# Input-Restricted Stability of Continuous and Discrete Time Nonlinear Feedback Systems

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***Abstract*—In this article, we develop the concept of input-restricted stability, which determines whether a feedback interconnection remains stable only for inputs in a given subset of all possible inputs in a specified signal space. Graph separation concepts and continuity are employed to derive an input-restricted feedback stability theorem, which guarantees input-restricted stability of a feedback interconnection if both systems in the interconnection fulfil some given criteria related to their input–output relationships. Significantly, this result is applicable to both continuous and discrete time systems, unlike many existing local stability results. This theorem is then specialized into simpler-to-compute corollaries and expanded to additional theorems which provide useful additional insights. This article ends with two salient specializations of key results developed herein: one is a type of input-restricted small-gain stability theorem with one system bounded by a linear gain and the other by a quadratic gain; and the other is a type of input-restricted passivity theorem. For both of these specializations, which are not stable for all energy bounded inputs, an example is provided where the feedback interconnection is shown to be stable when the energy of exogenous inputs is below a given threshold.**

***Index Terms*—Graph separation, LMIs, nonlinear systems, stability of nonlinear systems.**

## I. Introduction

THE ability to determine the closed-loop stability of two open-loop systems before closing the loop is a fundamental area of research in control theory. Input–output feedback stability theory has a rich and detailed history [1], and includes classic stability tools still taught to undergraduate students today, such as the circle and Popov criteria, from absolute stability [2], and small-gain [3] and passivity [4] theory.

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Zames [3], Desoer and Vidyasagar [5], Safonov [6], and Teel [7], to name just a few, progressively proved that many disparate input–output stability techniques can be viewed as methods of proving the graphs of both systems in the loop are both separate and diverge as the input increases in size. This general, overarching theory is known as graph separation theory, and even more modern methods, such as the use of integral quadratic constraints (IQCs) [8], [9], can be shown to be equivalent to a separation of graphs under certain conditions [10]. Most recently, more connections between the IQC theory and dissipativity are revealed in [11] and [12], showing both the power of this method and the continued research into this vital field.

These graph separation methods prove closed-loop stability for inputs in a given signal space, which is most often a Lebesgue space. Indeed, general input–output feedback stability theory is often employed to determine whether a system will remain stable for all possible exogenous inputs. However, in engineering applications, it is unlikely that a system must be able to withstand every possible input in a given signal space. Instead, some partial information about the incoming signals is often known a priori, such as the maximum energy or maximum absolute value the input can have, and so there is inherent conservatism in requiring that stability is guaranteed for the whole space of inputs. Therefore, this article asks the question: can stability be proven for only a subset of a given input signal space, even if input–output feedback stability methods that consider the full signal space fail? Not only would this guarantee safety of real-world systems, which otherwise would be deemed unsafe, but it could also allow the implementation of less conservative controllers, which previously would be ruled-out as they are unable to guarantee stability of the closed-loop system.

Lyapunov theory allows a similar approach for state-space systems, proving that if the initial states remain within a certain region of the state space, the system will always return to a specific local equilibrium [13]. This approach can be extended to systems with inputs via dissipativity theory, where Lypaunov functions are replaced with storage functions. Local passivity and dissipativity related results can be found in [14], [15], and [16], for example.

A few attempts have been made previously to either define or prove some type of local or input-bounded stability: the authors in [17] and [18] provided definitions of local input–output stability, which they relate to Lyapunov stability, but both papers only focus on open-loop systems; and the authors in [19] and [20] give local closed-loop input–output stability results but both are only linear, small-gain conditions for systems with additive inputs and both place additional requirements on the spaces of the signals they are dealing with. In addition, only results in [21] are related to system graphs, but there local stability has already been assumed. More recently however, the authors in [15] and [22] presented local stability results based on IQCs, applicable in the continuous-time domain. Taking a different approach, Iwasaki et al. [23] provided stability results for when the system inputs are restricted in the frequency domain instead of the time domain. Notably, all of these existing results are applicable in only the continuous time domain.

Previously Hilborne and Lanzon [24] introduced a new definition of local input–output stability, called $M$-local boundedness, and proved $M$-local boundedness for a feedback interconnection of two continuous-time systems via the local separation of their graphs. The work herein both generalizes $M$-local boundedness to the concept of input-restricted stability, and it generalizes the stability result to discrete time systems, with some small improvements of the result. This article also provides useful extensions of the main stability theorem and illuminating applications which extend small-gain and passivity results to input-restricted stability results.

The rest of this article is organized as follows. Section II provides the preliminary notation and definitions used throughout. Section III gives a technical lemma before introducing the concept of input-restricted stability, then states and proves the first input-restricted stability result for a feedback interconnection, Theorem 1, before providing a number of specializing corollaries to this theorem. Section IV introduces the concept of a static linear mapping before using this to extend the first input-restricted stability result, allowing the input to be restricted to a subset other than a simple ball. This section also extends the input-restricted stability result to systems with additive disturbances. Section V develops two specialisations of the corollaries of the previous section: the first proves input-restricted stability of a feedback interconnection where one system has a linear bounded gain and one has a quadratically bounded gain; and the second proves input-restricted stability of the negative feedback interconnection of a passive linear time-invariant (LTI) system and a nonlinearity, which is very strictly passive for all inputs with a restricted maximum value. Numerical examples are given for each of these specializations. Finally, Section VI concludes this article.

## II. Preliminaries

### A. Notations

The spaces of reals and vector-valued reals are denoted by $\mathbb{R}$ and $\mathbb{R}^n$, respectively; the space of natural numbers is denoted by $\mathbb{N}$; the space of nonnegative real numbers is denoted by $\bar{\mathbb{R}}_+$; the space of natural numbers excluding 0 is denoted by $\mathbb{N}_+$; and the union of space of natural numbers excluding 0 and infinity is denoted by $\mathbb{N}_\infty = \mathbb{N}_+ \cup \{\infty\}$. The space of matrix valued reals is denoted by $\mathbb{R}^{n\times m}$.

A vector space $\mathcal{X}$ is a normed vector space if there exists a norm, denoted by $\|x\|$ for all $x \in \mathcal{X}$, which maps $x \in \mathcal{X}$ to $\|x\| \in \bar{\mathbb{R}}_+$. Some spaces or subsets of spaces are equipped with multiple norms, often defined by an integer. In this case, we explicitly denote this norm using a subscript, i.e., $\|x\|_p$ for some $p \in \mathbb{N}_\infty$. We also denote the norm $\|x\|_{p\vee q} = \max\{\|x\|_p, \|x\|_q\}$.

The kernel of a mapping $P : \mathcal{X} \to \mathcal{Y}$ between normed vector spaces is denoted by $\ker(P)$.

We equip the space of vector valued reals with the Euclidean norm and denote this as $|x|$ for all $x \in \mathbb{R}^n$ to easily differentiate this from other vector norms defined herein. Denoting $x^T$ as the transpose of any such $x$, we then define the Euclidean norm as $|x| = (x^T x)^{\frac{1}{2}}$.

The norm of any $M \in \mathbb{R}^{n\times m}$ is $\|M\| = \max_{x\in\mathbb{R}^m, x\neq 0} \|Mx\|/\|x\|$. This is equal to the maximum singular value of $M$.

We denote the closure of a space (or subset/subspace of a space) $\mathcal{X}$ as $\bar{\mathcal{X}}$, and we denote the interior of $\mathcal{X}$ as $\mathrm{Int}(\mathcal{X})$. The boundary of $\mathcal{X}$, denoted by $\partial\mathcal{X}$, is defined as $\bar{\mathcal{X}} \setminus \mathrm{Int}(\mathcal{X})$.

We denote the ball of radius $r$ in a normed vector space $\mathcal{X}$ as $\mathcal{B}^r_{\mathcal{X}} = \{ x \in \mathcal{X} \mid \|x\| < r \}$; we denote the closure of this ball as $\bar{\mathcal{B}}^r_{\mathcal{X}} = \{ x \in \mathcal{X} \mid \|x\| \leq r \}$; and we denote the sphere in $\mathcal{X}$ of radius $r$ as the boundary of the ball of radius $r$, i.e. $\partial\mathcal{B}^r_{\mathcal{X}} = \{ x \in \mathcal{X} \mid \|x\| = r \}$. If $\mathcal{X}$ is equipped with more than one norm, we explicitly denote $(\mathcal{B}^r_{\mathcal{X}})_p = \{ x \in \mathcal{X} \mid \|x\|_p < r \}$ and equivalently for $(\bar{\mathcal{B}}^r_{\mathcal{X}})_p$ and $(\partial\mathcal{B}^r_{\mathcal{X}})_p$.

The space of proper, real rational (possibly matrix-valued) transfer functions with no poles on the imaginary axis is denoted $\mathcal{RL}_\infty$. The subspace of $\mathcal{RL}_\infty$ consisting of functions with no poles in the closed right-half plane is denoted $\mathcal{RH}_\infty$.

Let $(A, B, C, D)$ or $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ denote a state-space realisation of a real, rational, proper, linear, time-invariant $G$ with associated transfer function denoted by $G(s) = D + C(sI - A)^{-1}B$.

Finally, for brevity, we sometimes write $\max\{a, b\}$ as $a \vee b$ for some $a, b \in \mathbb{N}$.

### B. Definitions

Here, we give various definitions used in the sequel, including the canonical system used in this article.

*Definition 1 (Norm for Cartesian product spaces):* Let $y = (y_1, y_2, \ldots, y_n) \in \mathcal{Y}_1 \times \mathcal{Y}_2 \times \cdots \times \mathcal{Y}_n$ where $\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_n$ are normed vector spaces. Then, the norm of $y$, $\|y\|_p$, is given by $\|y\|_p = \|(y_1, y_2, \ldots, y_n)\| = (\|y_1\|^p + \|y_2\|^p + \cdots + \|y_n\|^p)^{\frac{1}{p}}$ for some $p \in \mathbb{N}_+$.

When $p = \infty$, the norm is instead given by

$$\|y\|_\infty = \max_{i\in[0,n]} \|y_i\|. \quad (1)$$

As with norms equipped to a space, the value of $p$ chosen for the norm of $y$ may not be explicitly specified using a subscript, e.g., $\|y\|$. Note that $\|y\|$ and $\|y_i\|$ may have different $p$-norms.

*Definition 2 (Cartesian product space norm notation):* Let $y = (y_1, y_2) \in \mathcal{Y}_1 \times \mathcal{Y}_2$ where $\mathcal{Y}_1, \mathcal{Y}_2$ are normed vector spaces each equipped with multiple norms. In this case, subscripts may be used to define which norms of $y_1$ and $y_2$ are used in the norm of $y$, i.e. $\|y\|_{p_1,p_2} = (\|y_1\|^p_{p_1} + \|y_2\|^p_{p_2})^{\frac{1}{p}}$ is the $p$-norm of $y$, acting on the $p_1$-norm of $y_1$ and the $p_2$-norm of $y_2$.

*Definition 3 (Continuity of a function):* Let $f : \mathcal{W}_1 \times \cdots \times \mathcal{W}_n \to \mathcal{Y}$ be a function mapping multiple inputs to an output, where each input lives in a different normed vector space. Then, $f$ is continuous if, $\forall(w^{(2)}_1, \ldots, w^{(2)}_n) \in \mathcal{W}_1 \times \cdots \times \mathcal{W}_n$ and all $\epsilon > 0, \exists\delta > 0$ such that $\forall(w^{(1)}_1, \ldots, w^{(1)}_n) \in \mathcal{W}_1 \times \cdots \times \mathcal{W}_n$:

$$\|(w^{(1)}_1 - w^{(2)}_1, \ldots, w^{(1)}_n - w^{(2)}_n)\| < \delta$$
$$\implies \|f(w^{(1)}_1, \ldots, w^{(1)}_n) - f(w^{(2)}_1, \ldots, w^{(2)}_n)\| < \epsilon.$$

*Definition 4 (Lebesgue spaces):* The Lebesgue space $\mathcal{L}_p$ of vector valued continuous-time functions is defined for any $p \in \mathbb{N}_\infty$. For $p \in \mathbb{N}_+$, it is given by

$\mathcal{L}_p = \{\, y : [0,\infty) \to \mathbb{R}^n \;\mid\; \int_0^\infty |y(t)|^p \mathrm{d}t < \infty \,\}$. The $\mathcal{L}_p$-norm is defined as $\|f\|_p = (\int_0^\infty |f(t)|^p \mathrm{d}t)^{\frac{1}{p}}$ for all $f \in \mathcal{L}_p,\ p \in \mathbb{N}_+$. The space $\mathcal{L}_\infty$ is defined as $\mathcal{L}_\infty = \{\, y : [0,\infty) \to \mathbb{R}^n \;\mid\; \text{ess sup}_{t\in[0,\infty)}|y(t)| < \infty \,\}$, with the $\mathcal{L}_\infty$-norm defined as $\|f\|_\infty = \text{ess sup}_{t\in[0,\infty)}|f(t)|$ for all $f \in \mathcal{L}_\infty$.

Similarly, for $p \in \mathbb{N}_+$, $\ell_p$ is the Lebesgue space of vector valued discrete time functions defined by $\ell_p = \{\, y : \mathbb{N}_+ \to \mathbb{R}^n \mid \sum_{k=0}^\infty |y(k)|^p < \infty \,\}$. The $\ell_p$-norm is defined as $\|f\|_p = (\sum_{k=0}^\infty |f(k)|^p)^{\frac{1}{p}}$ for all $f \in \ell_p,\ p \in \mathbb{N}_+$. The space $\ell_\infty$ is defined as $\ell_\infty = \{\, y : \mathbb{N}_+ \to \mathbb{R}^n \mid \text{ess sup}_{k\in\mathbb{N}}|y(k)| < \infty \,\}$, with the $\ell_\infty$-norm defined as $\|f\|_\infty = \text{ess sup}_{k\in\mathbb{N}}|f(k)|_\infty$ for all $f \in \ell_\infty$.

If a function is described as "a Lebesgue function", the function could be in either Lebesgue space $\mathcal{L}_p$ or $\ell_p$; the context is such that it makes no difference. If the norm of a function in a Lebesgue space can be determined from context, or the value of $p$ does not affect the result, the subscript $p$ will be omitted.

A superscript may sometimes be explicitly used to denote the dimensions of the Euclidean space which underlies a Lebesgue space, e.g., $y \in \mathcal{L}^m \implies y(t) \in \mathbb{R}^m$.

*Definition 5 (Truncation of a Lebesgue function):* The truncation of a Lebesgue signal $y \in \mathcal{L}_p$ at time $\tau \in [0,\infty)$ is given by $y_\tau = P_\tau y$, where

$$P_\tau y(t) = \begin{cases} y(t) & \text{for } t \le \tau \\ 0 & \text{for } t > \tau \end{cases}$$

and similarly for $y \in \ell_p$, $\tau \in \mathbb{N}_+$ we have

$$P_\tau y(k) = \begin{cases} y(k) & \text{for } k \le \tau \\ 0 & \text{for } k > \tau. \end{cases}$$

If the domain of $y$ is undefined, we say that $\tau \in \mathcal{T}$ where $\mathcal{T}$ is understood to be the correct space for $\tau$ as given here corresponding to the space of $y$.

