

Feedback Stability Analysis via Frequency Dependent Constraints

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Abstract—We establish robust stability of feedback interconnections of linear time-invariant systems with potentially marginally stable open-loop poles via frequency-varying quadratic constraints. These constraints may be used to characterise numerous system properties, such as small-gain, passivity, negative imaginarity, and their weighted variants, that are manifested on different band-limited frequencies. Such systems arise in practical mechanical settings where force actuators and displacement/velocity sensors are not colocated. We demonstrate that frequency-limited system properties may be characterised using the powerful Iwasaki–Hara lemma in terms of convex feasibility problems involving linear matrix inequalities, thereby complementing the utility of the Iwasaki–Hara lemma for robust feedback stability analysis involving mixed system properties on the frequency axis.

Index Terms—Feedback stability, frequency dependent inequalities, integral quadratic constraints (IQCs), Iwasaki–Hara lemma.

I. INTRODUCTION

Linear time-invariant (LTI) systems that display varying properties across different frequency bands are ubiquitous in the field of systems and control [1], [2]. Of particular interest are systems that are passive [3], [4] or negative imaginary [5], [6] in certain frequency ranges, in which imaginary-axis poles may be present. Negative imaginary dynamics naturally arise in mechanical systems with colocated force actuators and position sensors [5], otherwise they are only manifested in limited frequency ranges [1], [2]. This article examines robust feedback stability of LTI systems with potential poles on the imaginary axis via frequency-varying quadratic constraints. Various system properties across distinct frequency ranges, such as weighted passivity, small-gain, and negative imaginarity, may be characterized by such quadratic constraints.

The theory of integral quadratic constraints (IQCs) [7] was originally developed for open-loop stable systems, and is hence inapplicable to the setting in this article. To accommodate potential imaginary-axis poles, either hard IQCs [8], [9], [10], [11] or soft IQCs equipped with homotopy in the gap metric [11], [12], [13], [14], [15] may be used. A third method [16], [17] employs a modification of the integration contour of the L_2 -space in the frequency domain. All these approaches involve the use of Hilbert signal spaces, which, from the perspective

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of LTI systems, is indirect and opaque. This article establishes robust feedback stability of LTI systems with possible imaginary-axis poles *directly* in the frequency domain via frequency-dependent constraints and the multivariable Nyquist criterion [18], [19], [20], without recourse to hard IQCs or the graph topology. The technical difficulty involved in applying the latter methods will also be laid out.

The frequency-varying constraints considered in this article may be used to characterise various system properties on the different frequency bands. They incorporate passivity [21], [22], [23], small-gain [24], negative imaginarity [25], [26], [27], [28], and many others [29], [30]. Related mixed system properties have been investigated in [31], [32], and [33] from the input–output perspective and [34], [35] from the viewpoint of dissipativity. In particular, Griggs et al. [32] fall under the general framework of hard IQCs in [11] while the authors in [31] and [33] established L_∞ stability of the closed-loop transfer function. By contrast, this article provides frequency-dependent conditions under which the stronger notion of H_∞ feedback stability can be guaranteed by making use of certain structured robustness results in [36] and the Nyquist stability criterion.

The content of this article is strongly related with the influential works by Iwasaki and Hara. In particular, the equivalence between feedback stability and quadratic graph separation is established in [36] for matrices, upon which the results in this article are built. Furthermore, these robust stability results rely on mixing frequency-dependent properties on the imaginary axis. Various properties on distinct frequency bands, such as weighted small-gain and passivity properties, may be characterized in a computationally tractable manner via linear matrix inequalities (LMIs) by the seminal Iwasaki–Hara lemma [37]. The Iwasaki–Hara lemma importantly generalises the classical Kalman–Yakubovich–Popov lemma [38] in numerous aspects by allowing for quadratic constraints defined on curves on the complex plane. It has its roots in [39], and also [2], where lightly damped mechanical systems with noncolocated force actuators and position sensors (e.g., a magnetic storage device with a swing-arm) are considered. Such systems exhibit negative imaginarity only in a certain frequency range, and feedback stabilisation of such systems may be achieved with the frequency-dependent results in this article; see also [40], [41] for studies of control performance of these systems. It is noteworthy that time-domain interpretation of such dynamical behaviours has been considered in [42]. A specialization to the negative imaginary setting appeared in [43].

The rest of this article is organized as follows. Section II introduces the notation used in this article and contains mathematical preliminaries. The main results on robust feedback stability based on frequency-dependent quadratic constraints for LTI systems with potentially marginally stable poles are derived in Section III. Application of the Iwasaki–Hara lemma for characterizing system properties on specified frequency bands is detailed in Section IV. Systems having mixed positive real, negative imaginary, and bounded real properties are considered in Section V. Section VI contains an example of a flexible mechanical structure that motivates the utility of the main results. Finally, Section VII concludes this article.

II. NOTATIONS AND PRELIMINARIES

Denote by \mathbb{R} , \mathbb{R}_+ , $\bar{\mathbb{R}}_+$, $j\mathbb{R}$, \mathbb{C} , \mathbb{C}_- , \mathbb{C}_+ , and $\bar{\mathbb{C}}_+$ the reals, the positive reals, the nonnegative reals, the imaginary axis, the complex plane, the open left-half complex plane, the open right-half complex plane, and the closed right-half complex plane, respectively. The real part and imaginary part of $s \in \mathbb{C}$ are denoted by $\text{Re}(s)$ and $\text{Im}(s)$, respectively.

A. Matrix Theory

For $M \in \mathbb{C}^{n \times m}$, denote by $\bar{\sigma}(M)$ (respectively, $\underline{\sigma}(M)$) the largest (respectively, smallest) singular value of M . Denote by \bar{M} the (element-wise) complex conjugate and M^* the complex conjugate transpose of M . Given $x \in \mathbb{C}^n$, let $|x| = \sqrt{x^*x}$. An $M \in \mathbb{C}^{n \times n}$ is said to be Hermitian if $M = M^*$. The set of $n \times n$ Hermitian matrices is denoted by \mathbb{H}_n . An $M \in \mathbb{H}_n$ is said to be positive semidefinite (respectively, definite), denoted $M \geq 0$ (respectively, $M > 0$), if $v^*Mv \geq 0$ (respectively, > 0) for all nonzero $v \in \mathbb{C}^n$. Given $M \in \mathbb{C}^{n \times n}$, define its field of values (a.k.a. numerical range) [44, Def. 1.1.1] as

$$F(M) = \{x^*Mx : x \in \mathbb{C}^n, |x| = 1\}$$

and the angular field of values [44, Def. 1.1.2] as

$$F'(M) = \{x^*Mx : x \in \mathbb{C}^n, x \neq 0\}.$$

The angular field of values is invariant to Hermitian congruence transformation, i.e., $F'(TMT^*) = F'(M)$ for all invertible $T \in \mathbb{C}^{n \times n}$ [44, Property 1.2.12]. Denote by $\lambda(M)$ the spectrum of M , i.e., the set of its eigenvalues. Denote by $\lambda_i(M)$, $i \in \{1, \dots, n\}$ the elements in $\lambda(M)$. Note that $\lambda(M) \subset F(M)$ [44, Property 1.2.6]. The following results are useful.

