma 5 is applied to compute the nonzero eigenvalues of the matrix pencil $s\bar{T} - \bar{S}$, in view of the method in Lemma 4.

Lemma 5: Considering the matrices $\bar{T} = I - ee^T$ and $\bar{S} = T(I - ee^T) - b_1e^T$ in (34). Partitioning $\bar{S}$ as the form in (35), then the three cases and the special case in Lemma 4 are also applicable to address the nonzero eigenvalues of the matrix pencil $s\bar{T} - \bar{S}$, by replacing matrices $\tilde{T}$ and $\tilde{S}$ into matrices $\bar{T}$ and $\bar{S}$.

Proof: The proof is the same with that proof in Lemma 4.□

Next, we analyze how the eigenvalue computation of $\lambda M + N$ is replaced by the eigenvalue computation of a matrix. For $\lambda > 0$

$$\det(\lambda M + N) = \lambda^{n-1} \det(\lambda^{-1}\bar{M} + \bar{N}) \quad (36)$$

where

$$\bar{M} = I - ee^T, \; \bar{N} = T^2(I - ee^T) + Tb_1e^T. \quad (37)$$

Thus, the real positive eigenvalues of the matrix pencil $\lambda M + N$ are the inverse of the real positive eigenvalues of the matrix pencil $\lambda\bar{M} + \bar{N}$. $\bar{M}$ in (37) is equivalent to M in Section IV. Then, by partitioning $\bar{N}$ as

$$\bar{N} = \begin{bmatrix} n_{11} & n_{12}^T \\ n_{21} & N_1 \end{bmatrix}$$

the same method of computing $\lambda M + N$ in Section IV can be applied to compute the eigenvalues of the matrix pencil $\lambda\bar{M} + \bar{N}$. Therefore, in Algorithm 3, the similar steps in Algorithm 1 can be used to test the NI properties of scalar transfer functions of descriptor systems, which avoid the eigenvalue computations of matrix pencils.

Remark 7: Similar to Remark 4, if the matrix $\bar{N} \in \mathbb{R}^{2\times 2}$ and $n_{11} = 0$ in Algorithm 3, then $\det(\lambda^{-1}\bar{M} + \bar{N}) = -n_{21}n_{12}^T$ is a constant. In this case, Steps 7–9 in Algorithm 3 can be omitted. If $\det(\lambda^{-1}\bar{M} + \bar{N}) = -n_{21}n_{12}^T = 0$, then $h(s)$ is lossless NI. If $n_{21}n_{12}^T > 0$, then $h(s)$ is not NI. If $n_{21}n_{12}^T < 0$, whether $h(s)$ is NI or SNI depends on the judgment of the pole condition of $h(s)$ in Step 4 of Algorithm 3 and the properness determination of $h(s)$.

Remark 8: If $\det(E) \neq 0$, both Algorithms 1 and 3 are applicable to test NI properties of $h(s)$. Compared with Algorithm 1, an advantage of Algorithm 3 is the availability for nonproper descriptor systems. Compared with Algorithm 2, the method in Algorithm 3 only involves the eigenvalue computations of matrices that cost less. Obviously, Algorithm 3 also has a limitation on $\det(A) \neq 0$.

Remark 9: If the given transfer function $h(s)$ is stable, all the poles of $h(s)$ lie in Re$[s] < 0$, and hence, Step 4 of Algorithms 1–3 can be omitted. If the realization (E, A, b, c) is minimal, then the discussion of zero-pole cancellation is not required, and hence, Step 4 of Algorithms 1–3 can be simplified.

Remark 10: The eigenvalues of matrices in Algorithms 1 and 3 can be computed by the QR algorithm, and the eigenvalues of matrix pencils in Algorithm 2 can be computed by the QZ algorithm. The costs of the QZ and QR algorithms are $30n^3 + \mathcal{O}(n^2)$ and $10n^3 + \mathcal{O}(n^2)$ operations, respectively [45]. Similar to the computational complexity analysis of the positive real test algorithms in [10] and [11], the overall operation count involving eigenvalue computations of matrices in Algorithm 1 or 3 is approximately $30n^3 + \mathcal{O}(n^2)$, while the overall operation count involving eigenvalue computations of matrix pencils in Algorithm 2 is approximately $90n^3 + \mathcal{O}(n^2)$. Moreover, the eigenvalue computations in Algorithms 1–3 based on the QR or QZ algorithm, can be implemented in MATLAB or by using off-the-shelf linear algebra libraries, such as LAPACK [45].