*Definition 6 (Extended Lebesgue spaces):* The extension of a continuous time Lebesgue space $\mathcal{L}_p$ is given by $\mathcal{L}_{pe} = \{y : [0,\infty) \to \mathbb{R}^n] \mid y_\tau \in \mathcal{L}_p \forall \tau \in \mathcal{T}\}$.

For discrete time, $\ell_{\infty e} = \ell_\infty$ and the extension of $\ell_p$ for $1 \le p < \infty$ is given by $\ell_{pe} = \{\, y : \mathbb{N}_+ \to \mathbb{R}^n \mid y_\tau \in \ell_\infty \forall \tau \in T \,\}$.

In the sequel, any vector space with a subscript $e$, e.g. $\mathcal{Y}_e$, is assumed to be an extended Lebesgue space of the underlying Lebesgue space, i.e., $\mathcal{Y}$, if not otherwise stated.

*Definition 7 (Truncations of a set of Lebesgue functions):* The truncation of $\mathcal{X}_e$, denoted $P_\tau \mathcal{X}_e$, is defined as $P_\tau \mathcal{X}_e = \{x_\tau | x \in \mathcal{X}_e\}$ for any given $\tau \in \mathcal{T}$.

*Definition 8 (Systems as operators):* A system $\Sigma$ is defined as an operator which acts on an input to generate an output, often described by $\Sigma : \mathcal{W}_e \to \mathcal{Y}_e$. An operator associates every input $w \in \mathcal{W}_e$ with a unique output $y = \Sigma(w) \in \mathcal{Y}_e$.

Similarly, for systems with multiple vector-valued inputs, i.e. $\Sigma : \mathcal{W}_{1e} \times \mathcal{W}_{2e} \times \cdots \times \mathcal{W}_{ne} \to \mathcal{Y}_e$ for some $n \in \mathbb{N}_+$, the notation $y = \Sigma(w_1, w_2, \ldots, w_n)$ denotes that $y$ is the output of $\Sigma$ to inputs $w_1, w_2, \ldots, w_n$, which is again unique.

We can now state that the feedback interconnection in Fig. 1, with $\Sigma_1 : \mathcal{W}_{1e} \times \mathcal{Y}_{1e} \to \mathcal{Y}_{2e}$ and $\Sigma_2 : \mathcal{W}_{2e} \times \mathcal{Y}_{2e} \to \mathcal{Y}_{1e}$, is defined by the following set of equations:

$$y_2 = \Sigma_1(w_1, y_1) \tag{2a}$$

$$y_1 = \Sigma_2(w_2, y_2) \tag{2b}$$

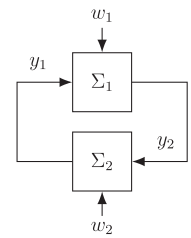


Fig. 1. Canonical feedback system.

with inputs $w = (w_1, w_2) \in \mathcal{W}_{1e} \times \mathcal{W}_{2e} = \mathcal{W}_e$ and outputs $y = (y_1, y_2) \in \mathcal{Y}_{1e} \times \mathcal{Y}_{2e} = \mathcal{Y}_e$.

*Definition 9 (Causality of a system):* A system $\Sigma : \mathcal{W}_e \to \mathcal{Y}_e$ is causal if $P_\tau(\Sigma(w)) = P_\tau(\Sigma(w_\tau))\ \forall \tau \in \mathcal{T}$.

*Definition 10 (Local continuity of a system):* A system $\Sigma : \mathcal{W}_e \to \mathcal{Y}_e$ is a locally continuous mapping if, $\forall w_2 \in \mathcal{W}_e, \epsilon > 0, \tau \in \mathcal{T}, \exists \delta > 0$ such that $\forall w_1 \in \mathcal{W}_e$,

$$\|P_\tau(w_1 - w_2)\| < \delta \Rightarrow \|P_\tau[\Sigma(w_1) - \Sigma(w_2)]\| < \epsilon.$$

*Remark 1:* Note that the definition of local continuity given above is true for all $\tau \in \mathcal{T}$, but not uniformly in $\tau$. This allows for the definition to apply to possibly unstable systems. Since it holds for all $\tau \in \mathcal{T}$, but does not hold when $\tau = \infty$, we denote this specific continuity as local continuity. It is a less strict version of local Lipschitz continuity defined by Willems [25].

*Definition 11 (Graph of a system):* The graph of a system $\Sigma : \mathcal{W}_e \times \mathcal{X}_e \to \mathcal{Y}_e$ for a given $w \in \mathcal{W}_e$ is the set of all possible input-output pairs $(x, y) \in \mathcal{X}_e \times \mathcal{Y}_e$ that the system $\Sigma$ can permit. This is defined as

$$G(w) = \{(x, y) \in \mathcal{X}_e \times \mathcal{Y}_e \mid y = \Sigma(w, x)\}.$$

The inverse graph, denoted $G'(w)$, is defined as

$$G'(w) = \{(y, x) \in \mathcal{Y}_e \times \mathcal{X}_e \mid y = \Sigma(w, x)\}.$$

*Definition 12 (Graph at zero):* The graph of a system $\Sigma : \mathcal{W}_e \times \mathcal{X}_e \to \mathcal{Y}_e$ at zero is the graph of that system with zero input $w$, i.e., $G(0)$.

In the sequel we will denote the graph of $\Sigma_1$ as $G_1(w_1) \subset \mathcal{Y}_{1e} \times \mathcal{Y}_{2e}$ and the graph of $\Sigma_2$ as $G_2(w_2) \subset \mathcal{Y}_{2e} \times \mathcal{Y}_{1e}$. It is then clear from the above definition that $y \in G_1(w_1) \cap G_2'(w_2)$ is a solution to the system in Fig. 1 described by (2) for any input $w = (w_1, w_2)$.

## C. System Properties

Now that we have specified the spaces and systems we are working with, we define some standard properties of those systems. Strong well-posedness of a feedback interconnection is also introduced, which adds local input–output continuity to the usual definition of well-posedness. This property is vital in proving the input-restricted stability result for a feedback interconnection.

*Definition 13 (Class $\kappa$-functions):* A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is in class $\kappa$, denoted by $\gamma \in \kappa$, if $\gamma$ is continuous, nondecreasing and $\gamma(0) = 0$.

*Definition 14 (Global stability):* A system $\Sigma : \mathcal{W}_e \to \mathcal{Y}_e$ is said to be globally stable if there exists $\gamma \in \kappa$ such that $\|y_\tau\| \le \gamma(\|w_\tau\|)\ \forall w \in \mathcal{W}_e, y = \Sigma(w), \tau \in \mathcal{T}$.

*Remark 2:* This definition of stability we denote "global" to differentiate it from our new "input-restricted stability" definition. However, it is the same stability definition as is commonly used in graph separation theory, e.g., [7].

*Definition 15 (Global stability of systems with multiple inputs):* A system $\Sigma : \mathcal{W}_e \times \mathcal{Z}_e \to \mathcal{Y}_e$ is globally stable if there exists a functional $\gamma \in \kappa$ such that, for all $w \in \mathcal{W}_e$ and $z \in \mathcal{Z}_e$ with $y = \Sigma(w, z)$, we have $\|y_\tau\| \leq \gamma(\|(w_\tau, z_\tau)\|)\ \forall \tau \in \mathcal{T}$.

*Definition 16 (Well-posedness and strong well-posedness of a feedback interconnection):* A feedback interconnection described by (2) and illustrated in Fig. 1 is *well-posed* if, for each input $w \in \mathcal{W}_e$, there is a unique $y \in \mathcal{Y}_e$, given by $y = G_1(w_1) \cap G_2'(w_2)$, which is a solution to (2) and the map from $w$ to $y$ is causal.

In addition, such a feedback system is *strongly well-posed* if it is well-posed and the map from $w$ to $y$ is locally continuous as per Definition 3.

*Remark 3:* The definition of strong well-posedness given here is weaker than Willems' well-posedness definition [25], as Willems requires local Lipschitz continuity of the feedback system. Strong well-posedness given here holds as a consequence of Willems' well-posedness, which can be ensured by the open-loop systems being locally Lipschitz continuous and the product of their instantaneous gains being smaller than unity. These two requirements are trivially satisfied for many real-world systems (for example, if one of the two systems is strictly proper LTI and bounded on $j\mathbb{R}$), and as such the requirement that the feedback system is strongly well-posed is not difficult to fulfil. This is demonstrated in the following lemma.

*Lemma 1:* Consider the feedback interconnection in Fig. 1 governed by (2). If $\Sigma_1$ is locally Lipschitz continuous and $\Sigma_2 \in \mathcal{RL}_\infty$ is strictly proper, then the feedback interconnection is strongly well-posed.

*Proof:* If $\Sigma_2$ is strictly proper, then the instantaneous gain $\|\Sigma_2(j\infty)\| = 0$, and as $\Sigma_2 \in \mathcal{RL}_\infty$, it is locally Lipschitz continuous. The product of the instantaneous gains of $\Sigma_1$ and $\Sigma_2$ must be 0 and therefore the feedback interconnection is strongly well-posed [25]. ∎

## III. Input-Restricted Stability

In this section, the novel concept of input-restricted stability of a system is defined and we present a theorem which guarantees input-restricted stability of a feedback interconnection as in Fig. 1, which holds in both continuous and discrete time settings. As with classic input–output feedback stability and graph separation results, this theorem uses open-loop properties of both systems to determine the behavior of the closed-loop system.

### A. Technical Lemma

Before we present our main stability theorem, we introduce a technical result in Lemma 2, which is needed in subsequent proofs. Notably, this Lemma relies on continuity of the system and is therefore applicable in the discrete time domain, unlike results in, e.g., [15] and [22].

First, we introduce the concept of a star shaped subset. This is used in the proof of Lemma 2, giving us the implication $\Sigma(w) \in \mathcal{Q} \implies \Sigma(\lambda w) \in \mathcal{Q}\ \forall \lambda \in [0, 1]$, as $\mathcal{Q}$ is star shaped and allowing a homotopy-like argument to be applied using $\lambda$.

*Definition 17 (Star shaped subset):* Let $\mathcal{S} \subset \mathcal{X}$ be a subset of a normed vector space. Then, $\mathcal{S}$ is star shaped if $\mathcal{S}$ contains the origin and if, $\forall s \in \mathcal{S}$, we have $ks \in \mathcal{S}\ \forall k \in [0, 1]$, i.e., the line segment from the origin to any point $s \in \mathcal{S}$ lies in $\mathcal{S}$.

*Lemma 2:* Let the system $\Sigma : \mathcal{W}_e \to \mathcal{Y}_e$ be a locally continuous mapping. Let $\mathcal{S} \subset \mathcal{W}_e$ and $\mathcal{Q} \subset \mathcal{Y}_e$ be some star shaped subsets of $\mathcal{W}_e$ and $\mathcal{Y}_e$ respectively, and $\mathcal{Q}$ be open. Suppose $\Sigma(0) = 0$ and $P_\tau(\Sigma(w)) \notin \partial\mathcal{Q}\ \forall \tau \in \mathcal{T}, w \in \mathcal{S}$. Then, $w \in \mathcal{S} \implies P_\tau(\Sigma(w)) \in \mathcal{Q}\ \forall \tau \in \mathcal{T}$.

*Proof:* See Appendix. ∎

*Remark 4:* Lemma 2 is not really intended to be a result to be taken on its own; it is instead a tool to be used to prove the main theorems. However, it can be understood as follows: given a system $\Sigma$ such that $\Sigma(0) = 0$, if we can prove that an input of $\Sigma$ within a certain subset of the input space which contains the origin cannot generate an output of $\Sigma$ in a subset of the output space which is continuous and encloses, but does not contain, the origin, and that $\Sigma$ is continuous from the input to the output, then continuity of $\Sigma$ ensures that any input as described above must generate an output of $\Sigma$ which is enclosed by the subset of the output space given above. This is due to input–output continuity of $\Sigma$.

As the requirements of the closed-loop input-bounded stability theorems in this article set up scenarios in which the requirements of Lemma 2 are fulfilled, we present this result separately here to be referred back to.

### B. Input-Restricted Stability

Here, we introduce the novel concept of input-restricted stability. Note that this is a generalization of $M$-local boundedness found in [24] as there the input was restricted to a ball of radius $M$, whereas here the input can be restricted to any arbitrary subset of the input space.

*Definition 18 (Input-restricted stability):* Let there be some $\mathcal{S} \subset \mathcal{W}_e$. A system, or more generally a mapping, $\Sigma : \mathcal{W}_e \to \mathcal{Y}_e$ is said to be "input-restricted stable for all inputs in $\mathcal{S}$" if there exists $\gamma \in \kappa$ such that $\|y_\tau\| \leq \gamma(\|w_\tau\|)\ \forall w \in \mathcal{S}, y = \Sigma(w), \tau \in \mathcal{T}$.

*Definition 19 (Input-restricted stability of a feedback interconnection):* Consider a feedback interconnection described by (2) and illustrated in Fig. 1, and let there be some $\mathcal{S} \subset \mathcal{W}_e$. This feedback interconnection is then said to be "input-restricted stable for all inputs in $\mathcal{S}$" if it is both well-posed and there exists $\gamma \in \kappa$ such that $\|y_\tau\| \leq \gamma(\|w_\tau\|)\ \forall w \in \mathcal{S}, y = \Sigma(w), \tau \in \mathcal{T}$, where $\Sigma : \mathcal{W}_e \to \mathcal{Y}_e$ denotes the closed-loop operator corresponding to (2).

We now present the first major result of this article. The following theorem proves input-restricted stability of a feedback interconnection using properties of the graphs of the two systems. The unique aspect of this theorem is that these graph properties need only hold for a subset of the inputs and outputs, rather than for all possible signals in the input and output spaces. As the proof of this theorem relies on continuity of the feedback system and not continuity in time, it is applicable in the discrete time domain, an improvement on the authors' previous work [24, Thm. 2].

*Theorem 1 (Input-restricted stability):* Consider the feedback interconnection in Fig. 1, governed by (2), and assume the interconnection is strongly well-posed. Suppose there exist some functionals $\alpha,\ \beta : \mathcal{W} \times \mathcal{Y} \to \mathbb{R}, \gamma \in \kappa$ and some $M,\ N > 0$

with $\gamma(M) < N$ such that, $\forall \tau \in \mathcal{T}$ and $\forall w \in \bar{\mathcal{B}}_{\mathcal{W}}^M$, the following holds:

1) $y \in G_1(w_1)$ and $\|y_\tau\| \leq N \Longrightarrow \alpha(w_\tau, y_\tau) \leq 0$;
2) $y \in G_2'(w_2)$ and $\|y_\tau\| \leq N \Longrightarrow \beta(w_\tau, y_\tau) \leq 0$;
3) $\alpha(w_\tau, y_\tau) \leq 0$ and $\beta(w_\tau, y_\tau) \leq 0$ $\Longrightarrow \|y_\tau\| \leq \gamma(\|w_\tau\|)$.

Furthermore, suppose that $G_1(0) \cap G_2'(0) = 0$, i.e., the origin is a solution of the feedback interconnection with zero exogenous signals. Then, the feedback system is input-restricted stable for all inputs in $\bar{\mathcal{B}}_{\mathcal{W}}^M$.

