Negative Imaginariness Indices in Feedback Stability Analysis

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***Abstract*—We introduce the notions of negative imaginariness input and output indices for describing systems that do not manifest the negative imaginary (NI) property on certain frequency bands, thereby characterizing classes of linear time-invariant systems that are larger than those of negative imaginary systems without poles at the origin. We show that if the total frequency-dependent negative imaginariness in a feedback interconnection is positive as measured by the NI indices, whereby any deficiency in negative imaginariness in one open-loop component can be compensated for by a surplus of negative imaginariness in another, then the feedback interconnection is stable if and only if the static (a.k.a. dc) loop gain is less than unity. The result covers the feedback interconnection of a negative imaginary system and a strictly negative imaginary system, which naturally gives rise to positive total negative imaginariness. Importantly, we derive the NI indices-based condition from a more general robust stability result established herein involving quadratic frequency dependent inequalities that may be used to characterize system properties beyond negative imaginariness. The proof relies on the multivariable Nyquist stability criterion and does not make use of state-space realizations of the underlying systems.**

***Index Terms*—Feedback stability, multipliers, negative imaginariness indices, negative imaginary systems, Nyquist criterion.**

I. Introduction

The negative imaginary system property is known to arise in various practical applications [1], [2], and is particularly relevant in the dynamics of a lightly damped structure with colocated force actuators and position sensors [3], [4]. The notion of negative imaginary systems was introduced in [5] and then extended to include poles on the imaginary axis in [6], [7], and [8]. Various investigations of negative imaginary systems have been conducted in the literature. A severely nonexhaustive list includes studies of irrational transfer functions [9], [10], output consensus [11], output strict negative imaginariness [12], and connections to integral–quadratic constraints [13] and dissipativity [14], [15], [16]. In [17], state-feedback equivalence of linear time-invariant systems to negative imaginary systems is examined in detail. It is also noteworthy that negative imaginariness is closely related to counterclockwise dynamics in the nonlinear setting [18].

Mechanical systems with colocated force actuators and position sensors typically exhibit the negative imaginary property, i.e., their corresponding transfer functions are negative imaginary over all nonzero positive frequencies. Nevertheless, in applications where the force actuator and position sensor are not colocated, such as the control of swing-arm positioning in hard-disk drives [19], [20], the corresponding transfer functions may not be negative imaginary at all nonzero positive frequencies, but only on certain frequency bands. This renders the aforementioned standard negative imaginary theory inapplicable to feedback stability analysis of such systems.

In this article, we introduce the notions of negative imaginariness input and output indices (NI I/O-indices) for the purpose of characterizing negative imaginariness surplus and deficit, in a similar spirit to the role of passivity indices [21], [22]; see also applications of passivity index theory to the stability of grid-converter interactions in [23] and [24]. We note, however, that NI I/O indices serve different purposes to passivity indices, which are used to quantify the level of passivity in a system, much akin to the intrinsic difference between negative imaginary systems and passive systems. We show that the feedback interconnection of an open-loop system with negative imaginariness deficit on certain frequency bands and another open-loop system with excess negative imaginariness on the same frequency bands can be guaranteed to be stable provided that the sum of the negative imaginariness is positive on these bands. In other words, a surplus of negative imaginariness in one open-loop system can be used to offset a lack of negative imaginariness in another from the perspective of ensuring feedback stability. The main result is a necessary and sufficient condition for closed-loop stability in terms of the static loop gain, which reminisces and recovers that for the stability of a feedback interconnection of standard negative imaginary systems. In particular, under the assumptions that the eigenvalues of the instantaneous loop gain are real and less than unity (which incorporates the case of strictly proper loop transfer function) and the total frequency-varying negative imaginariness in the loop is positive, we show that the feedback system is stable if and only if the largest real eigenvalue of the static loop gain is less than unity, should it exist? We illustrate the utility of this result with an example on the control of flexible structure with noncolocated force actuator and position sensor.

In order to derive the aforementioned NI I/O indices-based robust stability result, we establish a generalization of the main result in [25] by incorporating a more general form of characterization of system property via quadratic frequency-dependent inequalities and allowing for imaginary-axis poles in both the open-loop systems in a feedback interconnection. Such inequalities may be used to capture a broad range of system properties, including passivity, small-gain, and negative imaginariness as well as their weighted versions and combinations. The proof for this main result makes use of the multivariable Nyquist stability criterion [26], [27], [28], as well as quadratic graph separation ideas from the integral–quadratic constraint (IQC) literature [29], [30].

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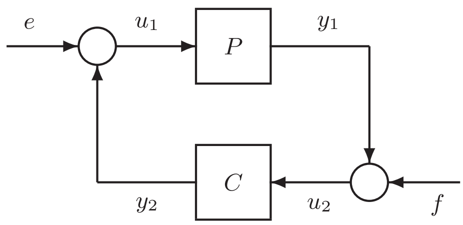

Fig. 1. Standard feedback configuration $[P, C]$.

As a consequence, the proof conveniently circumvents cumbersome algebraic manipulations of state-space matrices derived from the negative imaginary lemma [6] that are commonly practiced in the negative imaginary literature [5], [8], [15].

The contributions of the theoretical results in this article include a novel NI I/O-indices-based approach to handling flexible systems equipped with noncolocated force actuators and position sensors. Furthermore, by accommodating imaginary-axis poles in both the open-loop systems in a feedback interconnection, one may synthesize controllers for servo systems so that they track (respectively, reject) sinusoidal references (respectively, disturbances) via the internal model principle.

The rest of the article is organized as follows. Notations and mathematical preliminaries are provided in Section II. In Section III, a general feedback stability result, involving frequency-varying quadratic inequalities, is established. The notions of NI I/O-indices are defined in Section IV, where a main result on robust closed-loop stability, involving trading of negative imaginariness between open-loop systems, is provided. Specializations of the NI I/O-indices-based result to negative imaginary systems are supplied in Section V. Section VI contains an example on the control of a flexible structure that demonstrates the usefulness of NI I/O-indices and their corresponding robust stability result. Finally, Section VII concludes this article.

II. Notations and Preliminaries

Denote by $\mathbb{R}, \mathbb{C}, \mathbb{C}_+$, and $\bar{\mathbb{C}}_+$ the reals, the complex plane, the open right-half complex plane, and the closed right-half complex plane, respectively. The Euclidean norm in $\mathbb{C}^n$ is denoted by $|\cdot|$.