Remark 11: A key feature of this article is the transformation of the NI transfer function definitions conditions into eigenvalue computations, leading to the development of Algorithms 1–3. However, for very large systems, particularly those with high-dimensional or poorly conditioned matrices, eigenvalue computations can be impractical. This impracticality arises from several factors. For instance, the availability of computational resources, such as memory and processing power, can limit eigenvalue computations. In addition, for extremely

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Algorithm 3: Test of Negative Imaginariness for $E \neq I$.

INPUT: The matrices $E, A \in \mathbb{R}^{n\times n}$, $b, c \in \mathbb{R}^n$ and $d \in \mathbb{R}$ of a transfer function $h(s) = c^T(sE - A)^{-1}b + d$ with $\det(A) \neq 0$.

OUTPUT: not NI, lossless NI, NI or SNI.

1: If $b = 0$ or $c = 0$, then $h(s)$ is lossless NI.

2: Compute $\gamma = \|c\|_2$, and construct a Householder transformation matrix Q such that $Qc = \gamma e$.

3: Set $T = Q^{-T}A^{-1}EQ^T$ and $b_1 = Q^{-T}A^{-1}b$.

4: Compute the non-zero eigenvalues of T. If all the non-zero eigenvalues of T have negative real part, go to Step 5. Otherwise, use Lemma 5 to compute the non-zero eigenvalues of the matrix pencil $s\bar{T} - \bar{S}$, which is the matrix pencil obtained by (34). If T has the eigenvalues with positive real part, check if they are also the non-zero eigenvalues of the matrix pencil $s\bar{T} - \bar{S}$. If no, stop: $h(s)$ is not NI. If T has purely imaginary eigenvalues which are not the non-zero eigenvalues of the matrix pencil $s\bar{T} - \bar{S}$, check if the inverse of these purely imaginary eigenvalues satisfies the conditions 3 and 5 of Definition 1. If no, stop: $h(s)$ is not NI.

5: Set $\bar{N} = T^2(I - ee^T) + Tb_1e^T$. Partition $\bar{N}$ as $\begin{bmatrix} n_{11} & n_{12}^T \\ n_{21} & N_1 \end{bmatrix}$.

6: If $n_{11} < 0$, stop: $h(s)$ is not NI. If $n_{11} = 0$, go to Step 7. If $n_{11} > 0$, set $\hat{N} = (1/n_{11})n_{21}n_{12}^T - N_1$ and go to Step 8.

7: If $n_{21} = \mathbf{0}$, stop: $h(s)$ is lossless NI. Otherwise, compute a Householder transformation matrix Q_2 such that $Q_2n_{21} = -\|n_{21}\|_2 e$. Set $\bar{N} = (I - ee^T)Q_2N_1Q_2^{-1} + en_{12}^TQ_2^{-1}$. Repartition $\bar{N}$ as $\begin{bmatrix} n_{11} & n_{12}^T \\ n_{21} & N_1 \end{bmatrix}$.Then, go to Step 6.

8: Compute the eigenvalues of $\hat{N}$. If there has any real positive eigenvalue with odd multiplicity, stop: $h(s)$ is not NI.

9: Select any $\lambda_0 > 0$ that is not an eigenvalue of $\hat{N}$. If $f(\lambda_0) = \det(\lambda_0^{-1}\bar{M} + \bar{N}) < 0$, then $h(s)$ is not NI. Otherwise, if the order of the numerator polynomial of $h(s)$, is higher than the order of the denominator polynomial of $h(s)$, $h(s)$ is NI. Otherwise, go to Step 10.

10: If all the non-zero eigenvalues of T which are not the non-zero eigenvalues of the matrix pencil $s\bar{T} - \bar{S}$ have negative real part, and if all the eigenvalues of $\hat{N}$ are not real positive eigenvalues, $h(s)$ is SNI. Otherwise, $h(s)$ is NI.