Lemma 1 ([44, Property 1.2.5, Th. 1.7.8, and Problem 12 on page 15]): Let $M, N \in \mathbb{C}^{n \times n}$. Then, the following holds:

- (i) $F'(M) \subset \bar{\mathbb{C}}_+$ if and only if $M + M^* \geq 0$;
- (ii) $F'(M) \subset \mathbb{C}_+$ if and only if $M + M^* > 0$;
- (iii) If $0 \notin F(N)$, then $\lambda(MN) \subset F'(M)F'(N)$;
- (iv) $F'(M) \subset \mathbb{R}$ if and only if M is Hermitian.

Lemma 2 ([36, Corollary 1]): Let $M \in \mathbb{C}^{p \times q}$ and $N \in \mathbb{C}^{q \times p}$. Then

$$\det(I - \tau MN) \neq 0 \quad \text{for all } \tau \in [0, 1]$$

if there exists $\Pi \in \mathbb{H}_{q+p}$ such that

$$\begin{bmatrix} I \\ \tau M \end{bmatrix}^* \Pi \begin{bmatrix} I \\ \tau M \end{bmatrix} \geq 0 \quad \text{for all } \tau \in [0, 1] \quad (1)$$

and

$$\begin{bmatrix} N \\ I \end{bmatrix}^* \Pi \begin{bmatrix} N \\ I \end{bmatrix} < 0.$$

The following corollary is a direct consequence from Lemma 2 when the multiplier Π is specialized to capture bounded real, positive real and negative imaginary conditions.

Corollary 3: Let $M \in \mathbb{C}^{p \times q}$ and $N \in \mathbb{C}^{q \times p}$. Then

$$\det(I - MN) \neq 0$$

if

$$\begin{bmatrix} I \\ M \end{bmatrix}^* \Pi \begin{bmatrix} I \\ M \end{bmatrix} \geq 0$$

and

$$\begin{bmatrix} N \\ I \end{bmatrix}^* \Pi \begin{bmatrix} N \\ I \end{bmatrix} < 0$$

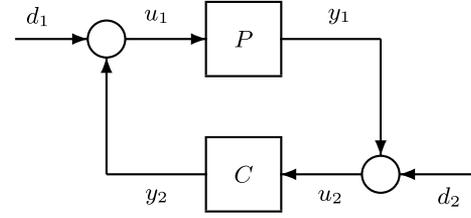


Fig. 1. Standard feedback configuration $[P, C]$.

with $\Pi = \begin{bmatrix} \gamma^2 I & 0 \\ 0 & -I \end{bmatrix}$ for some $\gamma > 0$, or $\Pi = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$, or $\Pi = \begin{bmatrix} 0 & jI \\ -jI & 0 \end{bmatrix}$, or any conic combination of them.

Proof: The result is trivially obtained directly from Lemma 2 by taking Π to be the specified forms provided in this corollary. ■

B. Transfer Functions

Let \mathbf{L}_∞ consist of transfer functions G satisfying $\text{ess sup}_{\omega \in \mathbb{R}} \bar{\sigma}(G(j\omega)) < \infty$. Let $\mathbf{R}^{n \times m}$ denote the set of real-rational proper transfer function matrices of dimensions $n \times m$ and $\mathbf{RH}_\infty^{n \times m}$ be its stable subset containing elements with no poles in $\bar{\mathbb{C}}_+$. Every $G \in \mathbf{R}^{n \times m}$ can be realised as $G(s) = C(sI - A)^{-1} + D$, and we denote such a realization by $G = (A, B, C, D)$. Denote by $\mathbf{M}^{n \times m} \subset \mathbf{R}^{n \times m}$ the set of stable and marginally stable transfer functions such that $G \in \mathbf{M}^{n \times m}$ if and only if

- (i) G has no poles in \mathbb{C}_+ and
- (ii) for any $\omega_0 \geq 0$, if $j\omega_0$ is a pole of G , then it is a simple pole.

Let $\mathbf{C}_m \subset \mathbf{L}_\infty$ be the class of functions $\Pi : \mathbb{R} \rightarrow \mathbb{C}^{m \times m}$ that are piecewise continuous and for which $\lim_{\omega \rightarrow \infty} \Pi(\omega)$ exists, $\Pi(\omega) = \Pi(\omega)^*$ for all $\omega \in [0, \infty)$, and $\Pi(-\omega) = \Pi(\omega)$ for all $\omega \in (0, \infty)$.

The positive feedback interconnection of two transfer functions $P \in \mathbf{R}^{n \times m}$ and $C \in \mathbf{R}^{m \times n}$, denoted by $[P, C]$, is described by

$$\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} I & -C \\ -P & I \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

This is illustrated in Fig. 1.

Definition 4: $[P, C]$ is said to be stable if

$$\begin{bmatrix} I & -C \\ -P & I \end{bmatrix}^{-1} = \begin{bmatrix} I + C(I - PC)^{-1}P & C(I - PC)^{-1} \\ (I - PC)^{-1}P & (I - PC)^{-1} \end{bmatrix} \in \mathbf{RH}_\infty.$$

The following three new definitions on frequency-band-limited positive real, negative imaginary, and bounded real properties will be critical to describing mixed positive real, negative imaginary, and bounded real systems in this article.

First, we define a class of systems that has a positive real-type property on a subset of positive frequencies. When considering potential poles on the imaginary axis, we have to restrict attention only to those poles inside the frequency set of interest.

Definition 5: Given $\Omega \subset \bar{\mathbb{R}}_+ \cup \{\infty\}$, a system $G \in \mathbf{R}^{m \times m}$ is said to be positive real (a.k.a. passive) on Ω if all of the following conditions hold.

- (i) G has no poles in the open right half plane.
- (ii) For all $\omega \in \Omega$ such that $j\omega$ is not a pole of G ,

$$G(j\omega) + G(j\omega)^* \geq 0.$$

- (iii) For any $\omega_0 \in \Omega \cap \bar{\mathbb{R}}_+$, if $j\omega_0$ is a pole of G , then it is a simple pole and

$$\lim_{s \rightarrow j\omega_0} (s - j\omega_0)G(s) \geq 0.$$

If G is positive real on the entire $\bar{\mathbb{R}}_+ \cup \{\infty\}$, it is said to be a positive real (a.k.a. passive) system.

Second, we define a class of systems that has a negative imaginary-type property on a subset of positive frequencies. When considering potential poles on the imaginary axis, we have to restrict attention only to those poles inside the frequency set of interest.

Definition 6: Given $\Omega \subset \bar{\mathbb{R}}_+ \cup \{\infty\}$, a system $G \in \mathbf{R}^{m \times m}$ is said to be negative imaginary on Ω if all of the following conditions hold.