*Proof:* See Appendix. ■

*Remark 5:* The assumption that $G_1(0) \cap G_2'(0) = 0$ is very mild as it is simply stating that the system stays at rest when there is no input, which is very common for real-world systems. This supposition can even be replaced by the milder assumption $G_1(0) \cap G_2'(0) \in \mathcal{B}_{\mathcal{Y}}^N$ and the results will still hold.

The proof of Theorem 1 uses a homotopy-like argument to show that by local continuity of the feedback interconnection, the point $[P_\tau G_1(0)] \cap [P_\tau G_2'(0)] = 0$ must be able to be continuously deformed to $[P_\tau G_1(w_1)] \cap [P_\tau G_2'(w_2)]$ $\forall \tau \in \mathcal{T}$ for any $w \in \mathcal{W}_e$. When $w \in \bar{\mathcal{B}}_{\mathcal{W}}^M$, suppositions 1–3 imply that $[P_\tau G_1(w_1)] \cap [P_\tau G_2'(w_2)] \cap \partial\mathcal{B}_{\mathcal{Y}}^N = \emptyset$ $\forall \tau \in \mathcal{T}$, and we must therefore have $[P_\tau G_1(w_1)] \cap [P_\tau G_2'(w_2)] \in \mathcal{B}_{\mathcal{Y}}^N$ $\forall \tau \in \mathcal{T}$ when $w \in \bar{\mathcal{B}}_{\mathcal{W}}^M$. Input-restricted stability then follows.

### C. Specialisations of Theorem 1

Theorem 1 gives a very general framework for determining input-restricted stability for a closed-loop system: those applying it must find feasible and sensible functionals $\alpha$, $\beta$, and $\gamma$ and choose appropriate values for $M$ and $N$. The following two corollaries specialise the result of this theorem, making the application of the theorem simpler in return for increasing the conservatism of the result by decoupling the functionals with the choices of $M$ and $N$.

*Corollary 1:* Consider the feedback interconnection in Fig. 1, described by (2), where $w_1 = 0$, $w = w_2 \in \mathcal{W}_{2e} = \mathcal{W}_e$ and $y = (y_1, y_2) \in \mathcal{Y}_{1e} \times \mathcal{Y}_{2e} = \mathcal{Y}_e$, and assume the interconnection is strongly well-posed. Suppose there exist some functionals $\tilde{\alpha} : \mathcal{Y} \to \mathbb{R}$, $\tilde{\beta} : \mathcal{W} \times \mathcal{Y} \to \mathbb{R}$, $\tilde{\gamma} \in \kappa$ and some $M$, $N > 0$ with $\tilde{\gamma}(M) < N$ such that, $\forall \tau \in \mathcal{T}$, the following holds:

1) $y \in G_1(0)$ and $\|y_\tau\| \leq N \Longrightarrow \tilde{\alpha}(y_\tau) \leq 0$;
2) $\forall w \in \bar{\mathcal{B}}_{\mathcal{W}}^M,\ y \in G_2'(w_2) \Longrightarrow \tilde{\beta}(w_\tau, y_\tau) \leq 0$;
3) $\forall w \in \bar{\mathcal{B}}_{\mathcal{W}}^M,\ \tilde{\alpha}(y_\tau) \leq 0$ and $\tilde{\beta}(w_\tau, y_\tau) \leq 0$ $\Longrightarrow \|y_\tau\| \leq \tilde{\gamma}(\|w_\tau\|)$.

Furthermore, suppose that $G_1(0) \cap G_2'(0) = 0$, i.e. the origin is a solution of the feedback interconnection with zero exogenous signals. Then, the feedback system is input-restricted stable for all inputs in $\bar{\mathcal{B}}_{\mathcal{W}}^M$.

*Proof:* Consider Theorem 1, and define the functionals $\alpha$, $\beta$ and $\gamma$ using the functionals $\tilde{\alpha}$, $\tilde{\beta}$, and $\tilde{\gamma}$ given in this corollary as

$$\alpha(w_\tau, y_\tau) = \tilde{\alpha}(y_\tau) \qquad \forall(\tau, w, y) \in \mathcal{T} \times \mathcal{W}_e \times \mathcal{Y}_e$$
$$\beta(w_\tau, y_\tau) = \tilde{\beta}(w_\tau, y_\tau) \qquad \forall(\tau, w, y) \in \mathcal{T} \times \mathcal{W}_e \times \mathcal{Y}_e$$
$$\gamma(x) = \tilde{\gamma}(x) \qquad \forall x \in \mathbb{R}_+.$$

Since $\tilde{\alpha}$ does not depend on $w$ and therefore $\alpha$ as defined above also does not depend on $w$, all $(\tau, y)$ which satisfy supposition 1in this corollary will satisfy supposition 1 in Theorem 1 regardless of $w_\tau$.

Supposition 2 in this corollary states that $\tilde{\beta}(w_\tau, y_\tau) \leq 0\, \forall \tau \in \mathcal{T}, w \in \bar{\mathcal{B}}_{\mathcal{W}}^M$ and $y \in G_2'(w_2)$, and supposition 2 in Theorem 1 additionally states that $\beta(w_\tau, y_\tau) \leq 0$ need only hold when $\|y_\tau\| \leq N$. Therefore, by definition of $\beta$ above, any $(\tau, w, y)$ which satisfy supposition 2 in this corollary will also satisfy supposition 2 in Theorem 1.

For supposition 3, the definitions of $\alpha$ and $\beta$ above give

$$\{(\tau, w, y) \in \mathcal{T} \times \bar{\mathcal{B}}_{\mathcal{W}}^M \times \mathcal{Y}_e | \alpha(w_\tau, y_\tau) \leq 0, \beta(w_\tau, y_\tau) \leq 0\} =$$
$$\{(\tau, w, y) \in \mathcal{T} \times \bar{\mathcal{B}}_{\mathcal{W}}^M \times \mathcal{Y}_e | \tilde{\alpha}(y_\tau) \leq 0, \tilde{\beta}(w_\tau, y_\tau) \leq 0\}$$

which proves that all $(\tau, w, y)$ that satisfy supposition 3 in this corollary also satisfy supposition 3 in Theorem 1.

Hence, if suppositions 1–3 in this corollary are satisfied, then suppositions 1–3 in Theorem 1 must also be satisfied. Input-restricted stability then follows directly from Theorem 1. ■

The usefulness of this corollary is that the region where suppositions 1 and 2 hold are independent of $M$ and $N$, respectively. In supposition 1, $\tilde{\alpha}$ can be chosen as a condition on the zero graph of $\Sigma_1$ such as an IQC, and in supposition 2, $\tilde{\beta}$ can be chosen as a condition on $\Sigma_2$ with no constraints on $y$. This is particularly useful when $\Sigma_2$ is an LTI system, as the behavior of an LTI system is invariant to a scaling of the inputs.

*Corollary 2:* Consider the feedback interconnection in Fig. 1, described by (2), where $w = (w_1, w_2) \in \mathcal{W}_{1e} \times \mathcal{W}_{2e} = \mathcal{W}_e$ and $y = (y_1, y_2) \in \mathcal{Y}_{1e} \times \mathcal{Y}_{2e} = \mathcal{Y}_e$, and assume the interconnection is strongly well-posed. Suppose there exist some functionals $\hat{\alpha} : \mathcal{W} \times \mathcal{Y} \to \mathbb{R}$, $\hat{\beta} : \mathcal{W} \times \mathcal{Y} \to \mathbb{R}$, $\hat{\gamma} \in \kappa$ and some $M > 0$ with $\hat{\gamma}(M) < \infty$ such that, $\forall \tau \in \mathcal{T}$, the following holds:

1) $\forall w \in \mathcal{W}_e,\ y \in G_1(w_1) \Longrightarrow \hat{\alpha}(w_\tau, y_\tau) \leq 0$;
2) $\forall w \in \mathcal{W}_e,\ y \in G_2'(w_2) \Longrightarrow \hat{\beta}(w_\tau, y_\tau) \leq 0$;
3) $\forall w \in \bar{\mathcal{B}}_{\mathcal{W}}^M,\ \hat{\alpha}(w_\tau, y_\tau) \leq 0$ and $\hat{\beta}(w_\tau, y_\tau) \leq 0$ $\Longrightarrow \|y_\tau\| \leq \hat{\gamma}(\|w_\tau\|)$.

Furthermore, suppose that $G_1(0) \cap G_2'(0) = 0$, i.e. the origin is a solution of the feedback interconnection with zero exogenous signals. Then, the feedback system is input-restricted stable for all inputs in $\bar{\mathcal{B}}_{\mathcal{W}}^M$.

*Proof:* Consider Theorem 1, and define the functionals $\alpha$, $\beta$, and $\gamma$ using the functionals $\hat{\alpha}$, $\hat{\beta}$, and $\hat{\gamma}$ given in Corollary 2 as

$$\alpha(w_\tau, y_\tau) = \hat{\alpha}(w_\tau, y_\tau) \qquad \forall(\tau, w, y) \in \mathcal{T} \times \mathcal{W}_e \times \mathcal{Y}_e$$
$$\beta(w_\tau, y_\tau) = \hat{\beta}(w_\tau, y_\tau) \qquad \forall(\tau, w, y) \in \mathcal{T} \times \mathcal{W}_e \times \mathcal{Y}_e$$
$$\gamma(x) = \hat{\gamma}(x) \qquad \forall x \in \mathbb{R}_+$$

and choose any $N$ such that $\hat{\gamma}(M) < N < \infty$, which ensures $\gamma(M) < N$ as required by Theorem 1.

By the definitions above, any $(\tau, w, y)$ which satisfy supposition 1 (respectively, supposition 2) of Corollary 2 also satisfy supposition 1 (respectively, supposition 2) of Theorem 1 regardless of the choice of $N$. Furthermore, the definitions of $\alpha$ and $\beta$ above also give

$$\{(\tau, w, y) \in \mathcal{T} \times \bar{\mathcal{B}}_{\mathcal{W}}^M \times \mathcal{Y}_e | \alpha(w_\tau, y_\tau) \leq 0, \beta(w_\tau, y_\tau) \leq 0\} =$$
$$\{(\tau, w, y) \in \mathcal{T} \times \bar{\mathcal{B}}_{\mathcal{W}}^M \times \mathcal{Y}_e | \hat{\alpha}(w_\tau, y_\tau) \leq 0, \hat{\beta}(w_\tau, y_\tau) \leq 0\}$$

which proves that all $(\tau, w, y)$ which satisfy supposition 3 in Corollary 2 also satisfy supposition 3 in Theorem 1.

Hence, if suppositions 1–3 in Corollary 2 are satisfied, then suppositions 1–3 in Theorem 1 must also be satisfied. Input-restricted stability then follows directly from Theorem 1. ■

*Remark 6:* Corollary 2 above is effectively a global version of Theorem 1, as suppositions 1 and 2 in the corollary must hold for all $y \in G_1(w_1)$ and $y \in G_2'(w_2)$, respectively, rather than only holding for a subset of these sets. Taking the limit as $M \to \infty$, this is identical to Teel's Proposition 5.1 in [7].

## IV. Extensions of the Input-Restricted Stability Theorem to Nonballs

Theorem 1 restricts both $y$ and $w$ to balls in their respective Lebesgue spaces. This is impractical in two ways. First, the rigid shape of a ball may not suit a given problem if not all elements of $y$ or $w$ need be restricted in the same way. Second, it may not be desirable or possible to restrict all elements of $y$ or $w$. Therefore, in this section we introduce the idea of a static linear mapping, which can warp these ball as fits the application and, through the choice of a nontrivial kernel, can select which elements of $y$ and $w$ are included in the restriction. This section also introduces a modified version of Theroem 1 for feedback interconnections with additive inputs.

### A. Static Linear Mappings

Here static linear mappings are defined and some technical results relating to them are given which are required for the later stability results.

*Definition 20 (Static linear mapping):* The operator $R$ defined as the mapping $R : \mathcal{Y}_e^m \to \mathcal{X}_e^n$, where $\mathcal{Y}_e^m, \mathcal{X}_e^n$ are $m$- and $n$-dimensional extended Lebesgue spaces, respectively, is a static linear mapping if there exists some matrix $\hat{R} \in \mathbb{R}^{n\times m}$ associated with $R$ such that

$$(Ry)(t) = \hat{R}y(t)\ \forall y \in \mathcal{Y}_e^m, t \in \mathcal{T}.$$

*Definition 21 (Combined static linear mapping between Cartesian product spaces):* Let $R_i : \mathcal{Y}_{ie}^{m_i} \to \mathcal{X}_{ie}^{n_i}$ be static linear mappings for $i \in \{1, 2\}$. Let $\mathcal{Y}_e = \mathcal{Y}_{1e}^{m_1} \times \mathcal{Y}_{2e}^{m_2}$ and $\mathcal{X}_e = \mathcal{X}_{1e}^{n_1} \times \mathcal{X}_{2e}^{n_2}$. Then, $R : \mathcal{Y}_e \to \mathcal{X}_e$ with $Ry = (R_1y_1, R_2y_2)$ is said to be a combined static linear mapping between $\mathcal{Y}_e$ and $\mathcal{X}_e$.

For some $\mathcal{S} \subset \mathcal{Y}_e$, we denote the image and pre-image[1] of $\mathcal{S}$ under the static linear mapping $R : \mathcal{Y}_e \to \mathcal{X}_e$ as $R[\mathcal{S}] = \{\, x \in \mathcal{X}_e \mid x = Ry, y \in \mathcal{S} \,\}$ and $R^{-1}[S] = \{\, y \in \mathcal{Y}_e \mid Ry \in \mathcal{S} \,\}$, respectively.

We now provide two technical lemmas, which will be useful in the proofs of the subsequent results.

*Lemma 3:* Let $R : \mathcal{Y}_e^m \to \mathcal{X}_e^n$ be a static linear mapping with associated matrix $\hat{R} \in \mathbb{R}^{n\times m}$. Then, $R$ is a locally continuous map.

*Proof:* First, we prove that the mapping $\hat{R}$ induced from $\mathbb{R}^m$ to $\mathbb{R}^n$ is continuous. Choosing $\delta = \epsilon/\|\hat{R}\|$, it is clear to see that $\forall x_2 \in \mathbb{R}^m$ and all $\epsilon > 0$, $\exists \delta > 0$ such that

$$|x_1 - x_2| < \delta \implies |\hat{R}x_1 - \hat{R}x_2| < \epsilon.$$

Hence, the mapping induced by $\hat{R}$ is continuous.

Then, by continuity of the mapping induced by $\hat{R}$, we have

$$\forall y_2(t) \in \mathbb{R}^m, \epsilon > 0, \exists\delta > 0 \text{ such that}$$
$$|y_1(t) - y_2(t)| < \delta \implies |\hat{R}y_1(t) - \hat{R}y_2(t)| < \epsilon$$

$$\Rightarrow \forall y_2 \in \mathcal{Y}_e^m, \epsilon > 0, \tau \in \mathcal{T}, \exists\delta > 0 \text{ such that}$$
$$\|y_{1\tau} - y_{2\tau}\| < \delta \implies \|R(y_{1\tau}) - R(y_{2\tau})\| < \epsilon.$$

■

*Lemma 4:* Let $R : \mathcal{Y}_e \to \mathcal{X}_e$ be a static linear mapping with associated matrix $\hat{R}$. Then, for any $r > 0$, the set $R^{-1}[\mathcal{B}_{\mathcal{X}}^r] \cap \mathcal{Y}$ is open and star shaped.