A. Matrices

For $M \in \mathbb{C}^{n\times m}$, denote by M^* the complex conjugate transpose of M. An $M \in \mathbb{C}^{n\times n}$ is said to be Hermitian, if $M = M^*$. A Hermitian M is said to be positive semidefinite (respectively, definite), denoted $M \geq 0$ (respectively, $M > 0$), if $v^* M v \geq 0$ (respectively, > 0) for all nonzero $v \in \mathbb{C}^n$. Denote by $\lambda(M)$ the spectrum of M, i.e., the set of its eigenvalues, and $\bar{\lambda}(M)$ (respectively, $\underline{\lambda}(M)$) the largest (respectively, smallest) real eigenvalue of M, if it exists. The $m \times m$ identity matrix is denoted as I_m, whose subscript is often omitted when the dimension is clear from the context.

Next, we state an important result on graph separation of two matrices that will be used repeatedly throughout the article.

Lemma II.1 (See [30, Corollary 1]): Let $M \in \mathbb{C}^{p\times q}$ and $N \in \mathbb{C}^{q\times p}$. Then

$$\det(I - \tau MN) \neq 0 \quad \text{for all } \tau \in [0, 1]$$

if there exists $\Pi = \Pi^*$ such that

$$\begin{bmatrix} I \\ \tau M \end{bmatrix}^* \Pi \begin{bmatrix} I \\ \tau M \end{bmatrix} \geq 0 \quad \text{for all } \tau \in [0, 1] \tag{1}$$

and

$$\begin{bmatrix} N \\ I \end{bmatrix}^* \Pi \begin{bmatrix} N \\ I \end{bmatrix} < 0.$$

B. Transfer Functions and Feedback Systems

Let $\mathbf{R}^{n\times m}$ denote the set of real-rational proper $n \times m$ transfer function matrices and $\mathbf{RH}_\infty^{n\times m}$ be its stable subset containing elements with no poles in $\bar{\mathbb{C}}_+$.

The positive feedback interconnection of two transfer functions $P \in \mathbf{R}^{n\times m}$ and $C \in \mathbf{R}^{m\times n}$, denoted by $[P, C]$, is described by

$$\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} I & -C \\ -P & I \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

see Fig. 1.

Definition II.2: A feedback interconnection of P and C is said to be *(internally) stable*, if the transfer function mapping from $\begin{bmatrix} e \\ f \end{bmatrix}$ to $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ is stable, i.e.,

$$[P, C] = \begin{bmatrix} I & -C \\ -P & I \end{bmatrix}^{-1} = \begin{bmatrix} I + C(I - PC)^{-1}P & C(I - PC)^{-1} \\ (I - PC)^{-1}P & (I - PC)^{-1} \end{bmatrix}$$

is an element in $\mathbf{RH}_\infty$.

Denote by $\mathbf{M}^{n\times m} \subset \mathbf{R}^{n\times m}$ the set of transfer functions such that $G \in \mathbf{M}^{n\times m}$ if and only if

(i) G has no poles in $\mathbb{C}_+ \cup \{0\}$ and

(ii) for any $\omega_0 > 0$, if $j\omega_0$ is a pole of G, then it is a simple pole.

Let $\mathbf{C}$ be the class of functions $\Pi : (0, \infty) \to \mathbb{C}^{m\times m}$ that are piecewise continuous and $\Pi(\omega) = \Pi(\omega)^*$ for all $\omega > 0$.

III. Frequency-Dependent Quadratic Constraints-Based Robust Stability

In this section, we derive a generalization of [25, Thm. III.1], which will be utilized in Section IV to establish an NI I/O-indices-based result. The main result here provides a necessary and sufficient condition for the stability of a feedback interconnection of two open-loop systems with possible poles on the imaginary axis manifesting complementary properties characterized by frequency-dependent quadratic constraints over nonzero finite frequencies. It generalizes [25, Thm. III.1], which considers only weighted positive realness, imposes equal dimensions on the input and output of the open-loop systems, and allows for only one of the open-loop systems to have imaginary-axis poles. The proof of the result below makes use of a quadratic graph separation result from [30], a homotopy argument in the frequency domain, together with the multivariable Nyquist stability criterion [28].

Theorem III.1: Given that $G_1 \in \mathbf{M}^{p\times q}$ and $G_2 \in \mathbf{M}^{q\times p}$, suppose that there exist $\epsilon > 0$ and $\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{bmatrix} \in \mathbf{C}$ such that the following holds:

(i) $\lambda(G_1(\infty)G_2(\infty)) \subset \mathbb{R}$, $\bar{\lambda}(G_1(\infty)G_2(\infty)) < 1$;

(ii) $\Pi_{11}(\omega) \geq 0$ and $\Pi_{22}(\omega) \leq 0$ for all $\omega > 0$;

(iii) For all $\omega > 0$ such that $j\omega$ is not a pole of G_1 or G_2, either

$$\begin{bmatrix} I \\ G_1(j\omega) \end{bmatrix}^* \Pi(\omega) \begin{bmatrix} I \\ G_1(j\omega) \end{bmatrix} \geq 0 \tag{2}$$

$$\begin{bmatrix} G_2(j\omega) \\ I \end{bmatrix}^* \Pi(\omega) \begin{bmatrix} G_2(j\omega) \\ I \end{bmatrix} < 0 \tag{3}$$

or

$$\begin{bmatrix} I \\ G_1(j\omega) \end{bmatrix}^* \Pi(\omega) \begin{bmatrix} I \\ G_1(j\omega) \end{bmatrix} > 0 \tag{4}$$

$$\begin{bmatrix} G_2(j\omega) \\ I \end{bmatrix}^* \Pi(\omega) \begin{bmatrix} G_2(j\omega) \\ I \end{bmatrix} \leq 0; \tag{5}$$

(iv) If $j\omega_0$ is a pole of either G_1 or G_2, then Π is continuous at ω_0 and

$$\Pi_{11}(\omega) = 0 \text{ and } \Pi_{22}(\omega) = 0 \quad \text{for all } |\omega - \omega_0| \leq \epsilon; \tag{6}$$