- (i) G has no poles in the open right half plane.
(ii) For all $\omega \in \Omega$ such that $j\omega$ is not a pole of G ,

$$j(G(j\omega) - G(j\omega)^*) \geq 0.$$

- (iii) For any $\omega_0 \in \Omega \cap \bar{\mathbb{R}}_+$, if $j\omega_0$ is a pole of G , then it is a simple pole and

$$\lim_{s \rightarrow j\omega_0} (s - j\omega_0)jG(s) \geq 0.$$

- (iv) If $0 \in \Omega$ and is a pole of G , then it is at most a double pole and

$$\lim_{s \rightarrow 0} s^2 G(s) \geq 0.$$

If G is negative imaginary on the entire $\bar{\mathbb{R}}_+ \cup \{\infty\}$, it is said to be a negative imaginary system.

Third and last, we define a class of systems that has a bounded real-type property (with a specified gain γ) on a subset of positive frequencies. Note that poles on the imaginary axis are permitted in this class of systems, but these imaginary axis poles must be outside the frequency band where the bounded real property holds.

Definition 7: Given $\Omega \subset \bar{\mathbb{R}}_+ \cup \{\infty\}$, a system $G \in \mathbf{R}^{m \times m}$ is said to be bounded real on Ω with gain $\gamma \in \bar{\mathbb{R}}_+$ if both the following conditions hold.

- (i) G has no poles in the open right half plane.
(ii) For all $\omega \in \Omega$,

$$\gamma^2 I - G(j\omega)^* G(j\omega) \geq 0.$$

If G is bounded real on the entire $\bar{\mathbb{R}}_+ \cup \{\infty\}$, it is said to be a bounded real system.

III. FEEDBACK STABILITY

The main result of this article is derived in this section. It is concerned with the robust stability of a feedback interconnection of LTI systems with marginally stable poles and mixed system properties on the frequency axis, as characterized by quadratic constraints that vary with the frequency parameter. The result covers the well-known passivity theorem as a special instance.

A. Open-Loop Stable Systems

First, the following known result for open-loop stable systems is reviewed and its limitations discussed.

Lemma 3: Given $P \in \mathbf{RH}_\infty^{p \times q}$ and $C \in \mathbf{RH}_\infty^{q \times p}$, $[P, C]$ is stable if there exists $\Pi \in \mathbf{C}_{p+q}$ such that for all $\omega \in [0, \infty]$, $\tau \in [0, 1]$,

$$\begin{bmatrix} I \\ \tau P(j\omega) \end{bmatrix}^* \Pi(\omega) \begin{bmatrix} I \\ \tau P(j\omega) \end{bmatrix} \geq 0$$

and

$$\begin{bmatrix} C(j\omega) \\ I \end{bmatrix}^* \Pi(\omega) \begin{bmatrix} C(j\omega) \\ I \end{bmatrix} < 0.$$

Proof: The claim may be established using the IQC theorem [7, Th. 1] together with the Plancherel–Parseval theorem as in [45, Th. 3.1]. A more direct frequency-domain proof may be found in [46, Prop. II.1]. ■

Lemma 3 guarantees feedback stability of $[P, C]$ via two frequency-dependent inequalities/constraints. It provides a powerful framework within which various system properties across different frequency bands may be captured via a frequency-varying multiplier Π . When Π is expressed as

$$\Pi(\omega) = \begin{bmatrix} Q(\omega) & S(\omega) \\ S(\omega)^* & R(\omega) \end{bmatrix},$$

it is often known as a dynamical $(Q(\omega), S(\omega), R(\omega))$ -multiplier in the theory of dissipativity, as is considered in [31] and [35]. A great number of commonly encountered uncertainty may be captured by such a multiplier; see [7, Sec. VI].

Lemma 3 is only applicable to open-loop stable P and C . In the case where P has poles on the imaginary axis, the Plancherel–Parseval theorem cannot be applied, in the proof of Lemma 3, to obtain the conditions in the IQC theorem [7, Th. 1]. One way to handle imaginary-axis poles involves using hard IQCs [11, Th. III.1] as in the passivity theorem [3, Ch. VI], where the integrals are taken over all finite intervals for extended \mathbf{L}_2 signals. This works well when the multiplier Π is static, as in the case of passivity, but when Π is dynamical, it is unclear how one may obtain a hard IQC from a frequency-domain inequality when the transfer function involved has imaginary-axis poles.

An alternative method to accommodating imaginary-axis poles makes use of homotopies that are continuous in the graph topology, as in [13] and [15]. This method, however, relies on the existence of a nominal feedback system that is stable and the satisfaction of an IQC along the entire homotopy from the nominal system to the perturbed system under study. The search for such a homotopy is often a challenge and varies from one case to another.

In the next section, feedback stability of systems with possible marginally stable poles is established directly in the frequency domain using the Nyquist stability criterion. Doing so bypasses the difficulty involved in employing hard IQCs or the graph topology.

B. Open-Loop Systems With Possible Marginally Stable Poles

Robust stability conditions for open-loop systems that admit marginally stable poles are derived next in terms of frequency-dependent quadratic constraints.

Theorem 2: Given $P \in \mathbf{M}^{n \times n}$ and $C \in \mathbf{RH}_\infty^{n \times n}$, suppose there exist $\epsilon > 0$ and $\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{bmatrix} \in \mathbf{C}_{2n}$ such that all of the following conditions hold.

- (i) For all $\tau \in [0, 1]$ and $\omega \in \bar{\mathbb{R}}_+$ such that $j\omega$ is not a pole of P ,

$$\begin{bmatrix} I \\ \tau P(j\omega) \end{bmatrix}^* \Pi(\omega) \begin{bmatrix} I \\ \tau P(j\omega) \end{bmatrix} \geq 0. \quad (2)$$

- (ii) For every $j\omega_0$ that is a pole of P , Π is continuous at ω_0 ,

$$\Pi_{11}(\omega) = 0 \text{ and } \Pi_{22}(\omega) = 0 \text{ for all } |\omega - \omega_0| \leq \epsilon \quad (3)$$

and

$$\Pi_{12}(\omega_0)K + K^* \Pi_{12}(\omega_0)^* \geq 0 \quad (4)$$

where

$$K = \lim_{s \rightarrow j\omega_0} (s - j\omega_0)P(s). \quad (5)$$

(iii) For all $\omega \in [0, \infty)$,

$$\begin{bmatrix} C(j\omega) \\ I \end{bmatrix}^* \Pi(\omega) \begin{bmatrix} C(j\omega) \\ I \end{bmatrix} < 0. \quad (6)$$

Then, $[P, C]$ is stable.

Proof: If P has no poles on the imaginary axis, then the claim follows straightforwardly from Lemma 3. Thus, denote by $j\Omega$ the set of imaginary-axis poles of P . Consider a Nyquist contour \mathcal{N} on the imaginary axis indented into \mathbb{C}_+ around every element of $j\Omega$ via sufficiently small semicircles. We show below that the eigenloci of PC along the aforementioned Nyquist contour, i.e., $\{\lambda_i(P(s)C(s)) : s \in \mathcal{N}\}$, $i \in \{1, \dots, n\}$, does not intersect with the real interval $[1, \infty)$ on the complex plane. This implies that $(I - PC)^{-1}$ has no poles in $\bar{\mathbb{C}}_+ \setminus j\Omega$ by the multivariable Nyquist stability criterion [20, Th. L3], since P and C have no poles in $\bar{\mathbb{C}}_+ \setminus j\Omega$ and the indexed family of circuits formed by the juxtaposition of the eigenloci of PC do not encircle the critical point $1 + j0$.