*Proof:* To prove that $R^{-1}[\mathcal{B}_{\mathcal{X}}^r] \cap \mathcal{Y}$ is open, it must be shown that for any $y \in R^{-1}[\mathcal{B}_{\mathcal{X}}^r] \cap \mathcal{Y}$ there exists some $\epsilon > 0$ such that all $s \in \mathcal{Y}$ satisfying $\|s - y\| < \epsilon$ are also contained within $R^{-1}[\mathcal{B}_{\mathcal{X}}^r] \cap \mathcal{Y}$.

Suppose $y \in R^{-1}[\mathcal{B}_{\mathcal{X}}^r] \cap \mathcal{Y}$ and $s \in \mathcal{Y}$ and the distance between these two vectors is less than some $\epsilon > 0$. Then, we have

$$\begin{aligned}
&\|s - y\| < \epsilon\\
\iff\ &\|R\|\|s - y\| < \|R\|\epsilon\\
\implies &\|Rs - Ry\| < \|R\|\epsilon\\
\implies &\|Rs\| - \|Ry\| < \|R\|\epsilon\\
\iff\ &\|Rs\| < \|R\|\epsilon + \|Ry\|\\
\implies &\|Rs\| < \|R\|\epsilon + r - \delta
\end{aligned}$$

where $\delta > 0$ is such that $\|Ry\| \le r - \delta < r$, as $y \in R^{-1}[\mathcal{B}_{\mathcal{X}}^r] \cap \mathcal{Y}$. Then, if $0 < \epsilon \le \delta/\|R\|$, we have that $s \in R^{-1}[\mathcal{B}_{\mathcal{X}}^r] \cap \mathcal{Y}$. Such an $\epsilon$ can always be found for any $y \in R^{-1}[\mathcal{B}_{\mathcal{X}}^r] \cap \mathcal{Y}$ as $\delta > 0$ for any such $y$, and as such $R^{-1}[\mathcal{B}_{\mathcal{X}}^r] \cap \mathcal{Y}$ is open.

To prove star shapedness, we require $\tau y \in R^{-1}[\mathcal{B}_{\mathcal{X}}^r] \cap \mathcal{Y}\ \forall\tau \in [0, 1], y \in R^{-1}[\mathcal{B}_{\mathcal{X}}^r] \cap \mathcal{Y}$. By definition, we have

$$\begin{aligned}
y \in R^{-1}[\mathcal{B}_{\mathcal{X}}^r] \cap \mathcal{Y} \iff\ &\|Ry\| < r\\
\iff\ &\tau\|Ry\| < \tau r \qquad \forall\tau \in (0, 1]\\
\iff\ &\|R(\tau y)\| < \tau r \quad \forall\tau \in (0, 1]\\
\implies &\|R(\tau y)\| < r \qquad \forall\tau \in [0, 1]\\
\iff\ &\tau y \in R^{-1}[\mathcal{B}_{\mathcal{X}}^r] \quad \forall\tau \in [0, 1] \qquad (3)
\end{aligned}$$

where (3) is by linearity of $R$. Hence, $R^{-1}[\mathcal{B}_{\mathcal{X}}^r]$ is star shaped. ■

### B. Graphs and Balls

The following definitions are useful when dealing with feedback interconnections with additive inputs.

*Definition 22 (Graph of a system with no exogenous signal):* The graph of a system with no exogenous signal for a system $\Sigma : \mathcal{X}_e \to \mathcal{Y}_e$ is the set of all possible input–output pairs $(x, y) \in \mathcal{X}_e \times \mathcal{Y}_e$ for that system. This is defined as

$$G = \{\, (x, y) \in \mathcal{X}_e \times \mathcal{Y}_e \ \mid\ y = \Sigma(x) \,\}.$$

Similarly to the graph of a system, the inverse graph of a system with no exogenous signal is defined as

$$G' = \{\, (y, x) \in \mathcal{Y}_e \times \mathcal{X}_e \ \mid\ y = \Sigma(x) \,\}.$$

*Definition 23 (Ball in Cartesian product space):* Let $W = \mathcal{W}_1 \times \mathcal{W}_2$ be a Cartesian product of two Lebesgue spaces. Then, the Cartesian product of two balls in $\mathcal{W}_1$ and $\mathcal{W}_2$ is defined as

$$\mathcal{B}_{\mathcal{W}}^{(M_1,M_2)} = \mathcal{B}_{\mathcal{W}_1}^{M_1} \times \mathcal{B}_{\mathcal{W}_2}^{M_2}$$

[1]Note that the preimage of a set under a mapping is well-defined even if the mapping is not invertible.

or, for balls in explicitly defined norms

$$(\mathcal{B}_{\mathcal{W}}^{(M_1,M_2)})_{p,q} = (\mathcal{B}_{\mathcal{W}_1}^{M_1})_p \times (\mathcal{B}_{\mathcal{W}_2}^{M_2})_q.$$

*Remark 7:* The notation introduced in Definition 23 can also be used for the Cartesian product of the closures of two balls, or the Cartesian product of the boundaries of two balls.

### C. Extensions to the Input-Restricted Stability Theorem

We are now in a position to apply these static linear mappings to extend the results of Theorem 1. Note that the static linear mappings are allowed a nontrivial kernel and hence can be used as selection and scaling operators.

*Theorem 2:* Consider the feedback interconnection in Fig. 1, governed by (2), and assume the interconnection is strongly well-posed. Suppose there exist some functionals $\alpha,\ \beta : \mathcal{W} \times \mathcal{Y} \to \mathbb{R}$, $\tilde{\gamma}, \gamma \in \kappa$, some $M,\ N > 0$ with $\tilde{\gamma}(M) < N$, some $p, q \in \mathbb{N}_\infty$ and some static linear mappings $R : \mathcal{Y}_e \to \mathcal{X}_e$ and $S : \mathcal{W}_e \to \mathcal{Z}_e$ such that, $\forall \tau \in \mathcal{T}$ and $\forall w \in S^{-1}[(\bar{\mathcal{B}}_{\mathcal{Z}}^M)_p]$, the following holds:

1) $y \in G_1(w_1)$ and $\|R(y_\tau)\|_q \leq N \implies \alpha(w_\tau, y_\tau) \leq 0$;
2) $y \in G_2'(w_2)$ and $\|R(y_\tau)\|_q \leq N \implies \beta(w_\tau, y_\tau) \leq 0$;
3) $\alpha(w_\tau, y_\tau) \leq 0$ and $\beta(w_\tau, y_\tau) \leq 0$
$\implies \|R(y_\tau)\|_q \leq \tilde{\gamma}(\|S(w_\tau)\|_p)$ and $\|y_\tau\| \leq \gamma(\|w_\tau\|)$.

Furthermore, suppose that $G_1(0) \cap G_2'(0) = 0$, i.e. the origin is a solution of the feedback interconnection with zero exogenous signals. Then, the feedback system is input-restricted stable for all inputs in $S^{-1}[(\bar{\mathcal{B}}_{\mathcal{Z}}^M)_p]$. ■

*Proof:* See Appendix. ■

*Remark 8:* Theorem 2 extending Theorem 1 to warped balls is only part of its benefit. As we have explicitly allowed the static linear mapping $R$ to have a nontrivial kernel, it can be applied to systems where not every signal in $y$ can be restricted, and/or not every signal in $y$ affects the behaviors of $\Sigma_1$ or $\Sigma_2$. Not having to restrict every element of $y$ therefore allows for Theorem 2 to generate potentially less conservative results than Theorem 1. This can be seen for example in the proof of Theorem 5 later on in this article where the choice of $\tilde{R} = [I\ 0\ 0]$ means that we can omit $x$ and $\dot{x}$ from $R\tilde{y}_2$.

*Remark 9:* In allowing the ball in $\mathcal{Y}_e$ to be morphed by some static linear mapping $R$ in Theorem 2, we end up needing two distinct gain relationships in supposition 3 of Theorem 2 compared to the single gain relationship in supposition 3 of Theorem 1.

In Theorem 1, the gain relationship $\|y_\tau\| \leq \gamma(\|w_\tau\|)$ in the third supposition played two roles. First, it ensured that $\|y_\tau\| < N$ for all $\tau \in \mathcal{T}$ when $\|w\| \leq M$, and hence via an application of Lemma 2 ensured that $y_\tau$ remained within a ball of radius $N$ for all $\tau \in \mathcal{T}$ when $\|w\| \leq M$. Second, it provided the input–output gain relationship required to guarantee input-restricted stability as given in Definition 19.

In Theorem 2, these two roles are played by the two different gain relationships in supposition 3. The gain relationship $\|R(y_\tau)\|_q \leq \tilde{\gamma}(\|S(w_\tau)\|_p)$ ensures that $y_\tau \in R^{-1}[(\mathcal{B}_{\mathcal{X}}^N)_q]$ for all $w \in S^{-1}[(\bar{\mathcal{B}}_{\mathcal{Z}}^M)_p]$ and $\tau \in \mathcal{T}$, which builds on suppositions 1 and 2, whereas the gain relationship $\|y_\tau\| \leq \gamma(\|w_\tau\|)$ provides the input–output gain relationship required for input-restricted stability. The reason these two gain relationships cannot be combined is because $\ker(R)$ may be nontrivial, and if this is the case then we have that $\|R(y_\tau)\|_q \leq \tilde{\gamma}(\|S(w_\tau)\|_p) \not\Longrightarrow \exists \gamma \in \kappa$ such that $\|y_\tau\| \leq \gamma(\|w_\tau\|)$.

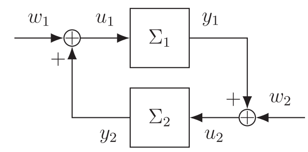


Fig. 2. Feedback interconnection with additive exogenous inputs.

Corollary 3 shows that $\ker(R) = \{0\}$ eliminates the requirement for the second gain relationship in the third supposition.

*Corollary 3:* Consider the feedback interconnection in Fig. 1, governed by (2), and assume the interconnection is strongly well-posed. Suppose there exist some functionals $\alpha,\ \beta : \mathcal{W} \times \mathcal{Y} \to \mathbb{R}$, $\tilde{\gamma} \in \kappa$, some $M,\ N > 0$ with $\tilde{\gamma}(M) < N$, some $p, q \in \mathbb{N}_\infty$ and some static linear mappings $S : \mathcal{W}_e \to \mathcal{Z}_e$ and $R : \mathcal{Y}_e \to \mathcal{X}_e$ with $\ker(R) = \{0\}$ such that, $\forall \tau \in \mathcal{T}$ and $\forall w \in S^{-1}[(\bar{\mathcal{B}}_{\mathcal{Z}}^M)_p]$, the following holds:

1) $y \in G_1(w_1)$ and $\|R(y_\tau)\|_q \leq N \implies \alpha(w_\tau, y_\tau) \leq 0$;
2) $y \in G_2'(w_2)$ and $\|R(y_\tau)\|_q \leq N \implies \beta(w_\tau, y_\tau) \leq 0$;
3) $\alpha(w_\tau, y_\tau) \leq 0$ and $\beta(w_\tau, y_\tau) \leq 0$
$\implies \|R(y_\tau)\|_q \leq \tilde{\gamma}(\|S(w_\tau)\|_p)$.

Furthermore, suppose that $G_1(0) \cap G_2'(0) = 0$, i.e. the origin is a solution of the feedback interconnection with zero exogenous signals. Then, the feedback system is input-restricted stable for all inputs in $S^{-1}[(\bar{\mathcal{B}}_{\mathcal{Z}}^M)_p]$.

*Proof:* If $\ker(R) = \{0\}$, $\ker(\hat{R}) = \{0\}$ and therefore $\hat{R}^{-1}$, and by implication $R^{-1}$, exist. Therefore, $\|R(y_\tau)\|_q \leq \tilde{\gamma}(\|S(w_\tau)\|_p) \implies \|y_\tau\|_q \leq \gamma(\|w_\tau\|_p)$ for some $\gamma \in \kappa$. Hence, supposition 3 of this corollary is equivalent to supposition 3 of Theorem 2, thus input-restricted stability follows via Theorem 2. ■

Theorem 2 and Corollary 3 enable a ball in $\mathcal{W}_e$ of size $M$ and a ball in $\mathcal{Y}_e$ of size $N$ to be warped by the arbitrary static linear mappings $S$ and $R$, respectively. This gives greater scope to Theorem 2 and allows for potentially less conservative choices of $\alpha$, $\beta$, and $\gamma$.

*Remark 10:* In Theorem 2 and Corollary 3, it is possible to set $N = 1$ (by absorbing it into $R$ and $\tilde{\gamma}$) and $M = 1$ (by absorbing it into $S$) without loss of generality. However, we prefer not to do this mathematical simplification for pedagogical reasons.

### D. Interconnections With Additive Exogenous Signals

The theorems and corollaries given in the previous two sections all apply to feedback systems where the exogenous inputs enter the individual subsystems in an undetermined way. Therefore, if those inputs were to enter the feedback system additively, this could be seen simply as a specialization of the previous feedback system and therefore the input-restricted stability results presented previously could be applied. However, since the vast majority of feedback systems which control engineers and researchers deal with have additive inputs, and since—in such a feedback interconnection—it makes little sense to restrict the output of each system $\Sigma_i$ (i.e., only the input of each system $\Sigma_i$ should be restricted; this point will be discussed further in Remark 11), it is salient to present a theorem which is explicitly and directly applicable to feedback interconnections with additive exogenous inputs. As such, we define a new canonical feedback interconnection with additive $w_1$ and $w_2$, shown in Fig. 2.

The positive feedback interconnection shown in Fig. 2 with additive exogenous inputs $w_1$ and $w_2$, with $\Sigma_1 : \mathcal{W}_{1e} \to \mathcal{W}_{2e}$ and $\Sigma_2 : \mathcal{W}_{2e} \to \mathcal{W}_{1e}$, is governed by the following equations:

$$u_1 = w_1 + y_2 \quad (4a)$$

$$u_2 = w_2 + y_1 \quad (4b)$$

$$y_1 = \Sigma_1(u_1) \quad (4c)$$

$$y_2 = \Sigma_2(u_2) \quad (4d)$$

where $u = (u_1, u_2)$ and $w = (w_1, w_2)$ are signals in $\mathcal{W}_e = \mathcal{W}_{1e} \times \mathcal{W}_{2e}$ and $y = (y_1, y_2) \in \mathcal{W}'_e = \mathcal{W}_{2e} \times \mathcal{W}_{1e}$.

In the sequel, we will denote the graph of $\Sigma_1$ from (4) as $G_1$ and the inverse graph of $\Sigma_2$ as $G'_2$. It is then clear from (4) that $(u_1, y_1) \in G_1$ and $(y_2, u_2) \in G'_2$. Finally, note that as with Theorem 1, the following theorem is applicable in both continuous and discrete time.