(v) If $j\omega_0$ is a pole of G_1, then $j\omega_0$ is not a pole of G_2, (2) holds for all $|\omega - \omega_0| \leq \epsilon, \omega \neq \omega_0$, (3) holds for all $|\omega - \omega_0| \leq \epsilon$, and

$$\Pi_{12}(\omega_0)K_1 + K_1^* \Pi_{12}(\omega_0)^* \geq 0 \tag{7}$$

where

$$K_1 = \lim_{s \to j\omega_0} (s - j\omega_0)G_1(s); \tag{8}$$

(vi) If $j\omega_0$ is a pole of G_2, then $j\omega_0$ is not a pole of G_1, (5) holds for all $|\omega - \omega_0| \leq \epsilon, \omega \neq \omega_0$, (4) holds for all $|\omega - \omega_0| \leq \epsilon$, and

$$\Pi_{12}(\omega_0)^* K_2 + K_2^* \Pi_{12}(\omega_0) \leq 0 \tag{9}$$

where

$$K_2 = \lim_{s \to j\omega_0} (s - j\omega_0)G_2(s). \tag{10}$$

Then, $[G_1, G_2]$ is stable if and only if either $\lambda(G_1(0)G_2(0)) \cap \mathbb{R} = \emptyset$ or $\bar{\lambda}(G_1(0)G_2(0)) < 1$.

Proof: The proof is provided in the appendix. ■

A few remarks are in order. Theorem III.1 generalizes [25, Thm. III.1] by allowing for more a general form of quadratic characterizations of system properties beyond weighted passivity, different input–output dimensions of the open-loop systems, and possible imaginary-axis poles in both open-loop systems. The reader is referred to [25] for various significant specializations of [25, Thm. III.1] as well as illustrative examples. In particular, the multipliers used in [25, Thm. III.1] exploit only phase-type information [31], [32] in the open-loop systems, whereas now with Theorem III.1, it is possible to make use of additional gain-type information [31] to reduce conservatism in feedback stability analysis. Connections to segmental phase [33] are also worth exploring.

Since Theorem III.1 allows for open-loop systems that may admit imaginary-axis poles, it may not be established using the classical IQC theorem [29], which handles only open-loop stable systems. IQCs may be used to analyze the stability of feedback interconnections of open-loop unstable systems via homotopies that are continuous in the gap from nominal feedback systems that are stable [34], [35], [36], [37]. Theorem III.1 circumvents the assumption of the existence of the latter. Alternatively, parts of the proof of Theorem III.1 may be established using the results in [38] and [39], in which modified Hilbert signal spaces on indented imaginary axis are used. Importantly, the more direct proof provided above is purely in the frequency domain and involves no operator-theoretic apparatus.

IV. Robust Stability Analysis via Negative Imaginariness Indices

A. Negative Imaginariness Indices

In this section, we define NI I/O-indices and show how they can be utilized in guaranteeing robust stability of a feedback interconnection of open-loop systems in which excess negative imaginariness in one is used to offset deficit in another.

Intuitively, a negative imaginary single-input-single-output system $G(s)$ is one whose transfer function $G(j\omega)$ has negative imaginary part for all positive frequencies ω, i.e., the Nyquist plot for $\omega > 0$ lies in the lower half complex plane. We define below the notions of NI I/O-indices and relate them to existing notions of negative imaginary systems.

Definition IV.1: A $G \in \mathbf{M}^{m\times m}$ is said to have negative imaginariness input index (NI I-index) $\nu(\omega)$, if the following holds:

(i) for all $\omega > 0$ such that $j\omega$ is not a pole of G,

$$j\left(G(j\omega) - G(j\omega)^*\right) \geq \nu(\omega) I;$$

(ii) for any $\omega_0 > 0$, if $j\omega_0$ is a pole of G, then its residual

$$\lim_{s \to j\omega_0} (s - j\omega_0) jG(s) \geq 0.$$

Definition IV.2: A $G \in \mathbf{M}^{m\times m}$ is said to have negative imaginariness output index (NI O-index) $\rho(\omega)$, if the following holds:

(i) for all $\omega > 0$ such that $j\omega$ is not a pole of G,

$$j\left(G(j\omega) - G(j\omega)^*\right) \geq \rho(\omega) G(j\omega)^* G(j\omega) \text{ and}$$

(ii) for any $\omega_0 > 0$, if $j\omega_0$ is a pole of G, then its residual

$$\lim_{s \to j\omega_0} (s - j\omega_0) jG(s) \geq 0.$$

Note that since $G \in \mathbf{M}^{m\times m}$ is real-rational, its NI I/O-indices can always be chosen to be piecewise continuous without loss of generality and verified by checking equivalent linear matrix inequalities (LMIs) obtained from applying the generalized Kalman–Yakubovich–Popov (KYP) lemma [40] as in [41]. Moreover, $\nu(\omega)$ and $\rho(\omega)$ are always well-defined for every $j\omega$ that is not a pole of the system under study, since they can be taken to be a negative number with a large magnitude. When $\nu(\omega) > 0$ (respectively, $\rho(\omega) > 0$), it quantifies the level of excess negative imaginariness in the input (respectively, output) sense at frequency ω. On the other hand, when $\nu(\omega) < 0$ or $\rho(\omega) < 0$, it quantifies the level of deficiency in negative imaginariness. A few notes on connections with existing notions of negative imaginary systems follow. In particular, if $G \in \mathbf{M}^{m\times m}$ has an NI I-index of $\nu(\omega) = 0$ or NI O-index of $\rho(\omega) = 0$ for all $\omega > 0$, then G is known to be a negative imaginary system [6], [8]. Moreover, if G has no poles on the imaginary axis and an NI I-index $\nu(\omega) > 0$ for all $\omega > 0$, then G is known to be a strictly negative imaginary system (SNI) [5], [6], which has been shown to be equivalent to being input strictly negative imaginary (ISNI) in [15]. Strict negative imaginariness has been shown to be necessary and sufficient from the perspective of stabilizing all negative imaginary systems with possible poles on the imaginary axis [42]. On the contrary, if G has no poles on the imaginary axis and an NI O-index $\rho(\omega) > 0$ for all $\omega > 0$, then G belongs to a class of systems that has not been studied in detail in the literature and is distinct from the class of output strictly passive systems defined in [15].

B. Main Result

The main result on robust stability of feedback interconnections of systems having NI I/O-indices is stated below.