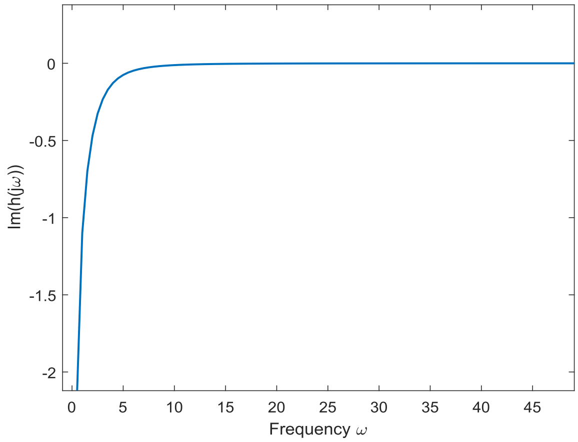

Fig. 1. Plot of the imaginary part of the transfer function $h(j\omega)$.

high-dimensional or poorly conditioned matrices, QR and QZ algorithms may become infeasible and numerically unstable, resulting in failures in MATLAB and LAPACK computations.

VI. Illustrative Examples

In this section, three examples are presented to determine negative imaginariness of transfer functions by using eigenvalue-based algorithms. Example 2 is provided to show how the eigenvalue computation is applied for testing negative imaginariness. Example 3 is a n-stage *RLC* circuit network, which shows that the eigenvalue computation is useful to determine negative imaginariness of large-dimensional systems. The last example is used to test negative imaginariness of nonproper descriptor systems.

Example 2: Consider the transfer function (25) and the minimal realization (26) in Example 1. Algorithm 1 is applied to test NI properties of (25), where γ, Q, S, and b_1 are given by Example 1. Then, according to Step 4 of Algorithm 1, the eigenvalues of matrix A are 0, -1, and -3. Obviously, system (25) has a pole at the origin and condition 4 of Definition 1 holds. Then, Step 5 of Algorithm 1 establishes the following matrix N with $n_{11} = 0$ and $n_{21} = [0 \quad 1]^T \neq \mathbf{0}$:

$$N = \begin{bmatrix} 0 & 1.44 & -2.6 \\ 0 & -1.08 & 3.2 \\ 1 & -7.2 & 13 \end{bmatrix}.$$

Thus, Step 7 establishes a new matrix N with $n_{11} = 2.6 > 0$

$$N = \begin{bmatrix} 2.6 & -1.44 \\ 3.2 & -1.08 \end{bmatrix}.$$

Then, $\hat{N} = -0.6923$ is obtained with multiplicity 1. In view of Step 9 of Algorithm 1, the transfer function (25) is NI, but not SNI. To illustrate this, the imaginary part of the transfer function (25) is plotted in Fig. 1, where

$$\begin{aligned} \mathrm{Im}[h(j\omega)] &= -\frac{1}{2} j [h(j\omega) - h(-j\omega)] \\ &= -\frac{13\omega^2 + 9}{\omega^5 + 10\omega^3 + 9\omega} < 0 \end{aligned}$$

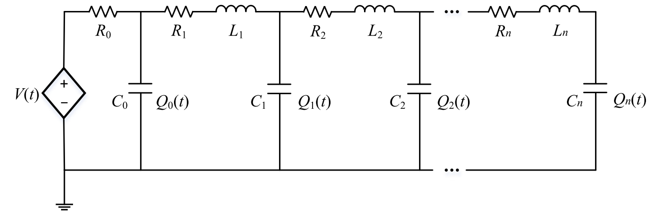

Fig. 2. n-stage *RLC* circuit network.

for all $\omega > 0$.

Moreover, since $\det(A) = 0$ and $c^T b = 0$ hold in this example, the existing methods of judging negative imaginariness, such as LMIs [12], [13], spectral conditions [32], and Riccati equations [15], [35], [37], are not applicable to the test of system (25). Compared with the existing methods in [12], [13], [15], [32], [35], and [37], Example 2 shows that Algorithm 1 has less restrictions in determining NI properties of SISO system. In addition, if the state-space realization in (26) is changed to an arbitrary nonminimal realization, Algorithm 1 is also applicable while the LMI condition in [15, Lemma 2] requires a minimal realization. In addition, the NI properties of (25) can be tested directly by using Definition 1 with plotting the imaginary part in Fig. 1. However, for a given state-space realization of high-dimensional systems, the test using NI definitions becomes complex. For example, the transfer function in following Example 3 is difficult to be obtained for a 2001-dimensional system, and the determination of the imaginary part is very complex.