First, for any $\omega \in [0, \infty)$ such that $j\omega$ is not a pole of P , (2) holding for all $\tau \in [0, 1]$ and (6) imply via Lemma 2 that $\det(I - \tau P(j\omega)C(j\omega)) \neq 0$ for all $\tau \in [0, 1]$, from which it follows that $\lambda(P(j\omega)C(j\omega)) \cap [1, \infty) = \emptyset$. For $\omega \in (-\infty, 0)$ such that $j\omega$ is not a pole of P , it also holds that $\lambda(P(j\omega)C(j\omega)) \cap [1, \infty) = \emptyset$ since $P, C \in \mathbf{R}^{n \times n}$, whereby $P(-j\omega) = \overline{P(j\omega)}$ and $C(-j\omega) = \overline{C(j\omega)}$.

Next, for every $j\omega_0$ that is a pole of P and the corresponding K defined in (5), note that $P(s) \approx \frac{1}{s-j\omega_0}K$ when s is sufficiently close to $j\omega_0$. By (2), (3), and Lemma 1(i), it holds that $F'(II_{12}(\omega)P(j\omega)) \subset \bar{\mathbb{C}}_+$ for $|\omega - \omega_0| \leq \epsilon$ and $\omega \neq \omega_0$, where continuity of Π at ω_0 has been used. Putting these together implies that $F'(II_{12}(\omega_0)K) \subset \mathbb{R}$. Together with (4), it follows that $F'(II_{12}(\omega_0)K) \subset \bar{\mathbb{R}}_+$, which is equivalent to $II_{12}(\omega_0)K$ being positive semidefinite Hermitian by Lemma 1(i) and (iv). Observe that (3) and (6) imply that $F'(II_{12}(\omega)^*C(j\omega)) \subset \mathbb{C}_-$ for $|\omega - \omega_0| \leq \epsilon$ by Lemma 1(ii). Applying Lemma 1(iii) then yields that for sufficiently small $\rho > 0$,

$$\begin{aligned} \lambda(P(s)C(s)) &= \lambda(II_{12}(\omega_0)P(s)C(s)II_{12}(\omega_0)^{-1}) \\ &\subset F'(II_{12}(\omega_0)P(s))F'(C(s)II_{12}(\omega_0)^{-1}) \\ &= F'(II_{12}(\omega_0)P(s))F'(II_{12}(\omega_0)^*C(s)) \\ &\approx F' \left(\frac{1}{s-j\omega_0} II_{12}(\omega_0)K \right) F'(II_{12}(\omega_0)^*C(j\omega_0)) \end{aligned}$$

for all $s = j\omega_0 + \rho e^{j\theta}$, $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, where by

$$\lambda(P(s)C(s)) \cap \mathbb{R}_+ = \emptyset.$$

Finally, we show that $(I - PC)^{-1}$ has no poles in $j\Omega$ under (2) and (6), so that $(I - PC)^{-1} \in \mathbf{RH}_\infty$ follows from the Nyquist-based analysis above. To this end, let $j\omega_0 \in j\Omega$ and note from (6) that for all $|\omega - \omega_0| \leq \epsilon$, $C(j\omega)$ is nonsingular and

$$-C(j\omega)^{-*}II_{12}(\omega)^* - II_{12}(\omega)C(j\omega)^{-1} > 0. \quad (7)$$

Summing (2) and (7) then yields the existence of $\nu > 0$ and $\eta > 0$ such that for all $|\omega - \omega_0| < \nu$ and $\omega \neq \omega_0$,

$$(P(j\omega) - C(j\omega)^{-1})^*II_{12}(\omega)^* + II_{12}(\omega)(P(j\omega) - C(j\omega)^{-1}) > \eta I,$$

by which

$$\begin{aligned} C(j\omega)^{-*}(P(j\omega)C(j\omega) - I)^*II_{12}(\omega)^* \\ + II_{12}(\omega)(P(j\omega)C(j\omega) - I)C(j\omega)^{-1} > \eta I. \end{aligned}$$

This implies the existence of $\gamma > 0$ such that

$$\underline{\sigma}(I - P(j\omega)C(j\omega)) > \gamma.$$

For all $|\omega - \omega_0| < \nu$ and $\omega \neq \omega_0$. Since $(I - PC)^{-1} \in \mathbf{R}$, it follows that

$$\sup_{|\omega - \omega_0| < \nu} \bar{\sigma}((I - P(j\omega)C(j\omega))^{-1}) < \infty$$

i.e., $j\omega_0$ is not a pole of $(I - PC)^{-1}$.

Since there are no pole-zero cancellations between P and C in $\bar{\mathbb{C}}$, the stability of $[P, C]$ is equivalent to $(I - PC)^{-1} \in \mathbf{RH}_\infty$ [47, Th. 5.7]. This completes the proof. ■

The condition in (3) signifies that the diagonal blocks of Π are zero when $j\omega$ is sufficiently close to an imaginary-axis pole of P , whereby Π reduces to a multiplier that corresponds to weighted passivity on its residue in (4) and no gain information about P would be exploited in establishing feedback stability.

Note that Theorem 2 is a robust stability result in that P and C may be arbitrary transfer functions that satisfy the conditions in the theorem for the stability of $[P, C]$ to hold. In other words, closed-loop stability is robust against perturbations that satisfy the frequency-domain inequalities in the theorem.

Remark 3: Note that if $\Pi = \begin{bmatrix} II_{11} & II_{12} \\ II_{21} & II_{22} \end{bmatrix} \in \mathbf{C}$ is such that $II_{11}(\omega) \geq 0$ and $II_{22}(\omega) \leq 0$ for all $\omega \in \bar{\mathbb{R}}_+$, then (2) holds for all $\tau \in [0, 1]$ if and only if it holds for $\tau = 1$.

IV. IWASAKI-HARA LEMMA

In this section, we elucidate how Theorem 2 nicely complements the Iwasaki-Hara lemma in [37, Th. 2]. In particular, by directly applying the Iwasaki-Hara lemma, we demonstrate that system properties that hold in a prespecified range of frequencies can be characterised via LMIs.