Theorem IV.3: For $i \in \{1, 2\}$, let $G_i \in \mathbf{M}^{m\times m}$ have NI I-index $\nu_i(\omega)$ and NI O-index $\rho_i(\omega)$. Suppose that the following hold:

(a) for all $\omega > 0$, if $j\omega$ is a pole of G_i, then it is not a pole of G_j and $\nu_j(\omega) > 0$, where $i \neq j$;

(b) for every $\omega > 0$, if $j\omega$ is neither a pole of G_1 nor G_2, then either

$$\rho_2(\omega) \geq 0 \quad \text{and} \quad \nu_1(\omega) + \rho_2(\omega) > 0$$

or

$$\rho_1(\omega) \geq 0 \quad \text{and} \quad \nu_2(\omega) + \rho_1(\omega) > 0;$$

(c) $\lambda(G_1(\infty)G_2(\infty)) \subset \mathbb{R}$, $\bar{\lambda}(G_1(\infty)G_2(\infty)) < 1$.

Then, $[G_1, G_2]$ is stable if and only if either $\lambda(G_1(0)G_2(0)) \cap \mathbb{R} = \emptyset$ or $\bar{\lambda}(G_1(0)G_2(0)) < 1$.

Proof: Let $\Pi_{12}(\omega) = j$ (hence, $\Pi_{21}(\omega) = \Pi_{12}(\omega)^* = -j$) for all $\omega > 0$. Let $\epsilon > 0$ be such that whenever $j\omega_0$ is a pole of G_i, $\nu_j(\omega) > 0$ for all $|\omega - \omega_0| < \epsilon$, where $i \neq j$. Such an ϵ exists by (a) and continuity of G_j at $j\omega_0$. Let $\Pi_{11}(\omega) = 0$ and $\Pi_{22}(\omega) = 0$ for all ω in

$$\Omega = \{\omega > 0 : |\omega - \omega_0| \leq \epsilon \text{ and } j\omega_0 \text{ is a pole of } G_1 \text{ or } G_2\}$$

from which it follows that Theorem III.1(iv) holds. Furthermore, in view of the residual properties in Definitions IV.1 and IV.2 as well as (a), Theorem III.1(v) and (vi) hold.

Next, using (b), for all $\omega > 0$ such that $\omega \notin \Omega$, let

$$\Pi_{11}(\omega) = \rho_2(\omega) \quad \text{and} \quad \Pi_{22}(\omega) = 0$$

whenever $\rho_2(\omega) \geq 0$ and $\nu_1(\omega) + \rho_2(\omega) > 0$, whereby (4) and (5) hold. For the remaining $\omega > 0$, let

$$\Pi_{11}(\omega) = 0 \quad \text{and} \quad \Pi_{22}(\omega) = -\rho_1(\omega)$$

whereby (2) and (3) hold, since $\rho_1(\omega) \geq 0$ and $\nu_2(\omega) + \rho_1(\omega) > 0$. Putting the above together yields Theorem III.1(ii) and (iii). The proof is thus completed by applying Theorem III.1. ∎

Theorem IV.3 allows for imaginary-axis poles in both the open-loop systems. This is useful even in the setting where G_1 is taken to be a plant and G_2 a controller to be designed by a user. Specifically, in the event where sinusoidal output disturbances are to be rejected or reference signals tracked using the internal model principle [43], the controller G_2 needs to be designed to have purely imaginary poles, and Theorem IV.3 provides conditions under which closed-loop stability may be concluded.

Theorem IV.3 demonstrates that even in the event where the NI I-index $\nu_i(\omega)$ is negative at a certain frequency ω, which corresponds to the case of $G_i(j\omega)$ being not negative imaginary but whose lack of negative imaginariness is lower bounded, feedback stability of $[G_1, G_2]$ may still be guaranteed provided that $G_j(j\omega)$ has excess NI O-index $\rho_j(\omega)$ so that the total negative imaginariness $\nu_i(\omega) + \rho_j(\omega)$ is positive. In other words, a surplus of negative imaginariness in one system may be used to offset a deficiency in another system at the same frequency from the perspective of establishing closed-loop stability. In the event where the trading of negative imaginariness is not feasible for showing closed-loop stability, Theorem III.1 may still be employed using other types of multipliers than those used in the proof of Theorem IV.3. In the next section, it will be demonstrated that Theorem IV.3 is a generalization of important negative imaginary type results in the literature.

V. Specializations of Theorem IV.3 to Negative Imaginary Results

The following two corollaries are important specializations of Theorem IV.3 to negative imaginary systems.

The first specialization of Theorem IV.3 considers a positive feedback interconnection of an NI system and an SNI system. Recall that an SNI system is equivalent to an ISNI system [15, Lemma 8]. This result is not previously known. It imposes very mild assumptions on the instantaneous gains of the two systems different from those assumed in the literature.

Corollary V.1: Let $G_1 \in \mathbf{M}^{m\times m}$ be an NI system and $G_2 \in \mathbf{RH}_\infty^{m\times m}$ be an SNI system. Suppose that $\lambda(G_1(\infty)G_2(\infty)) \subset \mathbb{R}$ and $\bar{\lambda}(G_1(\infty)G_2(\infty)) < 1$. Then, $[G_1, G_2]$ is stable if and only if either $\lambda(G_1(0)G_2(0)) \cap \mathbb{R} = \emptyset$ or $\bar{\lambda}(G_1(0)G_2(0)) < 1$.

Proof: Apply Theorem IV.3 with $\nu_2(\omega) > 0$ and $\rho_1(\omega) = 0$ for all $\omega > 0$. ∎

Since G_i is NI for $i \in \{1, 2\}$, it satisfies $G_i(0) = G_i(0)^T$, $G_i(\infty) = G_i(\infty)^T$, and $G_i(0) \geq G_i(\infty)$ [8, Lemma 8]. Without additional assumptions, we cannot conclude that the eigenvalues of $G_1(\infty)G_2(\infty)$ and $G_1(0)G_2(0)$ are real.

The next result contains two cases. In the first case, we specialize Corollary V.1 by imposing an additional assumption on either $G_1(\infty)$ or $G_2(\infty)$ to ensure that the eigenvalues of $G_1(\infty)G_2(\infty)$ and $G_1(0)G_2(0)$ are real, thus providing a result that is not previously known as the mild assumptions on the instantaneous gains of the two systems are different from the literature. In the second case, we directly specialize Theorem IV.3 to a positive feedback interconnection of an NI system and a strictly proper output strictly negative imaginary (OSNI) system to give another result that is not previously known. The reader is referred to [15, Def. 7] for a formal definition of an OSNI system and its associated characterizations in [15, Lemmas 6 and 7].