Example 3: Consider an n-stage *RLC* circuit network, as shown in Fig. 2, which is taken from [39]. Such ladder *RLC* networks are practically relevant, as they arise in real-world engineering modeling tasks, such as long transmission lines. The input is the voltage $V(t)$. The output is the total charge $Q(t)$ in the capacitances, where

$$Q(t) = \sum_{i=0}^{k} Q_i(t).$$

Let the state variable be

$$x(t) = \begin{bmatrix} u_0(t) & i_{L_1}(t) & u_1(t) & \cdots & i_{L_k}(t) & u_k(t) \end{bmatrix}^T$$

where $u_i(t)$ is the voltage of the capacitance C_i, and $i_{L_i}(t)$ is the current through the inductor L_i. Thus, the following system (38) with $u(t) = V(t)$ and $y(t) = Q(t)$ is obtained:

$$\begin{cases} \dot{x}(t) = Ax(t) + bu(t) \\ y(t) = c^T x(t) \end{cases} \tag{38}$$

where

$$A = \begin{bmatrix} -\frac{1}{C_0 R_0} & -\frac{1}{C_0} & 0 & 0 & 0 & \cdots & 0 \\ \frac{1}{L_1} & -\frac{R_1}{L_1} & -\frac{1}{L_1} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{C_1} & 0 & -\frac{1}{C_1} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{L_2} & -\frac{R_2}{L_2} & -\frac{1}{L_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{L_k} & -\frac{R_k}{L_k} & -\frac{1}{L_k} \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{C_k} & 0 \end{bmatrix}$$

$$b = \begin{bmatrix}\frac{1}{C_0R_0} & 0 & 0 & 0 & 0 & \cdots & 0\end{bmatrix}^T$$
$$c = \begin{bmatrix}C_0 & 0 & C_1 & 0 & C_2 & \cdots & C_k\end{bmatrix}^T.$$

Consider a 2001-dimensional system (38) with $k = 10^3$. Let $L_i = 1H$, $R_i = 0.5\Omega$, and $C_i = 1F$. By running Algorithm 1, it can be found that all the eigenvalues of A have negative real part and $n_{11} > 0$. Then, compute the eigenvalues of $\hat{N}$. It turns out that the eigenvalues of $\hat{N}$ are 2×10^3 different complex numbers. Thus, Step 9 of Algorithm 1 is used to found that the system (38) with $k = 10^3$ is SNI.

We also further test SNI properties by solving the LMIs conditions in [13]. In view of [13, Lemma 8], we need to find a matrix $Y > 0$ such that

$$AY + YA^T \leq 0, \text{and } b + AYc = 0 \tag{39}$$

and check whether $\text{rank}(M(j\omega)) = m$ for any $\omega \in (0, \infty)$, where $M(s) = LY^{-1}A^{-1}(sI - A)^{-1}b$, $m = \text{rank}(b) = 1$, and $L^TL = -AY - YA^T$. However, the solutions of the LMIs (39) are not found by using the solver SDPT3 or MOSEK, and the MATLAB toolbox YALMIP [46], which is designed for modeling and solving optimization problems. Moreover, the information "Maximum iterations or time limit exceeded" is returned. Obviously, the number of iterations and the solution time will increase with the expansion of the matrix dimension, which may lead to no solution or difficulty in solving. In contrast, Algorithm 1 in this article only involves the computation of matrix eigenvalues that can be applied to determine negative imaginariness of large dimensional systems.

Moreover, when the SNI lemma based on Riccati equations in [35] and [37], which can be converted to an eigenvalue problem for a Hamiltonian matrix, is taken, there exist 4002 eigenvalues of the Hamiltonian matrix and the system (38) is verified to be SNI. However, testing of SNI properties via Riccati equations or the eigenvalue computations for the Hamiltonian matrix in [35] and [37] is more expensive than Algorithm 1, which only involves the calculation of 2×10^3 eigenvalues.

In Example 3 with $k = 10^3$, the runtime of Algorithm 1 is 5.0477 s. In addition, for system (38), we further increased the value of k, resulting in system dimensions of 6001 and 10 001, with corresponding runtime of 174.6259 and 738.8268 s, respectively. As the dimension of system (38) increases, the runtime and computational costs increase rapidly. With further increases in the value of k, eigenvalue calculations will become difficult and infeasible. In fact, when we increased k to 2.5×10^4, resulting in a system dimension of 50 001, our hardware's memory was insufficient to support the creation of such a large matrix. This also aligns with the limitations of eigenvalue computations analyzed in Remark 11.