Consider the main Theorem 2. The inequality in (4) is straightforward to verify because it has no frequency dependence. Next, we demonstrate how the Iwasaki-Hara lemma may be used to characterize (2) and (3). For simplicity, we assume that $II_{11}(\omega) \geq 0$ and $II_{22}(\omega) \leq 0$ for all $\omega \in \bar{\mathbb{R}}_+$, so that Remark 3 is applicable and (2) only needs to be verified for $\tau = 1$. This is not a restrictive assumption because most multipliers, such as those corresponding to weighted small-gain, passivity, and negative imaginary properties, satisfy this assumption. The difficulty in verifying (2) lies in the fact that $\Pi \in \mathbf{C}$ is only piecewise continuous on \mathbb{R} , and thus may not admit a realisation $(A_\Pi, B_\Pi, C_\Pi, D_\Pi)$ on the entirety of $j\mathbb{R}$. Nonetheless, for any segment $[\omega_1, \omega_2] \subset \mathbb{R} \cup \{\infty\}$ on which Π is continuous, Π may be approximated arbitrarily closely by $\tilde{\Pi} \in \mathbf{RL}_\infty$ on this frequency segment [48, Lemma A.6.11]. Such a $\tilde{\Pi}$ should satisfy $\tilde{\Pi}(j\omega) = \tilde{\Pi}(j\omega)^*$ for all $\omega \geq 0$, in which case it may be factorised into $\tilde{\Pi}(j\omega) = \Psi(j\omega)^*M\Psi(j\omega)$ for all $\omega \geq 0$, where $M = M^T$ is a real matrix and $\Psi = [\Psi_1 \ \Psi_2] \in \mathbf{RH}_\infty$ [49, Sect. 7.3]. Such factorizations are not unique.

Suppose $\Pi(\omega) = \Psi(j\omega)^*M\Psi(j\omega)$ for $\omega \in [\omega_1, \omega_2] \subset [0, \infty)$ and one would like to verify (2) for $\omega \in [\omega_1, \omega_2]$ and $\tau = 1$.

Theorem 3: Let a minimal realisation of $\Psi \begin{bmatrix} I \\ P \end{bmatrix}$ be $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$. Then, (2) holds for all $\omega \in [\omega_1, \omega_2]$ and $\tau = 1$ if and only if there exist complex matrices $Y = Y^*$ and $Z = Z^* \geq 0$ such that the following holds:

(i) (for $\omega_2 < \infty$)

$$\begin{bmatrix} \hat{A} & \hat{B} \\ 0 & I \end{bmatrix}^T \begin{bmatrix} -Z & Y + j\omega_c Z \\ Y - j\omega_c Z & -\omega_1 \omega_2 Z \end{bmatrix} \begin{bmatrix} \hat{A} & \hat{B} \\ 0 & I \end{bmatrix} \leq \begin{bmatrix} \hat{C}^T \\ \hat{D}^T \end{bmatrix} M \begin{bmatrix} \hat{C} & \hat{D} \end{bmatrix}$$

where $\omega_c = (\omega_1 + \omega_2)/2$;

(ii) (for $\omega_2 = \infty$)

$$\begin{bmatrix} \hat{A} & \hat{B} \\ 0 & I \end{bmatrix}^T \begin{bmatrix} 0 & Y + jZ \\ Y - jZ & -2\omega_1 Z \end{bmatrix} \begin{bmatrix} \hat{A} & \hat{B} \\ 0 & I \end{bmatrix} \leq \begin{bmatrix} \hat{C}^T \\ \hat{D}^T \end{bmatrix} M \begin{bmatrix} \hat{C} & \hat{D} \end{bmatrix}.$$

Proof: This follows directly from the state-space nonstrict inequality version of the Iwasaki–Hara lemma [37, Th. 4]. ■

Observe that the LMIs above may also be used to verify (2) under (3). Finally, it is worth noting that (6) in Theorem 2 may be verified in a similar manner by alluding to [37, Th. 3] in lieu of [37, Th. 4]. In this case, minimality of realisations is no longer a concern and pole-zero cancellations in $\Psi \begin{bmatrix} C \\ \gamma \end{bmatrix}$ are allowed. The details are omitted for brevity.

V. MIXED SYSTEMS

In this section, we specialize the results from the earlier sections to a class of systems having mixed positive real, negative imaginary, and bounded real properties on the frequency axis.

First, we define what we mean by mixed systems.

Definition 8: $P \in \mathbf{R}^{m \times m}$ is said to be “mixed positive real, negative imaginary, and bounded real with gain γ ” if:

- (i) it is positive real on $\Omega^P \subset \mathbb{R}_+ \cup \{\infty\}$;
- (ii) it is negative imaginary on $\Omega^N \subset \mathbb{R}_+$;
- (iii) it is bounded real on $\Omega^B \subset \mathbb{R}_+ \cup \{\infty\}$ with gain $\gamma \in \mathbb{R}_+$;
- (iv) $\Omega^P \cup \Omega^N \cup \Omega^B = \mathbb{R}_+ \cup \{\infty\}$.

Observe that—for a mixed positive real, negative imaginary, and bounded real (with gain γ) system—the frequency sets Ω^P , Ω^N , and Ω^B are allowed to intersect in any combination. Indeed, an overlap of frequencies where systems properties occur is very common due to the inherent continuity of the frequency responses. Such an overlap of system properties is beneficial as it means that we do not need to know the transition frequencies between system properties with any precision.

The reason why we choose to exclude the values 0 and ∞ from the negative imaginary property in Definition 8 and insist that these two frequency values must be either positive real or bounded real is so that we can derive an unconditional stability result (see Theorem 2). Indeed, purely negative imaginary results (i.e., not mixed systems), such as those in [25] and [26], result in conditional stability statements with extra conditions imposed at 0 and ∞ .

Next, we provide a feedback stability result when two mixed positive real, negative imaginary, and bounded real (with gain γ) systems are interconnected in a positive feedback loop. One of the two systems (C in Theorem 2) must have strict properties. The result simply states that if the positive real, negative imaginary, and bounded real properties of the two mixed systems occur at the same frequency regions Ω^P , Ω^N , and Ω^B respectively, then the feedback interconnection of these two mixed systems is stable.

Theorem 2: Let $[a_i^P, b_i^P] \subset \mathbb{R}_+ \cup \{\infty\}$ with $a_i^P < b_i^P$ for all $i \in \{1, \dots, r_1\}$ and define $\Omega^P = \bigcup_{i=1}^{r_1} [a_i^P, b_i^P]$. Let $[a_i^N, b_i^N] \subset \mathbb{R}_+$ with $a_i^N < b_i^N$ for all $i \in \{1, \dots, r_2\}$ and define $\Omega^N = \bigcup_{i=1}^{r_2} [a_i^N, b_i^N]$. Let $[a_i^B, b_i^B] \subset \mathbb{R}_+ \cup \{\infty\}$ with $a_i^B < b_i^B$ for all $i \in \{1, \dots, r_3\}$ and define $\Omega^B = \bigcup_{i=1}^{r_3} [a_i^B, b_i^B]$. Suppose that $\Omega^P \cup \Omega^N \cup \Omega^B = \mathbb{R}_+ \cup \{\infty\}$.

Let $P \in \mathbf{R}^{m \times m}$ be positive real on Ω^P , negative imaginary on Ω^N , and bounded real on $[a_i^B, b_i^B] \subset \mathbb{R}_+ \cup \{\infty\}$ with gain $\gamma_i \in \mathbb{R}_+$ for all $i \in \{1, \dots, r_3\}$. Suppose also that the poles of P on $j\mathbb{R}_+$ do not coincide with the boundary points of $j\Omega^P$ and $j\Omega^N$.