Corollary V.2: Let $G_1 \in \mathbf{M}^{m\times m}$ be an NI system and $G_2 \in \mathbf{RH}_\infty^{m\times m}$. Let either Supposition I or Supposition II holds as follows:

I. Let G_2 be an SNI system, either $G_1(\infty) \geq 0$ or $G_2(\infty) \geq 0$, and $\bar{\lambda}(G_1(\infty)G_2(\infty)) < 1$.
II. Let G_2 be a strictly proper OSNI system and $\det(G_2(j\omega_0)) \neq 0$ for all $\omega_0 > 0$ such that $j\omega_0$ is a pole of G_1.

Then, $[G_1, G_2]$ is stable if and only if $\bar{\lambda}(G_1(0)G_2(0)) < 1$.

Proof: (Supposition I)—Since both G_1 and G_2 are NI, it follows from [8, Lemma 8] that $G_i(0) = G_i(0)^T \geq G_i(\infty) = G_i(\infty)^T$ for $i \in \{1, 2\}$. Now, since either $G_1(\infty) \geq 0$ or $G_2(\infty) \geq 0$, it follows that $\lambda(G_1(\infty)G_2(\infty)) \subset \mathbb{R}$ and $\lambda(G_1(0)G_2(0)) \subset \mathbb{R}$. The result then follows via direct application of Corollary V.1.

(Supposition II): Since G_2 is strictly proper, condition (c) in Theorem IV.3 is trivially fulfilled. Furthermore, $\lambda(G_1(0)G_2(0)) \subset \mathbb{R}$ since $G_2(0) \geq G_2(\infty) = 0$ and $G_1(0) = G_1(0)^T$ from [8, Lemma 8]. In addition, since G_2 is strictly proper OSNI, $j[G_2(j\omega_0) - G_2(j\omega_0)^*] > 0$ if and only if $\det(G_2(j\omega_0)) \neq 0$. Let $\nu_1(\omega) = 0$ and $\rho_2(\omega) = \delta\omega$ with $\delta > 0$ for all $\omega > 0$. The result follows by applying Theorem IV.3. ∎

Corollary V.2 (Supposition I) captures several known results in the literature on closed-loop stability of NI systems. In particular, [5, Thm. 5] and [6, Thm. 1], wherein $G_1(\infty)G_2(\infty) = 0$ and $G_2(\infty) \geq 0$, can be obtained from Corollary V.2 (Supposition I). Likewise, [8, Corollary 13], wherein $G_1(\infty) = 0$, can also be obtained from Corollary V.2 (Supposition I). Corollary V.2 (Supposition II) captures [15, Corollary 9] when the OSNI system is strictly proper as the result does not prohibit common zeros between $[G_1(j\omega) - G_1(j\omega)^*]$ and $[G_2(j\omega) - G_2(j\omega)^*]$ for $\omega \in (0, \infty)$. However, Corollary V.2 (Supposition II) is silent in the case of biproper OSNI systems unlike [15, Corollary 9]. Nonetheless, the more general NI stability results, namely, [8, Thms. 9 and 14], cannot be derived from either Corollary V.1 or Corollary V.2.

VI. Noncolocated Sensor/Actuator Flexible Structure Example

This section provides an example on a flexible structure with noncolocated force actuator and position sensor to illustrate the usefulness of the negative imaginariness indices-based robust closed-loop stability result derived in Section IV.

Consider a single-input-single-output flexible structure with force actuator and position sensor [19] described by

$$P(s) = \sum_{i=1}^{m} \frac{k_i}{s^2 + 2\zeta_i\omega_i s + \omega_i^2}$$

where $0 \leq \zeta_i < 1$, and $0 < \omega_i < \omega_{i+1}$ for all $i \in \{1, \dots, m-1\}$. P is said to be in-phase, if the coefficients of the flexible modes k_i are all positive, i.e., $k_i > 0$ for all $i \in \{1, \dots, m\}$. This happens when the force actuator is colocated with the position sensor, and in this case, P is a negative imaginary system that is amenable to the negative imaginary theory [8]. However, in situations where the control authority is limited, especially when the force actuator is not colocated with the position sensor, then finite-frequency positive realness or negative imaginariness turns out to be a crucial property for good control performance; see [44, Sect. II] for a myriad of justifications for this property from practical perspectives. Noncolocated force actuator and position sensor typically render some of the k_is to be negative, for which the corresponding mode at ω_i is said to be reversed-phase. Under this circumstance, P may not manifest negative imaginariness on the entire frequency band, which causes the negative imaginary theory [8] to be inapplicable. Nevertheless, feedback control design for such a P may still be performed using the main NI I/O-indices-based result in this article, which we demonstrate with a simple example below. It is noteworthy that useful $\mathbf{H}_\infty$ loop-shaping-based design procedures for tackling similar issues may be found in [20], [45], and [46].

Consider a simple but illustrative model for a flexible structure with noncolocated force actuator and position sensor

$$G_1(s) = \frac{1}{s^2+1} + \frac{0.8}{s^2+0.2s+2^2} + \frac{-0.25}{s^2+0.1s+4^2}$$

where there are imaginary-axis poles at $\pm j1$, and the numerator of third term is negative, meaning that the second flexible mode $\omega_2 = 4$ is reversed-phase. Observe that

$$j(G_1(j\omega) - G_1(j\omega)^*) = \frac{0.32\omega}{\omega^4 - 7.96\omega^2 + 16} - \frac{0.05\omega}{\omega^4 - 31.99\omega^2 + 256}$$

from which it may be calculated that

$$j(G_1(j\omega) - G_1(j\omega)^*) > 0 \quad \forall \omega \in (0,1) \cup (1, 3.55) \cup (4.89, \infty)$$

and

$$j(G_1(j\omega) - G_1(j\omega)^*) > -1.25 \quad \forall \omega \in [3.55, 4.89].$$

The inequalities above may also be verified via LMIs obtained from the generalized KYP lemma in [40].