*Theorem 3 (Input-restricted stability for feedback interconnections with additive inputs):* Consider the feedback interconnection in Fig. 2, governed by (4), and assume the interconnection is strongly well-posed. Suppose that, for $i \in \{1, 2\}$, there exist some functionals $\alpha,\ \beta : \mathcal{W}_1 \times \mathcal{W}_2 \to \mathbb{R}$, $\tilde{\gamma}_i, \gamma \in \kappa$; some reals $M_i,\ N_i > 0$ with $M = \|(M_1, M_2)\|$ and $\tilde{\gamma}_i(M) < N_i$; some $p_i, q_i \in \mathbb{N}_\infty$; and some static linear mappings $R_i : \mathcal{W}_{ie} \to \mathcal{X}_{ie}$, $S_i : \mathcal{W}_{ie} \to \mathcal{Z}_{ie}$ and combined static linear mappings $R : \mathcal{W}_e \to \mathcal{X}_e$, $S : \mathcal{W}_e \to \mathcal{Z}_e$ such that, $\forall \tau \in \mathcal{T}$, the following holds:

1) $(u_1, y_1) \in G_1$ and $\|R_1(u_{1\tau})\|_{q_1} \le N_1 \implies \alpha(u_{1\tau}, y_{1\tau}) \le 0$;
2) $(y_2, u_2) \in G'_2$ and $\|R_2(u_{2\tau})\|_{q_2} \le N_2 \implies \beta(y_{2\tau}, u_{2\tau}) \le 0$;
3) $\forall w \in S^{-1}[(\bar{\mathcal{B}}_{\mathcal{Z}}^{(M_1,M_2)})_{p_1,p_2}]$, $\alpha(u_{1\tau}, u_{2\tau} - w_{2\tau}) \le 0$, $\beta(u_{1\tau} - w_{1\tau}, u_{2\tau}) \le 0$ $\implies \|R_1(u_{1\tau})\|_{q_1} \le \tilde{\gamma}_1(\|S(w_\tau)\|_{p_1,p_2})$, $\|R_2(u_{2\tau})\|_{q_2} \le \tilde{\gamma}_2(\|S(w_\tau)\|_{p_1,p_2})$, and $\|u_\tau\| \le \gamma(\|w_\tau\|)$.

Furthermore, suppose that $G_1 \cap G'_2 = 0$, i.e., the origin is a solution of the feedback interconnection with zero exogenous signals. Then, the feedback interconnection is input-restricted stable for all inputs in $S^{-1}[(\bar{\mathcal{B}}_{\mathcal{Z}}^{(M_1,M_2)})_{p_1,p_2}]$. ■

*Proof:* See Appendix. ■

*Remark 11:* Suppositions 1 and 2 of Theorem 3 restrict only the size of the inputs of $\Sigma_1$ and $\Sigma_2$ for the functionals $\alpha$ and $\beta$ to be nonpositive over all signals in the graph of $\Sigma_1$ and $\Sigma_2$. This is motivated by how this theorem will be used in practice. Note that this is a fundamental difference in philosophy from the approach taken previously, where both the inputs and outputs of $\Sigma_1$ and $\Sigma_2$ were restricted, and as such this result is a new theorem rather than a simple corollary of Theorem 2.

*Remark 12:* Again, in Theorem 3, it is possible to set $N_1 = N_2 = 1$ (by absorbing in $R_i$ and $\tilde{\gamma}_i$) and $M_1 = M_2 = 1$ (by absorbing in $S_i$) without loss of generality. However, we again prefer not to do this mathematical simplification due to pedagogical reasons.

By choosing $R = S = I$ (the identity operator), we reach an additive exogenous input corollary which is close to Theorem 1 except for the fundamental differences laid out in Remark 11 which still hold.

*Corollary 4:* Consider the feedback interconnection in Fig. 2, governed by (4), and assume the interconnection is strongly well-posed. Suppose that, for $i \in \{1, 2\}$, there exist some functionals $\alpha,\ \beta : \mathcal{W}_1 \times \mathcal{W}_2 \to \mathbb{R}$, $\gamma_i \in \kappa$; some $p_i, q_i \in \mathbb{N}_\infty$; and some reals $M_i,\ N_i > 0$ with $M = \|(M_1, M_2)\|$ and $\gamma_i(M) < N_i$ such that, $\forall \tau \in \mathcal{T}$, the following holds:

1) $(u_1, y_1) \in G_1$ and $\|u_{1\tau}\|_{q_1} \le N_1 \implies \alpha(u_{1\tau}, y_{1\tau}) \le 0$;
2) $(y_2, u_2) \in G'_2$ and $\|u_{2\tau}\|_{q_2} \le N_2 \implies \beta(y_{2\tau}, u_{2\tau}) \le 0$;
3) $\forall w \in (\bar{\mathcal{B}}_{\mathcal{W}}^{(M_1,M_2)})_{p_1,p_2}$, $\alpha(u_{1\tau}, u_{2\tau} - w_{2\tau}) \le 0$, $\beta(u_{1\tau} - w_{1\tau}, u_{2\tau}) \le 0$ $\implies \|u_{1\tau}\|_{q_1} \le \gamma_1(\|w_\tau\|_{p_1,p_2})$ and $\|u_{2\tau}\|_{q_2} \le \gamma_2(\|w_\tau\|_{p_1,p_2})$.

Furthermore, suppose that $G_1 \cap G'_2 = 0$, i.e., the origin is a solution of the feedback interconnection with zero exogenous signals. Then, the feedback system is input-restricted stable for all inputs in $(\bar{\mathcal{B}}_{\mathcal{W}}^{(M_1,M_2)})_{p_1,p_2}$.

*Proof:* Apply Theorem 3, setting $R_i = S_i = I$ (the identity operator) and $\tilde{\gamma}_i = \gamma_i$ for $i \in \{1, 2\}$. Then, we have $R_i u_i = u_i$ and $S_i w_i = w_i$ for $i \in \{1, 2\}$ and any $u, w \in \mathcal{W}_e$. Furthermore, $\|u_{1\tau}\|_{q_1} \le \gamma_1(\|w_\tau\|_{p_1,p_2})$ and $\|u_{2\tau}\|_{q_2} \le \gamma_2(\|w_\tau\|_{p_1,p_2}) \implies \exists \gamma \in \kappa$ such that $\|u_\tau\| \le \gamma(\|w_\tau\|)$, hence supposition 3 of this corollary is equivalent to supposition 3 of Theorem 3. Input-restricted stability for inputs in $(\bar{\mathcal{B}}_{\mathcal{W}}^{(M_1,M_2)})_{p_1,p_2}$ then follows directly from Theorem 3. ■

The next corollary to the general input-restricted stability result, Theorem 3, allows the input to one of the systems in the closed-loop with additive exogenous inputs to be unrestricted. Note that a similar corollary to Theorem 2 can be developed for the nonadditive exogenous input case through an appropriate choice of $R$.

*Corollary 5:* Consider the feedback interconnection in Fig. 2, governed by (4), and assume the interconnection is strongly well-posed. Suppose that there exist some functionals $\alpha,\ \beta : \mathcal{W}_1 \times \mathcal{W}_2 \to \mathbb{R}$, $\tilde{\gamma}, \gamma \in \kappa$; some reals $M_1, M_2, N > 0$ with $M = \|(M_1, M_2)\|$ and $\tilde{\gamma}(M) < N$; some integers $p_1, p_2, q \in \mathbb{N}_\infty$; and some static linear mappings $R : \mathcal{W}_{1e} \to \mathcal{X}_e$, $S_1 : \mathcal{W}_{1e} \to \mathcal{Z}_{1e}$, $S_2 : \mathcal{W}_{2e} \to \mathcal{Z}_{2e}$ and a combined static linear mapping $S : \mathcal{W}_e \to \mathcal{Z}_e$ such that, $\forall \tau \in \mathcal{T}$, the following holds:

1) $(u_1, y_1) \in G_1$ and $\|R(u_{1\tau})\|_q \le N \implies \alpha(u_{1\tau}, y_{1\tau}) \le 0$;
2) $(y_2, u_2) \in G'_2 \implies \beta(y_{2\tau}, u_{2\tau}) \le 0$;
3) $\forall w \in S^{-1}[(\bar{\mathcal{B}}_{\mathcal{Z}}^{(M_1,M_2)})_{p_1,p_2}]$, $\alpha(u_{1\tau}, u_{2\tau} - w_{2\tau}) \le 0$, $\beta(u_{1\tau} - w_{1\tau}, u_{2\tau}) \le 0$ $\implies \|R(u_{1\tau})\|_q \le \tilde{\gamma}(\|S(w_\tau)\|_{p_1,p_2})$ and $\|u_\tau\| \le \gamma(\|w_\tau\|)$.

Furthermore, suppose that $G_1 \cap G'_2 = 0$, i.e., the origin is a solution of the feedback interconnection with zero exogenous signals. Then, the feedback system is input-restricted stable for all inputs in $S^{-1}[(\bar{\mathcal{B}}_{\mathcal{Z}}^{(M_1,M_2)})_{p_1,p_2}]$.

*Proof:* Apply Theorem 3 setting $R_1 = R$ and $R_2 = 0$ (the null operator), which implies $\|R_2(u_{2\tau})\|_{q_2} = 0$ for all $\tau \in \mathcal{T}$ and $u \in \mathcal{W}_e$. Therefore, suppositions 2 and 3 in Theorem 3 are equivalent to suppositions 2 and 3 in this corollary. Hence, input-restricted stability for inputs in $S^{-1}(\bar{\mathcal{B}}_{\mathcal{Z}}^{(M_1,M_2)})_{p_1,p_2}$ follows directly from Theorem 3. ■

*Remark 13:* Corollary 5 is beneficial as it restricts the input to only one system, thus allowing supposition 2 to capture global input-output behavior of $\Sigma_2$. This is useful when $\Sigma_2$ is LTI, as the input–output behavior of an LTI system is invariant to a scaling of the input.

## V. Specializations and Applications

The input-restricted stability results presented in this article, while powerful, are not easily applicable to problems a control engineer might face. Therefore, this section aims to motivate the use of this theory through two specific applications, including numerical examples. The first application is a type of nonlinear small-gain result, and the second is a passivity result.

### A. Input-Restricted Nonlinear Small Gain With a Quadratic Gain Bound

The small-gain theorem is a classic and well-known feedback stability result, both for linear systems [3] and nonlinear systems [7]. Here, we present a type of input-restricted nonlinear small-gain stability theorem where one system is bounded by a linear gain and the other system is bounded by a quadratic gain. Global stability of this system cannot be proven using standard small-gain theory, but we can use Corollary 4 to prove input-restricted stability for some value of $M$. Note that as this theorem is based on Corollary 4, which is a corollary of Theorem 3, it is applicable in the discrete time domain as well as the continuous time domain unlike the similar specialization given in [24].

*Theorem 4 (Input-restricted nonlinear small gain with a quadratic gain bound):* Consider the positive feedback interconnection with additive exogenous signals as shown in Fig. 2, governed by (4). Let $w, u \in \mathcal{W}_{1e} \times \mathcal{W}_{2e} = \mathcal{W}_e$ and $y \in \mathcal{W}_{2e} \times \mathcal{W}_{1e} = \mathcal{W}'_e$, with $w = (w_1, w_2), u = (u_1, u_2), y = (y_1, y_2)$ and $\|x\| = \|(x_1, x_2)\| = \max\{\|x_1\|, \|x_2\|\}$ for all $x \in \mathcal{X}_e$ or $\mathcal{X}'_e$. Let the system $\Sigma_1$ have quadratic bound gain given by some $k_1 > 0$ and the system $\Sigma_2$ have linear gain bound given by some $k_2 > 0$, i.e.,

$$\|y_{1\tau}\| \leq k_1 \|u_{1\tau}\|^2 \qquad \forall \tau \in \mathcal{T}, (u_1, y_1) \in G_1 \tag{5}$$

$$\|y_{2\tau}\| \leq k_2 \|u_{2\tau}\| \qquad \forall \tau \in \mathcal{T}, (y_2, u_2) \in G'_2 \tag{6}$$

and assume that the feedback interconnection of $\Sigma_1$ and $\Sigma_2$ is strongly well-posed. Then, the system is input-restricted stable for any input $w$ in $\bar{\mathcal{B}}^M_{\mathcal{W}}$ where

$$M = \frac{1}{4k_1 k_2 (k_2 + 1)}. \tag{7}$$

*Proof:* We will apply Corollary 4 to prove this result. We will omit the subscripts $p_1, p_2, q_1$, and $q_2$; these are defined by the norms which $\mathcal{W}_{1e}$ and $\mathcal{W}_{2e}$, respectively, are equipped with.

First let us choose functionals $\alpha$ and $\beta$ as

$$\alpha(u_{1\tau}, y_{1\tau}) = \|y_{1\tau}\| - k_1 \|u_{1\tau}\|^2 \tag{8}$$

$$\beta(y_{2\tau}, u_{2\tau}) = \|y_{2\tau}\| - k_2 \|u_{2\tau}\| \tag{9}$$

which satisfy suppositions 1 and 2 of Corollary 4 for any $M_1, M_2, N_1, N_2 > 0$, and suppose we choose $M_1 = M_2 = M$ where $M$ is given by (7).