Let $C \in \mathbf{RH}_{\infty}^{m \times m}$ satisfy the following:

- (i) $-C(j\omega) - C(j\omega)^* > 0 \quad \forall \omega \in \Omega^P$;
- (ii) $j[C(j\omega) - C(j\omega)^*] > 0 \quad \forall \omega \in \Omega^N$;
- (iii) $I - \gamma_i^2 C(j\omega)^* C(j\omega) > 0 \quad \forall \omega \in [a_i^B, b_i^B], i \in \{1, \dots, r_3\}$.

Then, $[P, C]$ is stable.

Proof: Let $\Pi(\omega) = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ for all $\omega \in \Omega^P$, $\Pi(\omega) = \begin{bmatrix} 0 & jI \\ -jI & 0 \end{bmatrix}$ for all $\omega \in \Omega^N \setminus \Omega^P$ and $\Pi(\omega) = \begin{bmatrix} \gamma_i^2 I & 0 \\ 0 & -I \end{bmatrix}$ for all $\omega \in [a_i^B, b_i^B] \setminus (\Omega^P \cup \Omega^N)$ and $i \in \{1, \dots, r_3\}$. Then, $\Pi \in \mathbf{C}$ by definition. Furthermore, the suppositions on P guarantee that $P \in \mathbf{M}^{m \times m}$ because P has only simple poles on $j\mathbb{R}$ and no poles in \mathbb{C}_+ , and also guarantee that parts (i) and (ii) of Theorem 2 hold. Finally, the suppositions on C guarantee that part (iii) of Theorem 2 holds. The conclusion follows via application of Theorem 2. ■

The number of intervals as stated in Theorem 2 is restricted to be finite for computational purposes. Even though it is indeed possible to handle countably infinite closed intervals, doing so would amount to solving an infinite number of LMIs, which is not computationally tractable.

Since Theorem 2 considers a positive feedback interconnection, the strict positive real property of C on Ω^P in condition (i) has to be specified on $-C$ instead to compensate for the negative sign of the feedback loop in the passivity theorem.

Observe also that a different gain γ_i is allowed for each different continuous interval $[a_i^B, b_i^B]$ within Ω^B where the bounded real property holds as long as the controller C is bounded by gain $\frac{1}{\gamma_i}$ over the same connected frequency interval where P is bounded by gain γ_i .

Note again that the frequency regions Ω^P , Ω^N , and Ω^B —where the positive real, negative imaginary, and bounded real properties hold—can overlap in any combination. This is beneficial as it means that we do not need to know precisely the frequencies where system properties transition.

Next, we provide a convenient equivalent LMI characterisation via the Iwasaki–Hara lemma using a realisation of plant P to check whether P has mixed positive real, negative imaginary, and bounded real properties on the specified frequency regions Ω^P , Ω^N , and Ω^B . Note that the plant P can potentially have poles on the imaginary axis.

Theorem 3: Let $[a_i^P, b_i^P] \subset \mathbb{R}_+ \cup \{\infty\}$ with $a_i^P < b_i^P$ for all $i \in \{1, \dots, r_1\}$ and define $\Omega^P = \bigcup_{i=1}^{r_1} [a_i^P, b_i^P]$. Let $[a_i^N, b_i^N] \subset \mathbb{R}_+$ with $a_i^N < b_i^N$ for all $i \in \{1, \dots, r_2\}$ and define $\Omega^N = \bigcup_{i=1}^{r_2} [a_i^N, b_i^N]$. Let $[a_i^B, b_i^B] \subset \mathbb{R}_+ \cup \{\infty\}$ with $a_i^B < b_i^B$ for all $i \in \{1, \dots, r_3\}$ and define $\Omega^B = \bigcup_{i=1}^{r_3} [a_i^B, b_i^B]$.

Furthermore, let $P = (A_p, B_p, C_p, D_p) \in \mathbf{R}^{m \times m}$ with (A_p, B_p) controllable and (C_p, A_p) detectable. Define $\Omega_0 = \{\omega_0 \in \mathbb{R}_+ : \det(j\omega_0 I - A_p) = 0\}$.

Then, P is positive real on Ω^P , negative imaginary on Ω^N , and bounded real on $[a_i^B, b_i^B]$ with gain $\gamma_i \in \mathbb{R}_+$ for all $i \in \{1, \dots, r_3\}$ if and only if all of the following conditions hold:

- 1) A_p has at most simple eigenvalues on $j\mathbb{R}_+$ and there exists a real matrix $X = X^T \geq 0$ such that $A_p X + X A_p^T \leq 0$;
- 2) $\forall \omega_0 \in \Omega_0 \cap \Omega^P, \lim_{s \rightarrow j\omega_0} (s - j\omega_0)P(s) \geq 0$;
- 3) $\forall \omega_0 \in \Omega_0 \cap \Omega^N, \lim_{s \rightarrow j\omega_0} (s - j\omega_0)jP(s) \geq 0$;
- 4) $\forall i \in \{1, \dots, r_3\}$, there exist complex matrices $Y_i^P, Z_i^P \in \mathbb{H}_n$ with $Z_i^P \geq 0$ such that

$$\begin{bmatrix} A_p & B_p \\ I_n & 0 \end{bmatrix}^T \begin{bmatrix} -2\mu_i^P Z_i^P & Y_i^P + jZ_i^P \\ Y_i^P - jZ_i^P & -2\lambda_i^P Z_i^P \end{bmatrix} \begin{bmatrix} A_p & B_p \\ I_n & 0 \end{bmatrix} \leq \begin{bmatrix} 0 & I_m \\ C_p & D_p \end{bmatrix}^T \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix} \begin{bmatrix} 0 & I_m \\ C_p & D_p \end{bmatrix}$$

where $\mu_i^P = \frac{1}{a_i^P + b_i^P}$ and $\lambda_i^P = \frac{a_i^P b_i^P}{a_i^P + b_i^P}$;

- 5) $\forall i \in \{1, \dots, r_2\}$, there exist complex matrices $Y_i^N, Z_i^N \in \mathbb{H}_n$ with $Z_i^N \geq 0$ such that

$$\begin{bmatrix} A_p & B_p \\ I_n & 0 \end{bmatrix}^T \begin{bmatrix} -2\mu_i^N Z_i^N & Y_i^N + jZ_i^N \\ Y_i^N - jZ_i^N & -2\lambda_i^N Z_i^N \end{bmatrix} \begin{bmatrix} A_p & B_p \\ I_n & 0 \end{bmatrix} \\ \leq \begin{bmatrix} 0 & I_m \\ C_p & D_p \end{bmatrix}^T \begin{bmatrix} 0 & jI_m \\ -jI_m & 0 \end{bmatrix} \begin{bmatrix} 0 & I_m \\ C_p & D_p \end{bmatrix}$$

where $\mu_i^N = \frac{1}{a_i^N + b_i^N}$ and $\lambda_i^N = \frac{a_i^N b_i^N}{a_i^N + b_i^N}$;