By appealing to Theorem IV.3, it holds that $[G_1, G_2]$ is stable for any $G_2 \in \mathbf{M}$ such that the following holds:

(i) $G_1(0)G_2(0) < 1$;

(ii) G_2 has no poles at $\pm 1j$ and

$$j(G_2(j1) - G_2(j1)^*) > 0;$$

(iii) $j(G_2(j\omega) - G_2(j\omega)^*) \geq 0$ for all $\omega \in (0,1) \cup (1, 3.55) \cup (4.89, \infty)$ such that $j\omega$ is not a pole of G_2;

(iv) $j(G_2(j\omega) - G_2(j\omega)^*) \geq 1.25 G_2(j\omega)^* G_2(j\omega)$ for all $\omega \in [3.55, 4.89]$ such that $j\omega$ is not a pole of G_2.

Consider a controller G_2 of the form

$$G_2 = C_0 + C_p$$

which is a parallel connection of two controllers, C_0 and C_p. C_0 is a basic stabilizing controller for the term in G_1 corresponding to the lowest resonance frequency, i.e.,

$$G_0(s) = \frac{1}{s^2+1}$$

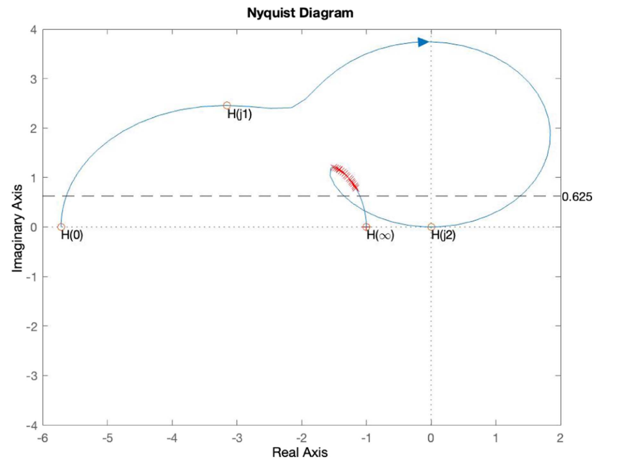

Fig. 2. Nyquist plot of $H(j\omega) = 1/G_2(j\omega)$ for $\omega \geq 0$, where the part for $\omega \in [3.55, 4.89]$ is marked with crosses.

to be designed to achieve low-frequency sensitivity reduction for G_1. Here, we have a simple first-order stable controller expressed as

$$C_0(s) = \frac{bs + c}{s + a}, \; a > 0.$$

C_p is a plug-in controller intended for rejecting a sinusoidal output disturbance at a certain prescribed frequency, say 2 rad/s for this example, and the simplest one is represented by

$$C_p(s) = \frac{k}{s^2 + 2^2}, \quad k > 0.$$

We are interested in the low-frequency sensitivity property of the feedback loop system governed by the sensitivity function

$$S(s) = \frac{1}{1 - G_1(s)G_2(s)}$$

where one of the fundamental necessary requirements is $|S(0)| < 1$ or $|1 - G_1(0)G_2(0)| > 1$. Note condition (i) above, which states that $G_1(0)G_2(0) < 1$. This then requires $G_1(0)G_2(0) < 0$, and in turn implies that $G_2(0) < 0$, whereby $k < -4c/a$. Since k and a are both positive, we have $c < 0$. As a guideline for determining the sign of b, we apply the Routh–Hurwitz criterion to a reduced-order feedback system $[G_0, C_0]$. The resulting stability criterion is $b < 1$ and $a - c < a(1-b) \iff c > ab$, from which it follows that $b < 0$. We can find an example of a G_2 that satisfies all of the requirements (i)–(iv) above as

$$G_2(s) = -\frac{s+1}{s+5} + \frac{0.1}{s^2+2^2}.$$

This G_2 is an NI system with imaginary-axis poles at $\pm j2$ that satisfies (i)–(iv) and gives rise to $S(0) = 0.8283$, thereby desensitizing the closed-loop system to low-frequency modeling uncertainty. See the inverse Nyquist plot of $G_2(j\omega) = 1/H(j\omega)$ in Fig. 2, which illustrates that all the sufficient requirements to guarantee the closed-loop stability are satisfied. Indeed, we can observe that $H(0) < G_1(0) \approx 1.184$, $j(H(j1) - H(j1)^*) < 0$, $j(H(j\omega) - H(j\omega)^*) \leq 0$ for all $\omega \in (0,1) \cup (1, 3.55) \cup (4.89, \infty)$, and $j(H(j\omega) - H(j\omega)^*) \leq -1.25$ (or, equivalently, $\mathrm{Im}(H(j\omega)) \geq 0.625$) for all $\omega \in [3.55, 4.89]$, i.e., conditions (i)–(iv) are satisfied. The impulse response of the closed-loop system is illustrated in Fig. 3, which clearly shows that the system is stable. Fig. 4 shows the closed-loop

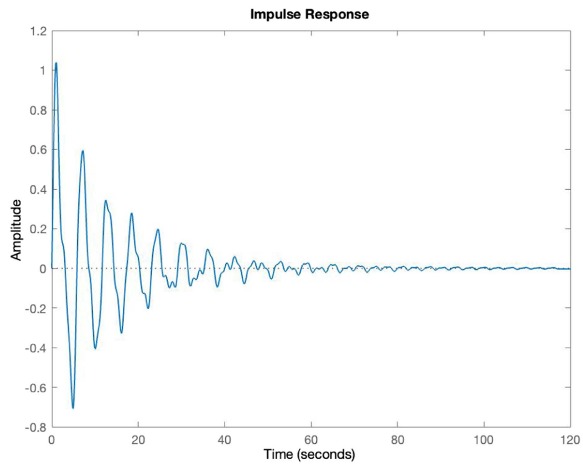

Fig. 3. Impulse response of $\frac{G_1}{1-G_1G_2}$.

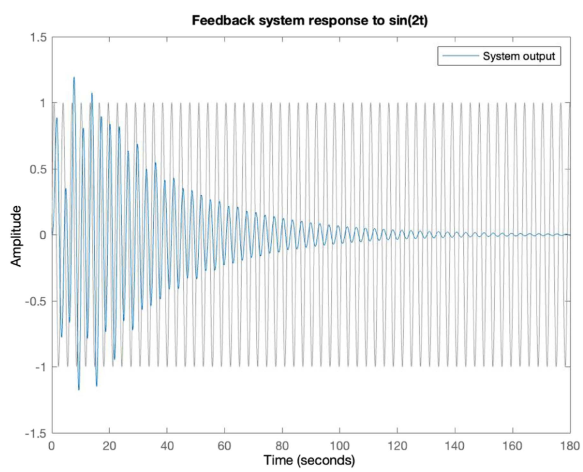

Fig. 4. Sinusoidal response of $\frac{G_1}{1-G_1G_2}$.

system's response to the disturbance signal $\sin(2t)$, which is evidently rejected as desired.