Next we show how these choices satisfy supposition 3 of Corollary 4. Using the triangle inequality, $\forall \tau \in \mathcal{T}$ and $w \in \bar{\mathcal{B}}^M_{\mathcal{W}}$ we have

$$\alpha(u_{1\tau}, u_{2\tau} - w_{2\tau}) \leq 0 \Longleftrightarrow \|u_{2\tau} - w_{2\tau}\| - k_1 \|u_{1\tau}\|^2 \leq 0$$
$$\Longrightarrow \|u_{2\tau}\| \leq \|w_{2\tau}\| + k_1 \|u_{1\tau}\|^2 \tag{10}$$

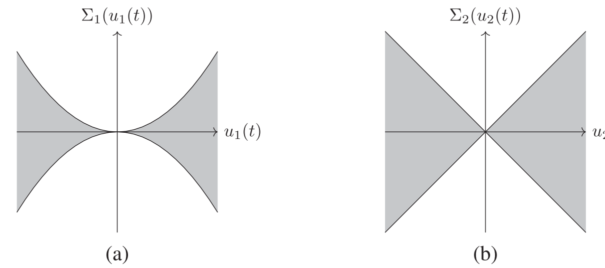


Fig. 3. Plots of (a) $u_1(t)$ against $\Sigma_1(u_1(t))$ and (b) $u_2(t)$ against $\Sigma_2(u_2(t))$ for static nonlinearities bounded by quadratic and linear gains, respectively. The solid black lines in both plots show the quadratic and linear boundaries of the system graphs, and the graphs must remain within the shaded gray regions.

and similarly for $\beta$, $\forall \tau \in \mathcal{T}$ and $w \in \bar{\mathcal{B}}^M_{\mathcal{W}}$ we have

$$\beta(u_{1\tau} - w_{1\tau}, u_{2\tau}) \leq 0 \Longleftrightarrow \|u_{1\tau} - w_{1\tau}\| - k_2 \|u_{2\tau}\| \leq 0$$
$$\Longrightarrow \|u_{1\tau}\| \leq \|w_{1\tau}\| + k_2 \|u_{2\tau}\|. \tag{11}$$

Supposition 3 of Corollary 4 must be satisfied when both $\alpha(u_{1\tau}, u_{2\tau} - w_{2\tau}) \leq 0$ and $\beta(u_{1\tau} - w_{1\tau}, u_{2\tau}) \leq 0 \; \forall \tau \in \mathcal{T}, w \in \bar{\mathcal{B}}^M_{\mathcal{W}}$. Therefore, (10) and (11) together imply that, $\forall \tau \in \mathcal{T}, w \in \bar{\mathcal{B}}^M_{\mathcal{W}}$, we have

$$\|u_{1\tau}\| \leq \|w_{1\tau}\| + k_2 \|w_{2\tau}\| + k_1 k_2 \|u_{1\tau}\|^2$$
$$\Longrightarrow \|u_{1\tau}\| - k_1 k_2 \|u_{1\tau}\|^2 \leq (k_2 + 1) \|w_\tau\|. \tag{12}$$

Let us define $\gamma_1 \in \kappa$ as

$$\gamma_1(x) = \frac{1 - \sqrt{\max\{0, 1 - 4k_1 k_2 (k_2 + 1) x\}}}{2k_1 k_2} = \frac{1 - \sqrt{\max\{0, 1 - x/M\}}}{2k_1 k_2}.$$

Now consider the quadratic equation $-k_1 k_2 z^2 + z - (k_2 + 1)x = 0$. This equation has roots at $z = (1 \pm \sqrt{1 - x/M})/2k_1 k_2$, hence for $0 \leq x \leq M$ it has a root at $z = \gamma_1(x)$. The quadratic inequality $-k_1 k_2 z^2 + z - (k_2 + 1)x \leq 0$ is therefore satisfied, for $0 \leq x \leq M$, when $z \leq \gamma_1(x)$ and $z \geq (1 + \sqrt{1 - x/M})/2k_1 k_2$. Therefore, by replacing $z$ with $\|u_{1\tau}\|$ and $x$ with $\|w_\tau\|$, we can see that (12) is satisfied when $\|u_{1\tau}\| \leq \gamma_1(\|w_\tau\|) \; \forall \tau \in \mathcal{T}, w \in \bar{\mathcal{B}}^M_{\mathcal{W}}$.

By (10), $\forall \tau \in \mathcal{T}, w \in \bar{\mathcal{B}}^M_{\mathcal{W}}$, we then have

$$\|u_{2\tau}\| \leq \|w_{2\tau}\| + k_1 \|u_{1\tau}\|^2 \tag{13}$$

$$\Longrightarrow \|u_{2\tau}\| \leq \|w_{2\tau}\| + k_1 \gamma_1(\|w_\tau\|)^2 = \gamma_2(\|w_\tau\|) \tag{14}$$

for some $\gamma_2 \in \kappa$.

As $\alpha$ and $\beta$ chosen earlier hold for any $N_1, N_2 > 0$, respectively, and any $M > 0$ (as they hold globally), we can choose any $N_1 > \gamma_1(M)$ and $N_2 > \gamma_2(M)$. Finally, from (5) and (6) we have that $0 \in G_1$ and $0 \in G'_2$ which together imply that $G_1 \cap G'_2 = 0$ by well-posedness. Hence, the system is input-restricted stable for any input $w$ in $\bar{\mathcal{B}}^M_{\mathcal{W}}$ for any $M = (4k_1 k_2 (k_2 + 1))^{-1}$. ■

*Remark 14:* If (5) included $\|u_{1\tau}\|$ instead of $\|u_{1\tau}\|^2$, we would then have $\gamma_1(x) = (k_2 + 1)/(1 - k_1 k_2)$ in the proof above, which clearly has no reliance on $x$ and $M$ and is the standard small-gain closed-loop gain [3, Thm. 1]. Hence, in this scenario Theorem 4 reduces to the classic small-gain theorem.

Fig. 3 shows the regions in which the graphs of $\Sigma_1$ and $\Sigma_2$ reside if both systems are static nonlinearities.

*Example 1:* As a numerical example, first we define the extended Lebesgue spaces $\mathcal{X}_e$ and $\mathcal{W}_e = \mathcal{X}_e \times \mathcal{X}_e$, where the underlying nonextended space $\mathcal{X}$ is equipped with the $\mathcal{L}_\infty$-norm and $\|(w_{1\tau}, w_{2\tau})\| = \max\{\|w_{1\tau}\|, \|w_{2\tau}\|\}$ $\forall \tau \in \mathcal{T}$ and all $w \in \mathcal{W}_e$. Then, consider the forced nonlinear differential equation with a quadratic nonlinearity, with state vector $x \in \mathcal{X}_e$ and inputs $w = (w_1, w_2) \in \mathcal{W}_e$, given by

$$\ddot{x} + \dot{x} + 100x - 0.88|x + w_1|(x + w_1) = 1.766 w_2.$$

Note that this system has no physical significance; it simply aims to give a simple example of a system with a quadratic nonlinearity.

By defining $y_2 = x$, $y_1 = 0.5|x + w_1|(x + w_1)$, $u_2 = y_1 + w_2$ and $u_1 = y_2 + w_1$, we see that this equation can be rewritten as the feedback interconnection of two systems $\Sigma_1$ and $\Sigma_2$, as per Fig. 2 and governed by (4), where $\Sigma_1$ is a static nonlinearity with a quadratic gain of $k_1 = 0.5$ as per (5) and $\Sigma_2$ is an LTI system with induced $\mathcal{L}_\infty$-to-$\mathcal{L}_\infty$ gain of $k_2 = 0.25$ as per (6).

Then, by applying Theorem 4, we can say that this nonlinear differential system will remain stable for all $w \in \bar{\mathcal{B}}_{\mathcal{W}}^M$ where $M = 1.6$.

This feedback interconnection was simulated with square waves injected at $w_1$ and $w_2$. Letting $\omega_0$ be the resonant frequency of the LTI system $\Sigma_2$, then: we chose the square wave at $w_1$ to have a frequency equal to $\omega_0/2$; the square wave at $w_2$ to have a frequency equal to $\omega_0$; the square wave at $w_1$ to have a $\pi/\omega_0$ second delay; and both had amplitudes of $1.6\lambda$. The frequencies of the square waves were chosen to maximize the output from the LTI system (as, when the quadratic nonlinearity is given a sinusoidal input, it will output a sinusoid of twice the frequency of the incoming signal). Theorem 4 states that stability will hold for all $\lambda \in [0, 1]$. This was confirmed in simulation. We then continue to increase $\lambda$ beyond unity to see when the feedback interconnection lost stability. We found that the feedback interconnection went unstable at $\lambda = 6.51$. Given that the result of Theorem 4 is a type of nonlinear small gain stability theorem, conservatism is expected in the result. However, a factor of 6.51 is less than one order of magnitude, and therefore shows promise for the application of this work.

## B. Input-Restricted Passivity

The next application of the input-restricted stability theory developed herein is a passivity-based result in which one system is an LTI system.

When one system in the loop is an LTI system, it is often useful to define $\alpha$ and $\beta$ as functionals operating on the states of the LTI system as well as on the exogenous inputs and signals around the loop. To achieve this, the LTI system can be modified to include the states and state derivatives as outputs. The following lemma shows how this can be achieved.

*Lemma 5:* Let $\Sigma_1 : \mathcal{W}_{1e} \times \mathcal{Y}_{1e} \to \mathcal{Y}_{2e}$ and $\Sigma_2 : \mathcal{W}_{2e} \times \mathcal{Y}_{2e} \to \mathcal{Y}_{1e}$ be in a feedback interconnection as per Fig. 1, governed by (2). Let $\Sigma_2$ be an LTI system given by

$$\Sigma_2(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C & D_1 & D_2 \end{array}\right]$$

with states and state derivatives $x, \dot{x} \in \mathcal{X}_e$ and initial state $x(0) = 0$. Then, this feedback interconnection can be equivalently redrawn as Fig. 4 where

$$y_2 = \hat{\Sigma}_1(w_1, \hat{y}_1) = \Sigma_1(w_1, y_1)$$

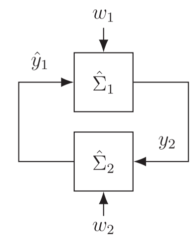


Fig. 4. Canonical feedback system redrawn as per Lemma 5.

$$\hat{y}_1 = \hat{\Sigma}_2(w_2, y_2)$$

$$\text{and } \hat{\Sigma}_2(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C & D_1 & D_2 \\ A & B_1 & B_2 \\ I & 0 & 0 \end{array}\right].$$

Furthermore, if $A$ is Hurwitz, then input-restricted stability of the feedback interconnection of $\Sigma_1$ and $\Sigma_2$ is equivalent to that of $\hat{\Sigma}_1$ and $\hat{\Sigma}_2$.

*Proof:* The first part of the result is trivial on noting that the extra outputs from $\hat{\Sigma}_2$ do not effect $\hat{\Sigma}_1$.

The second part of the result easily follows on noting that $\hat{y}_1 = \begin{bmatrix} y_1^T & \dot{x}^T & x^T \end{bmatrix}^T$ and $\|\dot{x}\|, \|x\|$ are bounded above by $\|w_2\|$, $\|y_2\|$ when $A$ is Hurwitz. ■

The following corollary applies Lemma 5 to the feedback interconnection with additive exogenous inputs shown in Fig. 2.

*Corollary 6:* Let $\Sigma_1 : u_1 \in \mathcal{W}_{1e} \mapsto y_1 \in \mathcal{W}_{2e}$ and $\Sigma_2 : u_2 \in \mathcal{W}_{2e} \mapsto y_2 \in \mathcal{W}_{1e}$ be in a feedback interconnection as per Fig. 2, described by (4). Let $\Sigma_2$ be an LTI system given by

$$\Sigma_2(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

with states and state derivatives $x, \dot{x} \in \mathcal{X}_e$ and initial state $x(0) = 0$. Then, this feedback interconnection can be equivalently recast as the feedback interconnection of $\hat{\Sigma}_1$ and $\hat{\Sigma}_2$ with $\hat{u}_1 = \hat{w}_1 + \hat{y}_2$, $u_2 = w_2 + y_1$, $y_1 = \hat{\Sigma}_1(\hat{u}_1) = \Sigma_1(\begin{bmatrix} I & 0 & 0 \end{bmatrix} \hat{u}_1)$, $\hat{y}_2 = \hat{\Sigma}_2(u_2)$ with

$$\hat{\Sigma}_2(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \\ A & B \\ I & 0 \end{array}\right]$$

where $\hat{w}_1 = \begin{bmatrix} w_1^T & 0 & 0 \end{bmatrix}^T \in \mathcal{W}_{1e} \times \mathcal{X}_e \times \mathcal{X}_e$. Furthermore, if $A$ is Hurwitz, then input-restricted stability of the feedback interconnection of $\Sigma_1$ and $\Sigma_2$ is equivalent to that of $\hat{\Sigma}_1$ and $\hat{\Sigma}_2$.

*Proof:* Trivial—Direct consequence of Lemma 5. ■

Before the passivity result can be presented, we define passivity, very strict passivity and the new concept of input-restricted passivity.

*Definition 24 (Input-restricted passive systems):* Let $\Sigma : \mathcal{W}_e \to \mathcal{Y}_e$ be a causal system mapping one extended Lebesgue space to another, and let $\mathcal{S} \subset \mathcal{W}_e$. The system $\Sigma$ is *input-restricted passive on* $\mathcal{S}$ if, $\forall w \in \mathcal{S}$ and $\tau \in \mathcal{T}$, we have

$$\langle w_\tau, (\Sigma(w))_\tau \rangle \geq 0,$$

and $\Sigma$ is *input-restricted very strictly passive on* $\mathcal{S}$ if there exists $\epsilon, \delta > 0$ such that, $\forall w \in \mathcal{S}$ and $\tau \in \mathcal{T}$, we have

$$\langle w_\tau, (\Sigma(w))_\tau \rangle \geq \epsilon \|w_\tau\|^2 + \delta \|(\Sigma(w))_\tau\|^2. \quad (15)$$

If $\mathcal{S} = \mathcal{W}_e$, instead the system is passive or very strictly passive, respectively.

We now present the input-restricted passivity result, in which one system is a passive LTI system and another is an input-restricted very strictly passive nonlinear system. What is notable in this result is that we restrict the $\mathcal{L}_2$-norm of the input while we restrict the $\mathcal{L}_\infty$-norm of the output of the LTI system.

*Theorem 5 (Input-restricted passivity):* Let $\check{\Sigma}_1 : \mathcal{W}_{1e} \to \mathcal{W}_{2e}$ be a locally Lipschitz continuous, static nonlinear system which is input-restricted very strictly passive on $(\bar{\mathcal{B}}^Q_{\mathcal{W}_1})_\infty$ for some $Q > 0$ and $\epsilon, \delta > 0$ satisfying (15). Let $\Sigma_2 : \mathcal{W}_{2e} \to \mathcal{W}_{1e}$ be a stable, passive, LTI and strictly proper system with minimal state space realisation $(A, B, C, 0)$ satisfying $A^T P + PA \leq 0$ and $PB = C^T$ for some $P = P^T \geq 0$. Let $\mathcal{W}_1$ and $\mathcal{W}_2$ (i.e., the nonextended spaces corresponding to $\mathcal{W}_{1e}$ and $\mathcal{W}_{2e}$) be both equipped with the $\mathcal{L}_\infty$- and $\mathcal{L}_2$-norms. Let $a, b > 0$ satisfy

$$b > \frac{1}{2\epsilon} \quad (16)$$

$$\begin{bmatrix} P & C^T a \\ Ca & b \end{bmatrix} \geq 0. \quad (17)$$

Then, the negative feedback interconnection of $\check{\Sigma}_1$ and $\Sigma_2$ is input-restricted stable for all $w_1 = 0$, $w_2 \in (\bar{\mathcal{B}}^M_{\mathcal{W}_2})_2$ with $M < \frac{Qa}{b}$.

*Proof:* First, we can, without loss of generality, consider the positive feedback interconnection in Fig. 2 between $\Sigma_1 = -\check{\Sigma}_1$ and $\Sigma_2$ instead of the negative feedback interconnection between $\check{\Sigma}_1$ and $\Sigma_2$.

Applying Corollary 6, we can replace $y_2$, $w_1$, $u_1$, $\Sigma_1$, and $\Sigma_2$ with $\hat{y}_2$, $\hat{w}_1$, $\hat{u}_1$, $\hat{\Sigma}_1$, and $\hat{\Sigma}_2$ such that

$$\hat{y}_2 = \begin{bmatrix} y_2 \\ \dot{x} \\ x \end{bmatrix}, \ \hat{w}_1 = \begin{bmatrix} w_1 \\ 0 \\ 0 \end{bmatrix}, \ \hat{u}_1 = \hat{w}_1 + \hat{y}_2 = \begin{bmatrix} u_1 \\ \dot{x} \\ x \end{bmatrix}$$

$$\hat{\Sigma}_1(\hat{u}_1) = \Sigma_1(\begin{bmatrix} I & 0 & 0 \end{bmatrix} \hat{u}_1) = -\check{\Sigma}_1(u_1)$$

$$\hat{\Sigma}_2(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \\ A & B \\ I & 0 \end{array}\right]$$

where $x, \dot{x} \in \mathcal{X}_e$ are the states and state derivatives of $\Sigma_2$ with $x(0) = 0$.