Example 4: Consider the following quarter-car model, which is taken from [42] and [47]:

$$m\ddot{q} + t(\dot{q} - \dot{u} - \dot{\omega}) + k(q - u - \omega) = 0 \tag{40}$$

where u is the active displacement of the suspension system, q is the vertical displacement of the vehicle chassis, ω is the vertical wheel displacement, m represents one quarter of the vehicle mass, t and k, respectively, represent the damping and stiffness related to the suspension and tires. Considering the case that the input is $(u + \omega)$ and the output is $(-\ddot{q})$, let the state variable be

$$x(t) = \begin{bmatrix}q & \dot{q} & u + \omega & \dot{u} + \dot{\omega}\end{bmatrix}^T.$$

Thus, the car dynamics system (40) has the following nonproper transfer function:

$$h(s) = c^T(sE - A)^{-1}b = \frac{-s^2(ts + k)}{(ms^2 + ts + k)} \tag{41}$$

where

$$E = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}, A = \begin{bmatrix}0 & 1 & 0 & 0\\ -\frac{k}{m} & -\frac{t}{m} & \frac{k}{m} & \frac{t}{m}\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{bmatrix}$$

$$b = \begin{bmatrix}0\\ 0\\ 0\\ -1\end{bmatrix}, c = \begin{bmatrix}\frac{k}{m}\\ \frac{t}{m}\\ -\frac{k}{m}\\ -\frac{t}{m}\end{bmatrix}.$$

Let $t = k = m = 1$. Then, in view of Algorithm 2, let $s_0 = 0$, $L = s_0E - A = -A$, and $U = I$ hold, where $\det(s_0E - A) \neq 0$. Furthermore, according to Steps 2 and 3 of Algorithm 2, $\gamma = \|U^{-T}c\|_2 = 2$, and the following matrices are computed:

$$G = \begin{bmatrix}1 & 1 & -1 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 0\end{bmatrix}$$

$$\bar{Q} = \begin{bmatrix}0.5 & 0.5 & -0.5 & -0.5\\ 0.5 & 0.5 & 0.5 & 0.5\\ -0.5 & 0.5 & 0.5 & -0.5\\ -0.5 & 0.5 & -0.5 & 0.5\end{bmatrix}$$

$$T = \begin{bmatrix}0.25 & 0.25 & 0.25 & 0.25\\ 0.75 & -0.25 & -0.25 & 0.75\\ -1.25 & -0.25 & 0.75 & -0.25\\ -0.75 & -0.75 & 0.25 & 0.25\end{bmatrix}$$

$$b_1 = \begin{bmatrix}0\\ 1\\ 0\\ -1\end{bmatrix}, S = -I.$$

According to Step 4 of Algorithm 2, the eigenvalues of $sT - S$ are $-0.5 \pm 0.8660j$ with negative real part. Then, Step 5 implies that the matrix pencil $\lambda M + N$ is

$$\lambda\begin{bmatrix}0 & -0.25 & 0.25 & 0.25\\ 0 & -0.25 & 0.25 & 0.25\\ 0 & -0.25 & 0.25 & -0.75\\ 0 & -0.25 & 0.25 & -0.75\end{bmatrix} + \begin{bmatrix}0 & 0 & 0 & 0\\ -1 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ -1 & 0 & 0 & 1\end{bmatrix}.$$

The eigenvalue of the matrix pencil $\lambda M + N$ is 0 with multiplicity 2. Thus, the transfer function (41) is NI according to Step 8 of Algorithm 2.

On the other hand, since $\det(A) \neq 0$, Algorithm 3 in Section V-B is also applicable to test negative imaginariness of system (41). First, according to Steps 2 and 3 of Algorithm 3, we have $\gamma = \|c\|_2 = 2$, and the following matrices:

$$Q = \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & 0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & -0.5 & 0.5 \end{bmatrix}$$

$$T = Q^{-T}A^{-1}EQ^T = \begin{bmatrix} -0.25 & -0.25 & -0.25 & -0.25 \\ -0.75 & 0.25 & 0.25 & -0.75 \\ 1.25 & 0.25 & -0.75 & 0.25 \\ 0.75 & 0.75 & -0.25 & -0.25 \end{bmatrix}$$