- 6) $\forall i \in \{1, \dots, r_3\}$, there exist complex matrices $Y_i^B, Z_i^B \in \mathbb{H}_n$ with $Z_i^B \geq 0$ such that

$$\begin{bmatrix} A_p & B_p \\ I_n & 0 \end{bmatrix}^T \begin{bmatrix} -2\mu_i^B Z_i^B & Y_i^B + jZ_i^B \\ Y_i^B - jZ_i^B & -2\lambda_i^B Z_i^B \end{bmatrix} \begin{bmatrix} A_p & B_p \\ I_n & 0 \end{bmatrix} \\ \leq \begin{bmatrix} 0 & I_m \\ C_p & D_p \end{bmatrix}^T \begin{bmatrix} \gamma_i^2 I_m & 0 \\ 0 & -I_m \end{bmatrix} \begin{bmatrix} 0 & I_m \\ C_p & D_p \end{bmatrix}$$

where $\mu_i^B = \frac{1}{a_i^B + b_i^B}$ and $\lambda_i^B = \frac{a_i^B b_i^B}{a_i^B + b_i^B}$.

Proof: Since the realization of P is stabilizable and detectable, hidden modes can only occur in \mathbb{C}_- . This ensures that there is no difference between the eigenvalues of A_p and the poles of P in $\bar{\mathbb{C}}_+$.

Now, note that condition (iv) in Definition 6 is automatically fulfilled since $0 \notin \Omega^N$. Next, note that conditions 1)–3) are equivalent to condition (i) in Definition 7 and conditions (i) and (iii) in Definitions 5 and 6.

Next, observe that condition 4) is equivalent to condition (ii) in Definition 5 via [37, Th. 4]. Similarly, condition 5) is equivalent to condition (ii) in Definition 6 via [37, Th. 4]. Finally, condition 6) is equivalent to condition (ii) in Definition 7 via [37, Th. 4]. This concludes the proof. \blacksquare

Remark 4: If $a_i^P = 0$ for some $i \in \{1, \dots, r_1\}$, then $\mu_i^P = \frac{1}{b_i^P} > 0$ and $\lambda_i^P = 0$. On the other hand, if $b_i^P = \infty$ for some $i \in \{1, \dots, r_1\}$, then $\mu_i^P = 0$ and $\lambda_i^P = a_i^P < \infty$. Indeed, it is easy to see that $\mu_i^P, \lambda_i^P \in \bar{\mathbb{R}}_+$ for all $i \in \{1, \dots, r_1\}$. Similarly for μ_i^B and λ_i^B .

On the other hand, $\mu_i^N, \lambda_i^N \in \bar{\mathbb{R}}_+$ for all $i \in \{1, \dots, r_2\}$ since $a_i^N, b_i^N \in \bar{\mathbb{R}}_+$ for all $i \in \{1, \dots, r_2\}$ (i.e., a_i^N and b_i^N cannot take values of 0 and ∞).

The LMIs in conditions 4)–6) involve complex variables due to the fact that the frequency regions Ω^P , Ω^N , and Ω^B are not symmetric about 0 (i.e., they admit one-sided frequencies) [37]. These complex LMIs can be equivalently transformed into real LMIs via the following equivalence: Given $R \in \mathbb{H}_n$, $R \leq 0$ (respectively, < 0) $\Leftrightarrow \begin{bmatrix} \text{Re}(R) & \text{Im}(R) \\ -\text{Im}(R) & \text{Re}(R) \end{bmatrix} \leq 0$ (respectively, < 0).

Next, we provide a convenient equivalent LMI characterization via the Iwasaki–Hara lemma using a realisation of controller C to check whether the chosen C fulfills the required mixed strict positive real, strict negative imaginary, and strict bounded real properties on the specified frequency regions Ω^P , Ω^N , and Ω^B . Stability of the controller C together with stabilizability and detectability of its realization are automatically guaranteed by the equivalent LMIs.

Theorem 5: Let $[a_i^P, b_i^P] \subset \bar{\mathbb{R}}_+ \cup \{\infty\}$ with $a_i^P < b_i^P$ for all $i \in \{1, \dots, r_1\}$ and define $\Omega^P = \bigcup_{i=1}^{r_1} [a_i^P, b_i^P]$. Let $[a_i^N, b_i^N] \subset \bar{\mathbb{R}}_+$ with $a_i^N < b_i^N$ for all $i \in \{1, \dots, r_2\}$ and define $\Omega^N = \bigcup_{i=1}^{r_2} [a_i^N, b_i^N]$. Let $\gamma_i \in \bar{\mathbb{R}}_+$ and $[a_i^B, b_i^B] \subset \bar{\mathbb{R}}_+ \cup \{\infty\}$ with $a_i^B < b_i^B$ for all $i \in \{1, \dots, r_3\}$ and define $\Omega^B = \bigcup_{i=1}^{r_3} [a_i^B, b_i^B]$.

Then, $C \in \mathbf{RH}_\infty^{m \times m}$ with stabilizable and detectable realization (A_c, B_c, C_c, D_c) satisfies the following:

- (i) $-C(j\omega) - C(j\omega)^* > 0 \quad \forall \omega \in \Omega^P$;
(ii) $j[C(j\omega) - C(j\omega)^*] > 0 \quad \forall \omega \in \Omega^N$;
(iii) $I - \gamma_i^2 C(j\omega)^* C(j\omega) > 0 \quad \forall \omega \in [a_i^B, b_i^B], i \in \{1, \dots, r_3\}$;
if and only if all of the following conditions hold:

- 1) \exists a real matrix $X = X^T > 0$ such that $A_c X + X A_c^T < 0$;
2) $\forall i \in \{1, \dots, r_1\}$, there exist complex matrices $Y_i^P, Z_i^P \in \mathbb{H}_n$ with $Z_i^P > 0$ such that

$$\begin{bmatrix} A_c & B_c \\ I_n & 0 \end{bmatrix}^T \begin{bmatrix} -2\mu_i^P Z_i^P & Y_i^P + jZ_i^P \\ Y_i^P - jZ_i^P & -2\lambda_i^P Z_i^P \end{bmatrix} \begin{bmatrix} A_c & B_c \\ I_n & 0 \end{bmatrix} \\ < \begin{bmatrix} 0 & I_m \\ C_c & D_c \end{bmatrix}^T \begin{bmatrix} 0 & -I_m \\ -I_m & 0 \end{bmatrix} \begin{bmatrix} 0 & I_m \\ C_c & D_c \end{bmatrix}$$

where $\mu_i^P = \frac{1}{a_i^P + b_i^P}$ and $\lambda_i^P = \frac{a_i^P b_i^P}{a_i^P + b_i^P}$;