Since $\Sigma_2$ is stable and the realisation is minimal, $A$ is Hurwitz. Then, input-restricted stability of the interconnection of $\Sigma_1$ and $\Sigma_2$ is equivalent to input-restricted stability of the interconnection of $\hat{\Sigma}_1$ and $\hat{\Sigma}_2$ via Corollary 6. We also define $\hat{G}_1$ and $\hat{G}'_2$ as the graph and inverse graph of $\hat{\Sigma}_1$ and $\hat{\Sigma}_2$, respectively.

To prove stability of the feedback interconnection of $\hat{\Sigma}_1$ and $\hat{\Sigma}_2$, we will apply Corollary 5. We will choose the matrix $\hat{R}$ associated with the static linear mapping $R$ as $\hat{R} = \begin{bmatrix} I & 0 & 0 \end{bmatrix}$ such that $R\hat{u}_1 = u_1$, $R\hat{y}_2 = y_2$, and $R\hat{w}_1 = w_1$, and therefore $\hat{\Sigma}_1(R\hat{u}_1) = \hat{\Sigma}_1(u_1)$. We also choose $S_1 = S_2 = I$ (the identity operator) such that $S^{-1}[(\bar{\mathcal{B}}^{(M_1,M_2)}_{\mathcal{Z}})_{p_1,p_2}] = (\bar{\mathcal{B}}^{(M_1,M_2)}_{\mathcal{W}})_{p_1,p_2}$ for any $M_1, M_2, p_1$ and $p_2$.

We choose $M_1 = 0$, $M_2 = M$, $N = Q$, $p_1 = 2$, $p_2 = 2$, and $q = \infty$. Finally, we will note that $w_1 = 0, w_2 \in (\bar{\mathcal{B}}^M_{\mathcal{W}_2})_2 \iff (\hat{w}_1, w_2) \in (\bar{\mathcal{B}}^{(0,M)}_{\mathcal{W}})_{2,2}$.

For supposition 1 of Corollary 5, choose

$$\begin{aligned} &\alpha(\hat{u}_{1\tau}, y_{1\tau}) \\ &\quad = \epsilon\|u_{1\tau}\|_2^2 + \delta\|(\check{\Sigma}_1(u_1))_\tau\|_2^2 - \langle u_{1\tau}, (\check{\Sigma}_1(u_1))_\tau\rangle \\ &\quad = \epsilon\|R\hat{u}_{1\tau}\|_2^2 + \delta\|(\hat{\Sigma}_1(\hat{u}_1))_\tau\|_2^2 - \langle R\hat{u}_{1\tau}, -(\hat{\Sigma}_1(\hat{u}_1))_\tau\rangle \\ &\quad = \epsilon\|R\hat{u}_{1\tau}\|_2^2 + \delta\|y_{1\tau}\|_2^2 + \langle R\hat{u}_{1\tau}, y_{1\tau}\rangle \end{aligned}$$

which by definition of $\check{\Sigma}_1$ and $R$ is less than or equal to 0 for all $\tau \in \mathcal{T}$ and all $(\hat{u}_1, y_1) \in \hat{G}_1$ when $\|R\hat{u}_1\|_\infty \leq Q$. Hence, supposition 1 in Corollary 5 is satisfied.

For supposition 2 of Corollary 5, choose

$$\begin{aligned} \beta(\hat{y}_{2\tau}, u_{2\tau}) = \max\{&x(\tau)^T P x(\tau) - 2\langle u_{2\tau}, Cx_\tau\rangle, \\ &\|\dot{x}_\tau - Ax_\tau - Bu_{2\tau}\|_2, -\|\dot{x}_\tau - Ax_\tau - Bu_{2\tau}\|_2, \\ &\|R\hat{y}_{2\tau} - Cx_\tau\|_2, -\|R\hat{y}_{2\tau} - Cx_\tau\|_2\} \end{aligned} \quad (18)$$

where $x$ and $\dot{x}$ can be included as they are part of $\hat{y}_2$. The first entry on the right hand side is less than or equal to 0 for all $\tau \in \mathcal{T}, (\hat{y}_2, u_2) \in \hat{G}'_2$ because, for all $t \in \mathcal{T}$, $A^T P + PA \leq 0$ and $PB = C^T$ imply

$$\begin{bmatrix} x(t) \\ u_2(t) \end{bmatrix}^T \begin{bmatrix} A^T P + PA & PB - C^T \\ B^T P - C & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ u_2(t) \end{bmatrix} \leq 0$$

$$\iff \frac{\mathrm{d}}{\mathrm{d}t}\left(x(t)^T P x(t)\right) - 2u_2(t)^T C x(t) \leq 0$$

which implies that $x(\tau)^T P x(\tau) - 2\langle u_{2\tau}, Cx_\tau\rangle \leq 0$ for all $\tau \in \mathcal{T}$. Furthermore, $(\hat{y}_2, u_2) \in \hat{G}'_2 \implies Cx_\tau = y_{2\tau} = R_1\hat{y}_{2\tau}$ and $\dot{x}_\tau = Ax_\tau + Bu_{2\tau}$ for all $\tau \in \mathcal{T}$, which implies (and is implied by) $\max\{\|R\hat{y}_{2\tau} - Cx_\tau\|_2, -\|R\hat{y}_{2\tau} - Cx_\tau\|_2\} \leq 0$ and $\max\{\|\dot{x}_\tau - Ax_\tau - Bu_{2\tau}\|_2, -\|\dot{x}_\tau - Ax_\tau - Bu_{2\tau}\|_2\} \leq 0$, respectively. Hence, (18) is less than or equal to zero for all $\tau \in \mathcal{T}, (\hat{y}_2, u_2) \in \hat{G}'_2$ and therefore supposition 2 in Corollary 5 is satisfied.

For supposition 3, we first find $\tilde{\gamma}$. Suppose (16) holds. Then, there always exists some $\sigma > 0$ large enough such that

$$\begin{bmatrix} \left(\frac{1}{2\delta} - \sigma\right) I & -I \\ -I & \left(\frac{1}{2\epsilon} - b\right) I \end{bmatrix} \leq 0.$$

Applying two Schur complements and a congruence transformation to this implies that

$$\begin{bmatrix} -2\epsilon I & 0 & -2\epsilon I & I \\ 0 & -2\delta I & -I & 0 \\ -2\epsilon I & -I & -(2\epsilon + \sigma)I & 0 \\ I & 0 & 0 & -bI \end{bmatrix} \leq 0.$$

Pre- and post-multiplying this LMI by $\begin{bmatrix} x(t)^T C^T & y_1(t)^T & [R\hat{w}_1(t)]^T & w_2^T \end{bmatrix}$ and its transpose respectively and then integrating from $t = 0$ to $t = \tau$ gives

$$\begin{aligned} &-2\epsilon\|Cx_\tau + R\hat{w}_{1\tau}\|_2^2 - 2\delta\|y_{1\tau}\|_2^2 - \sigma\|R\hat{w}_{1\tau}\|_2^2 - b\|w_{2\tau}\|_2^2 \\ &- 2\langle y_{1\tau}, R\hat{w}_{1\tau}\rangle + 2\langle w_{2\tau}, Cx_\tau\rangle \leq 0 \ \forall \tau \in \mathcal{T}. \end{aligned}$$

Grouping terms and noting that supposition 3 requires $(\hat{w}_1, w_2) \in (\bar{\mathcal{B}}_{\mathcal{W}}^{(0,2)})_{2,2}$ implies

$$-2(\langle y_{1\tau}, Cx_\tau\rangle + \delta\|y_{1\tau}\|_2^2 + \epsilon\|Cx_\tau\|_2^2) - b\|w_{2\tau}\|_2^2 + 2\langle w_{2\tau} + y_{1\tau}, Cx_\tau\rangle \le 0 \; \forall \tau \in \mathcal{T}, w \in (\bar{\mathcal{B}}_{\mathcal{W}}^{(0,M)})_{2,2}. \quad (19)$$

By applying $\beta(\hat{u}_{1\tau} - \hat{w}_{1\tau}, u_{2\tau}) \le 0$ and using the relations $u_2 = w_2 + y_1$ and $R\hat{u}_1 = R\hat{w}_1 + R\hat{y}_2 = R\hat{y}_2$ when $w \in (\bar{\mathcal{B}}_{\mathcal{W}}^{(0,M)})_{2,2}$, (19) implies

$$x(\tau)^T Px(\tau) - 2(\langle R\hat{u}_{1\tau}, y_{1\tau}\rangle + \epsilon\|R\hat{u}_{1\tau}\|_2^2 + \delta\|y_{1\tau}\|_2^2) - b\|w_{2\tau}\|_2^2 \le 0 \; \forall \tau \in \mathcal{T}, w \in (\bar{\mathcal{B}}_{\mathcal{W}}^{(0,M)})_{2,2}$$

which, via $\alpha(\hat{u}_{1\tau}, y_{1\tau}) \le 0$, implies

$$x(\tau)^T Px(\tau) \le b\|w_{2\tau}\|_2^2 = b\|w_\tau\|_{2,2}^2 \quad (20)$$

for all $\tau \in \mathcal{T}, w \in (\bar{\mathcal{B}}_{\mathcal{W}}^{(0,M)})_{2,2}$.

Applying Schur's complement to (17) and pre- and post-multiplying by $x(\tau)^T$ and $x(\tau)$, respectively, gives $x(\tau)^T Px(\tau) - \frac{a^2}{b}(Cx(\tau))^T Cx(\tau) \ge 0$ for all $\tau \in \mathcal{T}$. Substituting this into (20) and again applying $Cx_\tau = R\hat{y}_{2\tau} = R\hat{u}_{1\tau}$ (which is implied by $\alpha(\hat{u}_{1\tau}, \hat{y}_{1\tau}) \le 0$ and $w_1 = 0$), (20) implies

$$(R\hat{u}_1(\tau))^T R\hat{u}_1(\tau) \le \left(\frac{b}{a}\right)^2 \|w_\tau\|_{2,2}^2$$
$$\implies \|R\hat{u}_{1\tau}\|_\infty \le \frac{b}{a}\|w_\tau\|_{2,2} = \tilde{\gamma}(\|w_\tau\|_{2,2}) \quad (21)$$

for all $\tau \in \mathcal{T}, w \in (\bar{\mathcal{B}}_{\mathcal{W}}^{(0,M)})_{2,2}$.

Note that as $\tilde{\gamma}(x) = \frac{b}{a}x$ as chosen in (21), $M < \frac{Qa}{b}$ ensures that $N = Q > \tilde{\gamma}(M)$ as required by Corollary 5.

Next we find $\gamma$, beginning with a bound on $\|u_{2\tau}\|$. Since $\alpha(\hat{u}_{1\tau}, y_{1\tau}) \le 0 \; \forall \tau \in \mathcal{T}$ and $\hat{\Sigma}_1$ is static, we can consider only the integrand of $\alpha(\hat{u}_{1\tau}, y_{1\tau})$ so that $\alpha(\hat{u}_{1\tau}, y_{1\tau}) \le 0 \; \forall \tau \in \mathcal{T}$ implies, $\forall \tau \in \mathcal{T}$,

$$(R\hat{u}_1(\tau))^T y_1(\tau) + \epsilon(R\hat{u}_1(\tau))^T(R\hat{u}_1(\tau)) + \delta y_1(\tau)^T y_1(\tau) \le 0$$
$$\implies (R\hat{u}_1(\tau))^T y_1(\tau) + \delta y_1(\tau)^T y_1(\tau) \le 0$$
$$\implies \delta|y_1(\tau)|^2 \le |R\hat{u}_1(\tau)||y_1(\tau)|$$
$$\iff \delta|y_1(\tau)| \le |R\hat{u}_1(\tau)| \text{ when } |y_1(\tau)| \ne 0.$$

Applying $u_2 = w_2 + y_1$ and (21) gives, $\forall \tau \in \mathcal{T}, w \in (\bar{\mathcal{B}}_{\mathcal{W}}^{(0,M)})_{2,2}$,

$$\delta|u_2(\tau) - w_2(\tau)| \le |R\hat{u}_1(\tau)|$$
$$\implies \delta\|u_{2\tau}\|_\infty \le \|R\hat{u}_{1\tau}\|_\infty + \|w_{2\tau}\|_\infty$$
$$\implies \|u_{2\tau}\|_\infty \le \frac{1}{\delta}\left(\tilde{\gamma}(\|w_\tau\|_{2,2}) + \|w_{2\tau}\|_\infty\right) = \gamma_2(\|w_\tau\|_{2,2\vee\infty}). \quad (22)$$

In the sequel, we denote $\|w_\tau\| = \|w_\tau\|_{2,2\vee\infty}$. Noting that $A$ is Hurwitz, $\|x_\tau\|_\infty$ and $\|\dot{x}_\tau\|_\infty$ are bounded above by $\|u_{2\tau}\|_\infty$ and hence $\exists \gamma_x, \gamma_{\dot{x}}, \tilde{\gamma}_x, \tilde{\gamma}_{\dot{x}} \in \kappa$ such that $\|x_\tau\|_\infty \le \gamma_x(\|u_{2\tau}\|_\infty) \le \tilde{\gamma}_x(\|w_\tau\|)$ and $\|\dot{x}_\tau\|_\infty \le \gamma_{\dot{x}}(\|u_{2\tau}\|_\infty) \le \tilde{\gamma}_{\dot{x}}(\|w_\tau\|) \; \forall \tau \in \mathcal{T}$ and $w \in (\bar{\mathcal{B}}_{\mathcal{W}}^{(0,M)})_{2,2}$. Combining these gains gives us, $\forall \tau \in \mathcal{T}$ and $w \in (\bar{\mathcal{B}}_{\mathcal{W}}^{(0,M)})_{2,2}$

$$\|u_\tau\|_\infty \le \tilde{\gamma}(\|w_\tau\|) + \tilde{\gamma}_x(\|w_\tau\|) + \tilde{\gamma}_{\dot{x}}(\|w_\tau\|) + \gamma_2(\|w_\tau\|) = \gamma(\|w_\tau\|)$$

where $\gamma \in \kappa$, as required by supposition 3.

Finally, from the local Lipschitz continuity of $\check{\Sigma}_1$ and $D = 0$, Lemma 1 proves that the feedback interconnection is strongly well-posed, and $0 \in \hat{G}_1$, $0 \in \hat{G}'_2$ imply that $\hat{G}_1 \cap \hat{G}'_2 = 0$. Therefore, by Corollary 5 the feedback interconnection between $\hat{\Sigma}_1$ and $\hat{\Sigma}_2$ is input-restricted stable for all inputs $w_2 \in \bar{\mathcal{B}}_{\mathcal{W}_2}^M$ with $w_1 = 0$, and therefore by Corollary 6 the positive feedback interconnection between $\Sigma_1$ and $\Sigma_2$, and therefore the negative feedback interconnection between $\Sigma_1$ and $\Sigma_2$ is input-restricted stable for all inputs $w_2 \in \bar{\mathcal{B}}_{\mathcal{W}_2}^M$ with $w_1 = 0$. ■

*Remark 15:* It is simple to see that if the nonlinear system is very strictly passive with no input restriction (i.e., $Q = \infty$), then we have $M < \infty$ and therefore the result is simply the classic passivity theorem, specialized to a static nonlinearity and an LTI system.