More generally, one may consider

$$G_2(s) = \frac{\beta_1 s^{n-1} + \beta_2 s^{n-2} + \cdots + \beta_{n-1}s + \beta_n}{s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_{n-1}s + \alpha_n}$$

with $\alpha_i \geq 0$ chosen so that $G_2 \in \mathbf{M}$ and has no poles at $\pm j1$. The set of β_i's for which G_2 satisfies all of frequency-dependent quadratic conditions (ii)–(iv) above may then be equivalently stated in terms of LMIs via the generalized KYP lemma in [40]. In particular, since G_2 depends on β_i's affinely, conditions (iii) and (iv) may be transformed into equivalent LMIs as per [47, Prop. 1].

The example above demonstrates that even though the absence of colocated force actuator and position sensor causes a flexible structure to lose negative imaginariness in a certain frequency band, the lack of negative imaginariness in that band may be compensated for by excess negative imaginariness in the other system in the feedback loop, while still resulting in a stable closed-loop system overall. The NI I/O-indices defined in Definitions IV.1 and IV.2 are particularly suited for capturing deficiency and surplus in negative imaginariness, and their utility in robust feedback stability analysis or design is well demonstrated by Theorem IV.3, in a similar spirit to the well-studied passivity indices [21], [22], [48].

VII. Conclusion

We introduced the notions of NI I/O-indices and derived necessary and sufficient conditions for robust stability of a feedback interconnection of two open-loop systems in which the lack of negative imaginariness on certain frequency bands in one system may be compensated for by excess negative imaginariness in another. In particular, we showed that provided that the *total* negative imaginariness in the closed-loop system, as measured by the NI I/O-indices, is positive across all nonzero frequencies, then feedback stability is equivalent to a static loop gain condition, under an assumption on the instantaneous loop gain. This generalizes well-known results on negative imaginary systems that manifest negative imaginariness at all frequencies. We demonstrated the utility of the robust stability result with an example on the control of a flexible structure whose force actuator and position sensor are not colocated and hence does not exhibit negative imaginariness on the entire frequency axis. We established the NI I/O-indices-based robust stability result via a more general and powerful result in which the open-loop systems are characterized by frequency-dependent quadratic constraints.

Interesting future research directions include characterizing NI I/O-indices in a computationally tractable form via LMIs by making use of the generalized KYP lemma [40] and stabilizing feedback control synthesis along the lines of [49]. Generalizing the notions of NI I/O-indices to the nonlinear setting in a similar spirit to [50] is also a direction worth pursuing.

Appendix
Proof of Theorem III.1

For $i = 1, 2$, denote by $j\Omega_i$ the set of imaginary-axis poles of G_i and define $j\Omega = j\Omega_1 \cup j\Omega_2$. Consider a Nyquist contour $\mathcal{N}$ on the imaginary axis indented into $\mathbb{C}_+$ around every object of $j\Omega$ in the form of sufficiently small semicircles.

Step 1: We proceed to establish that the eigenloci of G_1G_2 along the Nyquist contour excluding the origin does not intersect with the real interval $[1, \infty)$ on the complex plane, i.e., $\lambda(G_1(s)G_2(s)) \cup [1, \infty) = \emptyset$ as s traverses along $\mathcal{N} \setminus \{0\}$.

First note that since $\Pi_{11}(\omega) \geq 0$ and $\Pi_{22}(\omega) \leq 0$ for all $\omega > 0$ by III.1

$$\begin{bmatrix} I \\ G_1(j\omega) \end{bmatrix}^* \Pi(\omega) \begin{bmatrix} I \\ G_1(j\omega) \end{bmatrix} \geq 0$$

is equivalent to

$$\begin{bmatrix} I \\ \tau G_1(j\omega) \end{bmatrix}^* \Pi(\omega) \begin{bmatrix} I \\ \tau G_1(j\omega) \end{bmatrix} \geq 0$$

for all $\tau \in [0, 1]$, and

$$\begin{bmatrix} G_2(j\omega) \\ I \end{bmatrix}^* \Pi(\omega) \begin{bmatrix} G_2(j\omega) \\ I \end{bmatrix} \leq 0$$

is equivalent to

$$\begin{bmatrix} \tau G_2(j\omega) \\ I \end{bmatrix}^* \Pi(\omega) \begin{bmatrix} \tau G_2(j\omega) \\ I \end{bmatrix} \leq 0$$

for all $\tau \in [0, 1]$. Consequently, by Lemma II.1, (iii) implies that

$$\det(I - \tau G_1(j\omega)G_2(j\omega)) \neq 0$$

for all $\tau \in [0,1]$ and $\omega > 0$ such that $j\omega$ is not a pole of G_1 or G_2. This in turn implies that

$$\lambda(G_1(j\omega)G_2(j\omega)) \cap [1,\infty) = \emptyset$$

for all $j\omega$ that is not a pole of G_1 or G_2.

Next, for every $j\omega_0$ that is a pole of G_1 and the corresponding K_1 defined in (8), note that $G_1(s) \approx \frac{1}{s-j\omega_0}K_1$ when s is sufficiently close to $j\omega_0$. Using (iv) and the fact that (2) holds for all $|\omega-\omega_0| \le \epsilon, \omega \ne \omega_0$ in (v), it follows that:

$$j\Pi_{12}(\omega_0)K_1 - jK_1^*\Pi_{12}(\omega_0)^* \ge 0$$

$$\text{and} \quad -j\Pi_{12}(\omega_0)K_1 + jK_1^*\Pi_{12}(\omega_0)^* \ge 0.$$

Together with (7), they imply that $\Pi_{12}(\omega_0)K_1 \ge 0$. Thus

$$\begin{aligned}&\Pi_{12}(\omega_0)\tau G_1(s) + \tau G_1(s)^*\Pi_{12}(\omega_0)^* \\ &\approx \Pi_{12}(\omega_0)\frac{\tau}{s-j\omega_0}K_1 + \left(\frac{\tau}{s-j\omega_0}K_1\right)^*\Pi_{12}(\omega_0)^* \ge 0\end{aligned}$$