$$b_1 = Q^{-T}A^{-1}b = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

Then, it can be found that the nonzero eigenvalues of T are $-0.5 \pm 0.8660j$ with negative real part, and hence, the matrix $\bar{N}$ in Step 5 is constructed as follows:

$$\bar{N} = \begin{bmatrix} 0 & -0.25 & 0.25 & 0.25 \\ -1 & -0.25 & 0.25 & 0.25 \\ 0 & -0.25 & 0.25 & -0.75 \\ -1 & -0.25 & 0.25 & -0.75 \end{bmatrix}$$

where $n_{11} = 0$ and $n_{21} = [-1 \quad 0 \quad -1]^T \neq \mathbf{0}$. By running Steps 6 and 7 of Algorithm 3 twice, a new matrix $\bar{N}$ is computed

$$\bar{N} = \begin{bmatrix} 0.4083 & -0.1444 \\ 0.2357 & -0.0833 \end{bmatrix}$$

where $n_{11} > 0$. It follows from Steps 8 and 9 that $\hat{N} = 0$ without any real positive eigenvalue and $\det(\lambda_0^{-1}\bar{M} + \bar{N}) > 0$ for $\lambda_0 > 0$. Obviously, the order of the numerator polynomial of the transfer function (41) is higher than the order of the denominator polynomial. Thus, (41) is a nonproper transfer function and (41) is NI according to Step 9 of Algorithm 3.

VII. Conclusion

The eigenvalue-based characterization of negative imaginariness of scalar transfer functions has been developed in this article. Based on the characterization, several algorithms, which involve eigenvalue computations of matrices or matrix pencils, were proposed for testing NI properties of scalar transfer functions. For a nonproper transfer function of descriptor systems, eigenvalue-based algorithms are also useful to test negative imaginariness. Finally, two numerical examples, an *RLC* circuit network as well as a car dynamics example were provided to illustrate the effectiveness of the proposed algorithms.

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Mei Liu received the B.Sc. degree in mathematics from the China University of Mining and Technology, Xuzhou, China, in 2012, and the Ph.D. degree in control science and engineering from the University of Science and Technology of China, Hefei, China, in 2017.

From 2017 to 2019, she was with The University of Hong Kong, Hong Kong, and The Hong Kong Polytechnic University, Hong Kong, as a Research Associate/Assistant. From March 2019 to June 2019, she was a Visiting Scholar with the Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, China. She is currently an Associate Professor with the Department of Automation, Tianjin University, Tianjin, China. Her current research interests include negative imaginary systems, positive real systems, optimal control, and robust control.

Dr. Liu is an early career Editorial Board Member for *Franklin Open*.

Kai Feng received the B.E. degree in automation from the School of Electrical and Information Engineering, Tianjin University, Tianjin, China, in 2021, and the M.E. degree in control science and engineering from Tianjin University in 2024. He is currently working toward the Ph.D. degree in mechatronics engineering with the Department of Electromechanical Engineering from University of Macau, Macau, China.

His current research interests include negative imaginary systems, positive real systems, and descriptor systems.

Alexander Lanzon (Senior Member, IEEE) received the B.Eng. (Hons). degree in electrical and electronic engineering from the University of Malta, Msida, Malta, in 1995, and the M.Phil. degree in robot control and the Ph.D. degree in control engineering from the University of Cambridge, Cambridge, U.K., in 1997 and 2000, respectively.

He has held research and academic positions with the Georgia Institute of Technology, Atlanta, GA, USA, and the Australian National University, Canberra, ACT, Australia, and industrial positions at ST-Microelectronics Ltd., Kirkop, Malta; Yaskawa Denki Ltd., Tokyo, Japan; and National ICT Australia Ltd., Eveleigh, NSW, Australia. In 2006, he joined the University of Manchester, Manchester, U.K., where he now holds the Chair in Control Engineering. His research interests include the fundamentals of feedback control theory, negative imaginary systems theory, robust control, nonlinear control (via input–output methods), and applying robust control methods to robotic, UAV, and motion control applications.

Dr. Lanzon is a Fellow of the Institute of Mathematics and its Applications, the Institute of Measurement and Control, and the Institution of Engineering and Technology. He was an Associate Editor for IEEE CONTROL SYSTEMS LETTERS (2023–2024) and IEEE TRANSACTIONS ON AUTOMATIC CONTROL (2013–2019), and a Subject Editor of the *International Journal of Robust and Nonlinear Control* (2012–2015).