- 3) $\forall i \in \{1, \dots, r_2\}$, there exist complex matrices $Y_i^N, Z_i^N \in \mathbb{H}_n$ with $Z_i^N > 0$ such that

$$\begin{bmatrix} A_c & B_c \\ I_n & 0 \end{bmatrix}^T \begin{bmatrix} -2\mu_i^N Z_i^N & Y_i^N + jZ_i^N \\ Y_i^N - jZ_i^N & -2\lambda_i^N Z_i^N \end{bmatrix} \begin{bmatrix} A_c & B_c \\ I_n & 0 \end{bmatrix} \\ < \begin{bmatrix} 0 & I_m \\ C_c & D_c \end{bmatrix}^T \begin{bmatrix} 0 & jI_m \\ -jI_m & 0 \end{bmatrix} \begin{bmatrix} 0 & I_m \\ C_c & D_c \end{bmatrix}$$

where $\mu_i^N = \frac{1}{a_i^N + b_i^N}$ and $\lambda_i^N = \frac{a_i^N b_i^N}{a_i^N + b_i^N}$;

- 4) $\forall i \in \{1, \dots, r_3\}$, there exist complex matrices $Y_i^B, Z_i^B \in \mathbb{H}_n$ with $Z_i^B > 0$ such that

$$\begin{bmatrix} A_c & B_c \\ I_n & 0 \end{bmatrix}^T \begin{bmatrix} -2\mu_i^B Z_i^B & Y_i^B + jZ_i^B \\ Y_i^B - jZ_i^B & -2\lambda_i^B Z_i^B \end{bmatrix} \begin{bmatrix} A_c & B_c \\ I_n & 0 \end{bmatrix} \\ < \begin{bmatrix} 0 & I_m \\ C_c & D_c \end{bmatrix}^T \begin{bmatrix} I_m & 0 \\ 0 & -\gamma_i^2 I_m \end{bmatrix} \begin{bmatrix} 0 & I_m \\ C_c & D_c \end{bmatrix}$$

where $\mu_i^B = \frac{1}{a_i^B + b_i^B}$ and $\lambda_i^B = \frac{a_i^B b_i^B}{a_i^B + b_i^B}$.

Proof: A stabilizable and detectable realization has no hidden modes in $\bar{\mathbb{C}}_+$. Therefore, condition 1) is equivalent to A_c is Hurwitz, which in turn is equivalent to $C = (A_c, B_c, C_c, D_c) \in \mathbf{RH}_\infty^{m \times m}$ and (A_c, B_c, C_c, D_c) is stabilizable and detectable.

Next, observe that condition 2) is equivalent to condition (i) via [37, Th. 3]. Similarly, condition 3) is equivalent to condition (ii) via [37, Th. 3]. Finally, condition 4) is equivalent to condition (iii) via [37, Th. 3]. This concludes the proof. \blacksquare

Theorem 3 is based on the state-space nonstrict inequality version of the Iwasaki–Hara lemma, namely [37, Th. 4] By contrast, Theorem 5 makes use of the state space strict inequality version of the Iwasaki–Hara lemma, i.e., [37, Th. 3]. The two results are distinct and built upon different assumptions on the state-space realizations.

VI. MOTIVATING EXAMPLE

Consider a single-input-single-output flexible structure with force actuator and position sensor described in [1] by

$$P(s) = \frac{k_0}{s} + \sum_{i=1}^m \frac{k_i}{s^2 + 2\zeta_i \omega_i s + \omega_i^2}$$

where $k_0 > 0$, $0 \leq \zeta_i < 1$, and $0 < \omega_i < \omega_{i+1}$ for all $i \in \{1, \dots, m-1\}$. P is said to be in-phase if the coefficients of the flexible modes k_i have the same sign as k_0 , i.e., $k_i > 0$ for all $i \in \{1, \dots, m\}$. This happens when the force actuator is colocated with the position sensor, and P is a negative imaginary system that is amenable to the negative imaginary theory [26]. The reader is referred to [2, Sec. II], where a variety of justifications is provided from practical perspectives for the finite-frequency positive realness or negative imaginarity to be a crucial property for good control performance in the case where the control authority is limited, especially when the force actuator is not colocated with the position sensor. This renders some of the k_i 's to be negative, and that P exhibits negative imaginarity only in a finite frequency band. Feedback control design for such a P may be performed with the aid of the main results in this paper, as demonstrated below. An alternative \mathbf{H}_∞ loop shaping-based design procedure may be found in [50].

Consider a simple, illustrative model for a flexible structure with noncolocated force actuator and position sensor

$$P(s) = \frac{1}{s} + \frac{1}{s^2 + 1} + \frac{-0.2}{s^2 + 0.1s + 5^2}.$$

Note that P has marginally stable poles (i.e., not a bounded real system) and is neither a passive system nor a negative imaginary system according to Definitions 5 and 6. Nonetheless, it may be verified with the aid of the Iwasaki–Hara lemma-based Theorem 3 from the preceding section that $P(j\omega)$ is positive real for $\omega \in [0, 0.1]$ as per Definition 5, negative imaginary for $\omega \in [0.1, 4]$ as per Definition 6, and $|P(j\omega)| \leq 0.5$ for $\omega \in [4, \infty]$ (cf., Definition 7). Note that in actual fact $P(j\omega)$ is negative imaginary also for $\omega \in [4, 4.94]$ and $|P(j\omega)| \leq 0.5$ for $\omega \in [2.3, \infty]$, i.e., there is an overlap of the frequencies where $P(j\omega)$ is negative imaginary and has a gain 0.5 or less. In accordance with Theorem 2 and Remark 3, by defining $\Pi(\omega) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ for $\omega \in [0, 0.1]$, $\Pi(\omega) = \begin{bmatrix} 0 & j \\ -j & 0 \end{bmatrix}$ for $\omega \in (0.1, 4)$, and $\Pi(\omega) = \begin{bmatrix} 0.5^2 & 0 \\ 0 & -1 \end{bmatrix}$ for $\omega \in [4, \infty]$, it follows that any $C \in \mathbf{RH}_\infty$ that satisfies $C(j\omega) + C(j\omega)^* < 0$ for $\omega \in [0, 0.1]$, $j(C(j\omega) - C(j\omega)^*) > 0$ for $\omega \in [0.1, 4]$, and $|C(j\omega)| < \frac{1}{0.5}$ for $\omega \in [4, \infty]$, would render $[P, C]$ stable. One such example is $C(s) = \frac{1}{s+1} - 1.1$.

VII. CONCLUSION

We derived frequency-varying conditions for robust feedback stability of systems with marginally stable open-loop poles. Such conditions allow the “mixing” of various useful system properties on the imaginary axis in a seamless fashion, and are particularly useful for stabilizing flexible structures with noncolocated force actuators and positions sensors, from which positive realness and negative imaginarity are manifested only on limited frequency bands. The results are intimately related to and complemented by the Iwasaki–Hara lemma, with which frequency-band-limited system properties may be characterized in a computationally effective manner using LMIs.

Performing control synthesis that results in a closed-loop system manifesting frequency-limited negative imaginarity via the Iwasaki–Hara lemma following the approaches in [51] is an interesting future research direction. Specifically, characterizations of strictly negative imaginary and output strictly negative imaginary systems on an open frequency band is of significant importance from the perspective of control design [52]. Nevertheless, these systems are not amenable to the Iwasaki–Hara lemma in its present form and further investigations are urgently needed.

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