for all $\tau \in [0,1]$, $s = j\omega_0 + \rho e^{j\theta}$, and $\theta \in [-\frac{\Pi}{2}, \frac{\Pi}{2}]$ when $\rho > 0$ is sufficiently small. On the other hand, since (3) holds for all $|\omega-\omega_0| \le \epsilon$ as in (v), it follows by continuity that

$$\Pi_{12}(\omega_0)^*G_2(s) + G_2(s)^*\Pi_{12}(\omega_0) < 0$$

for all $s = j\omega_0 + \rho e^{j\theta}, \theta \in [-\frac{\Pi}{2}, \frac{\Pi}{2}]$ when $\rho > 0$ is sufficiently small. Application of Lemma II.1 then yields that

$$\det(I - \tau G_1(s)G_2(s)) \ne 0$$

and hence

$$\lambda(G_1(s)G_2(s)) \cap [1,\infty) = \emptyset$$

for all $\tau \in [0,1]$ and $s = j\omega_0 + \rho e^{j\theta}$, $\theta \in [-\frac{\Pi}{2}, \frac{\Pi}{2}]$ when $\rho > 0$ is sufficiently small. For every $j\omega_0$ that is a pole of G_2, the arguments above may be repeated by using (vi) in lieu of (v) to reach the same conclusion. We have thus shown that the eigenloci of G_1G_2 along the Nyquist contour excluding the origin does not intersect with the real interval $[1,\infty)$ on the complex plane, i.e., $\lambda(G_1(s)G_2(s)) \cup [1,\infty) = \emptyset$ as s traverses along $\mathcal{N} \setminus \{0\}$.

Step 2: Here, it is shown that $(I-PC)^{-1}$ has no poles in $\bar{\mathbb{C}}_+ \setminus j\Omega$ if and only if either $\lambda(G_1(0)G_2(0)) \cap \mathbb{R} = \emptyset$ or $\bar{\lambda}(G_1(0)G_2(0)) < 1$. Given that G_1G_2 has no poles in $\mathbb{C}_+$, by the multivariable Nyquist stability criterion [28], $(I-G_1G_2)^{-1}$ has no poles in $\bar{\mathbb{C}}_+ \setminus j\Omega$ if and only if the circuits constructed from juxtaposing the continuous oriented eigenloci of $G_1(s)G_2(s)$ along $\mathcal{N}$ do not pass through or encircle the $+1$ point. Since $\lambda(G_1(\infty)G_2(\infty)) \subset \mathbb{R}$, $\bar{\lambda}(G_1(\infty)G_2(\infty)) < 1$ by (i) (i.e., all the eigenvalues of $G_1(\infty)G_2(\infty)$ are real and less than unity), and $\lambda(G_1(s)G_2(s)) \cap [1,\infty) = \emptyset$ as s traverses along $\mathcal{N} \setminus \{0\}$, as established in Step 1, there exists at least one circuit that passes through or encircles the $+1$ point if and only if $\lambda(G_1(0)G_2(0)) \cap \mathbb{R} \ne \emptyset$ and $\bar{\lambda}(G_1(0)G_2(0)) \ge 1$. That is, $(I-PC)^{-1}$ has no poles in $\bar{\mathbb{C}}_+ \setminus j\Omega$ if and only if either $\lambda(G_1(0)G_2(0)) \cap \mathbb{R} = \emptyset$ or $\bar{\lambda}$ $(G_1(0)G_2(0)) < 1$.

Step 3: We show next that $(I-G_1G_2)^{-1}$ has no poles in $j\Omega$ using (iv) and either (v) or (vi). Suppose $j\omega_0$ is a pole of G_1. From (iv) and (v), we have that

$$\Pi_{12}(\omega)G_1(j\omega) + G_1(j\omega)^*\Pi_{12}(\omega)^* \ge 0 \tag{11}$$

for all $|\omega-\omega_0| \le \epsilon, \omega \ne \omega_0$ and

$$G_2(j\omega)^*\Pi_{12}(\omega) + \Pi_{12}(\omega)^*G_2(j\omega) < 0 \tag{12}$$

for all $|\omega-\omega_0| \le \epsilon$. Suppose to the contrapositive that $(I-G_1G_2)^{-1}$ has a pole at $j\omega_0$, which means that there exists continuous $u(\omega) \in \mathbb{C}^q$ such that $u(\omega_0) \ne 0$ and

$$\lim_{\omega\to\omega_0}(I - G_1(j\omega)G_2(j\omega))u(\omega) = 0.$$

Premultiply and postmultiply (11) by $u(\omega)^*G_2(j\omega)^*$ and $G_2(j\omega)u(\omega)$, respectively, yield

$$\begin{aligned}&u(\omega)^*G_2(j\omega)^*\Pi_{12}(\omega)G_1(j\omega)G_2(j\omega)u(\omega) \\ &+ u(\omega)^*G_2(j\omega)^*G_1(j\omega)^*\Pi_{12}(\omega)^*G_2(j\omega)u(\omega) \ge 0.\end{aligned}$$

Taking $\omega \to \omega_0$ then results in

$$\begin{aligned}&u(\omega_0)^*G_2(j\omega_0)^*\Pi_{12}(\omega_0)u(\omega_0) \\ &+ u(\omega_0)^*\Pi_{12}(\omega_0)^*G_2(j\omega_0)u(\omega_0) \ge 0\end{aligned}$$

which violates (12). Hence, $(I-G_1G_2)^{-1}$ does not have a pole at $j\omega_0$.

On the contrary, if $j\omega_0$ is a pole of G_2, then the arguments above may be repeated using (vi) in place of (v) to reach the same conclusion that $(I-G_1G_2)^{-1}$ does not have a pole at $j\omega_0$.

Step 4: We establish here that $[G_1, G_2]$ is stable if and only if either $\lambda(G_1(0)G_2(0)) \cap \mathbb{R} = \emptyset$ or $\bar{\lambda}(G_1(0)G_2(0)) < 1$. Since there are no pole-zero cancellations in $\bar{\mathbb{C}}_+$ between G_1 and G_2, it holds that $[G_1, G_2]$ is stable if and only if $(I-G_1G_2)^{-1}$ is stable. The claim then follows from Steps 2 and 3 above.

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All research data supporting this publication are directly available within this publication.

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