*Remark 16:* Note that this result holds for any positive $\delta$, analogously to how the passivity theorem holds for any positive $\delta$ and $\epsilon$ [13, p.350]. It is only $\epsilon$ which affects $\tilde{\gamma}$ in the proof, and therefore the smaller $\epsilon$ is, the larger $\tilde{\gamma}$ is, and the smaller $M$ must be for a given $Q$.

*Remark 17:* The feedback system is robust to nonzero inputs at $w_1$. Let the induced $\mathcal{L}_2$-to-$\mathcal{L}_\infty$ gain of $\Sigma_2$ be $k$. Then, if we have $\|w_2\|_2 = M - \delta$ for some $0 < \delta < M$, the system will remain input-restricted stable for some nonzero $w_1$ with $\|w_1\|_\infty \le k\delta$, as this still ensures $\|u_{1\tau}\|_\infty < N = Q$ for all $\tau \in \mathcal{T}$.

Theorem 5 shows how the theory developed herein, while complex in its most general form, can be specialized to produce stability criteria which are both simple to test (i.e., by solving an LMI) and powerful in practice. If we choose $b$ incrementally above $\frac{1}{2\epsilon}$ and maximize $a$ when solving the LMI, we in turn maximize the upper bound on $M$, thus maximizing our bound on the energy that the input $w_2$ to the system can have before the feedback interconnection becomes unstable.

Let us apply Theorem 5 in a numerical example.

*Example 2:* Choose $\Sigma_2$ to be a stable, passive, SISO, LTI, and strictly proper system with minimal state space realization $(A, B, C, 0)$ where

$$A = \begin{bmatrix} -0.7 & 1.4 & 0.5 \\ -1.4 & -0.8 & -1.3 \\ -0.6 & 1.2 & -0.7 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 0.7 & 0 \end{bmatrix}. \quad (23)$$

Choose $\check{\Sigma}_1$ to be a SISO, continuous, piecewise linear, static mapping from $u_1 \in \mathcal{W}_{1e}$ to $y_1 \in \mathcal{W}_{2e}$ given by (24) and depicted in Fig. 5 , which we will denote by $\Phi$ as follows:

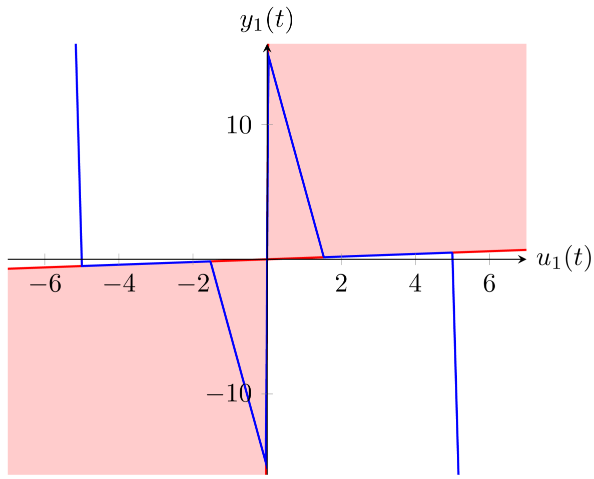


Fig. 5. Input–output relationship for $\Phi$. The blue line is the graph of $\Phi$ and the red shaded area includes the graphs of all static very strictly passive systems for $\epsilon > 0.099$ and $\delta > 0.00198$. The solid red lines are the solutions to $u_1(t)y_1(t) = 0.099u_1(t)^2 + 0.00198y_1(t)^2$.

$$y_1(t) = (\Phi u_1)(t) = \begin{cases} 505u_1(t) & \text{for } -0.03 < u_1(t) < 0.03 \\ 15.45 - 10u_1(t) & \text{for } 0.03 \le u_1(t) < \frac{15.45}{10.1} \\ -15.45 - 10u_1(t) & \text{for } -\frac{15.45}{10.1} < u_1(t) \le -0.03 \\ 0.1u_1(t) & \text{for } \frac{15.45}{10.1} \le |u_1(t)| < 5 \\ 500.5 - 100u_1(t) & \text{for } u_1(t) \ge 5 \\ -500.5 - 100u_1(t) & \text{for } u_1(t) \le -5. \end{cases} \tag{24}$$

It is easy to verify that $\Phi$ is locally very strictly passive on $(\bar{\mathcal{B}}^5_{\mathcal{W}_2})_\infty$ with $\epsilon = 0.099$ and $\delta = 0.00198$ by noting that the lines $y_1(t) = 505u_1(t)$, $\pm 15.45 - 10u_1(t)$ and $y_1(t) = 0.1u_1(t)$ satisfy $u_1(t)y_1(t) - \epsilon u_1(t)^2 - \delta y_1(t)^2 \ge 0$ for all $u_1(t)$ in the associated domains. Hence both $\check{\Sigma}_1$ and $\Sigma_2$ satisfy the prerequisites for applying Theorem 5.

By setting $b$ incrementally above $\frac{1}{2\epsilon}$ and maximizing $a$ as discussed earlier, we minimize $\gamma_2(x) = \frac{b}{a}x$ and in turn maximize the upper bound on $M$. Doing so gives this upper bound to be $\frac{Qa}{b} = \frac{5\times 1.8993}{5.0505} = 1.88$ rounded down to two decimal places. Hence, this feedback interconnection is input-restricted stable for all inputs $w_2 \in \bar{\mathcal{B}}^{1.88}_{\mathcal{W}_2}$ with $w_1 = 0$.

Simulations of this feedback interconnection reveal that the system becomes unstable when given a ramp input with gradient 118.1 and length of 0.3s at $w_1$. This signal has an $\mathcal{L}_2$-norm of 6.47, which is 3.44 times greater than the upper bound on $M$ given in this example. While this shows some expected conservatism in the value of $M$ given, it is just slightly larger than $M$ which is encouraging for the practical application of this theory.

## VI. Conclusion

In this article, the concept of input-restricted stability over a set was introduced, which determines whether a system is stable (i.e., there exists a gain relationship between the norms of the truncated system inputs and outputs) when subjected to inputs within the given restricted set. Graph separation concepts and system continuity were employed to develop Theorem 1, in which sufficient conditions on the open-loop systems guarantee input-restricted stability of a feedback interconnection of two nonlinear systems over a ball of exogenous input signals. Various corollaries of Theorem 1, and theorems extending the result, were also developed. Notable in these extensions are Theorems 2 and 3, along with their corollaries, which allow for the inputs and outputs to be restricted to a set other than a ball. This article concludes with two specializations of Corollaries 4 and 5, giving: an input-restricted small-gain theorem for a feedback interconnection comprising a nonlinear system with a linear gain bound and a nonlinear system with a quadratic gain bound; and an input-restricted passivity theorem for a feedback interconnection comprising a passive LTI system and a static nonlinearity, which is very strictly passive on a restricted set of input signals, respectively. These specializations were then applied to numerical examples which produced an upper bound on the energy of the input signal for which input-restricted stability is guaranteed if said bound is not violated. It is notable that both of these feedback interconnections are not globally stable as per traditional definitions of input–output stability [6], [7].

## Appendix

This Appendix includes proofs to Lemma 2 and to the main theorems presented in this article.

### A. Proof of Lemma 2

*Proof: Step 1:* First, we define a locally continuous path between the origin of $\mathcal{Y}_e$ and $\Sigma(w)$ for some $w \in \mathcal{W}_e$. Define a function $f : [0,1] \times \mathcal{W}_e \to \mathcal{Y}_e$ as

$$f[\lambda, w] = \Sigma(\lambda w).$$

We then define $f^\star[\lambda] = f[\lambda, w^\star]$ where $0 \ne w^\star \in \mathcal{W}_e$ is now fixed, rather than a variable of $f$.

Next, we prove that $f^\star$ is a locally continuous mapping from $\lambda \in [0,1]$ to $y = \Sigma(w^\star) \in \mathcal{Y}_e$. Let $y^{(i)} = \Sigma(w^{(i)})$ for $i \in \{1,2\}$. By local continuity of $\Sigma$, we have

$$\forall w^{(2)} \in \mathcal{W}_e, \epsilon > 0, \tau \in \mathcal{T}, \exists \delta > 0 \text{ such that, for } w^{(1)} \in \mathcal{W}_e,$$
$$\|P_\tau(w^{(1)} - w^{(2)})\| < \delta \Rightarrow \|P_\tau(y^{(1)} - y^{(2)})\| < \epsilon$$
$$\Rightarrow \forall \lambda_2 \in [0,1], \epsilon > 0, \tau \in \mathcal{T}, \exists \delta > 0 \text{ such that, for } \lambda_1 \in [0,1],$$
$$\|P_\tau(\lambda_1 w^\star - \lambda_2 w^\star)\| < \delta \Rightarrow \|P_\tau(f^\star[\lambda_1] - f^\star[\lambda_2])\| < \epsilon$$
$$\Rightarrow \forall \lambda_2 \in [0,1], \epsilon > 0, \tau \in \mathcal{T}, \exists \delta > 0 \text{ such that, for } \lambda_1 \in [0,1],$$
$$|\lambda_1 - \lambda_2| < \delta \Rightarrow \|P_\tau(f^\star[\lambda_1] - f^\star[\lambda_2])\| < \epsilon.$$

Hence, the mapping from $\lambda$ to $P_\tau(f^\star[\lambda])$ is locally continuous in [0,1] for all $\tau \in \mathcal{T}$, and therefore $f^\star$ is a locally continuous path between $f^\star[0] = \Sigma(0) = 0$ and $f^\star[1] = \Sigma(w^\star)$.

*Step 2:* We now show that, for $w^\star \in \mathcal{S}$, local continuity of $P_\tau(f^\star[\lambda])$ implies that $P_\tau(f^\star[\lambda]) \in \mathcal{Q}$ for all $\lambda \in [0,1], \tau \in \mathcal{T}$, and hence $P_\tau(\Sigma(w)) \in \mathcal{Q}\ \forall w \in S, \tau \in \mathcal{T}$.

We begin by fixing $w^\star \in \mathcal{S}$. From the star shapedness of $\mathcal{S}$, we infer that $\lambda w^\star \in \mathcal{S}\ \forall \lambda \in [0,1]$. From our original suppositions, we can then say that

$$P_\tau(f^\star[\lambda]) \notin \partial\mathcal{Q}\ \forall \lambda \in [0,1], \tau \in \mathcal{T}. \tag{25}$$

Let us now assume there exists some $\tau \in \mathcal{T}$ and $\lambda_1, \lambda_2 \in [0,1]$, where without loss of generality $\lambda_1 < \lambda_2$, such that $P_\tau(f^\star[\lambda_1]) \in \mathcal{Q}$ and $P_\tau(f^\star[\lambda_2]) \notin \bar{\mathcal{Q}}$. By our assumption that

$\mathcal{Q}$ is open, we have $\partial\mathcal{Q} \cap \mathcal{Q} = \emptyset$, and as such this ensures that $P_\tau(f^\star[\lambda_1]), P_\tau(f^\star[\lambda_2]) \notin \partial\mathcal{Q}$ as required by (25).

By local continuity of $P_\tau(f^\star[\lambda])$ in $\lambda$, there exists a $\theta \in (0,1)$ such that $P_\tau(f^\star[(1-\theta)\lambda_1 + \theta\lambda_2]) \in \partial\mathcal{Q}$, leading to a contradiction in (25). Therefore, our assumption that $P_\tau(f^\star[\lambda_1]) \in \mathcal{Q}$ and $P_\tau(f^\star[\lambda_2]) \notin \bar{\mathcal{Q}}$ at this value of $\tau$ contradicts local continuity of $P_\tau(f^\star[\lambda])$ in $\lambda$ and as such cannot hold. This proves we must either have

$$P_\tau(f^\star[\lambda]) \in \mathcal{Q}\ \forall\lambda \in [0,1], \tau \in \mathcal{T} \text{ or}$$
$$P_\tau(f^\star[\lambda]) \notin \bar{\mathcal{Q}}\ \forall\lambda \in [0,1], \tau \in \mathcal{T}. \tag{26}$$

By the assumption that $\Sigma(0) = 0$, $P_\tau(f^\star[0]) = 0$ for all $\tau \in \mathcal{T}$ and any $w^\star \in \mathcal{S}$. As $P_\tau(f^\star[0]) = 0 \in \mathcal{Q}$, (26) ensures that $P_\tau(f^\star[\lambda]) \in \mathcal{Q}\ \forall\lambda \in [0,1], \tau \in \mathcal{T}$ and $w^\star \in \mathcal{S}$. Finally, since $f^\star[1] = \Sigma(w^\star)$, it follows that $w \in \mathcal{S} \implies P_\tau(\Sigma(w)) \in \mathcal{Q}\,\forall\tau \in \mathcal{T}$. ■

### B. Proof of Theorem 1

*Proof: Step 1:* We begin by showing that, for any $w \in \bar{\mathcal{B}}^M_{\mathcal{W}}$, the resulting solution $y$ to the feedback interconnection described by (2) and shown in Fig. 1 satisfies $y_\tau \notin \partial\mathcal{B}^N_{\mathcal{Y}}\ \forall\tau \in \mathcal{T}$.

First, we define the set $\Gamma$ as

$$\Gamma = \left\{ (\tau, w, y) \in \mathcal{T} \times \bar{\mathcal{B}}^M_{\mathcal{W}} \times \mathcal{Y}_e \;\middle|\; \begin{array}{l} y = G_1(w_1) \cap G_2'(w_2), \\ y_\tau \in \bar{\mathcal{B}}^N_{\mathcal{Y}} \end{array} \right\}$$

This is the set of all truncation times $\tau$ and all inputs $w$ in the ball of radius $M$ which give an output $y$ of the closed-loop system such that the truncation of $y$ at time $\tau$ is in the ball of radius $N$.

Suppositions 1 and 2 together give

$$\alpha(w_\tau, y_\tau) \leq 0,\ \beta(w_\tau, y_\tau) \leq 0\ \forall(\tau, w, y) \in \Gamma.$$

Supposition 3 then gives that $\|y_\tau\| < N\ \forall(\tau, w, y) \in \Gamma$.

Given $w \in \bar{\mathcal{B}}^M_{\mathcal{W}}$, there is some $y$ which is a solution to (2), i.e., $y = G_1(w_1) \cap G_2'(w_2)$, due to well-posedness. Let us assume that, for some $\tau \in \mathcal{T}$, we have $\|y_\tau\| = N$. This means that, for such a $\tau$, $y_\tau \in \bar{\mathcal{B}}^N_{\mathcal{Y}}$, giving $(\tau, w, y) \in \Gamma$ which in turn (via suppositions 1–3) implies that $\|y_\tau\| < N$. This is a contradiction to our initial assumption, and therefore we must infer that we cannot have a $y$ which is a solution to (2) with input $w \in \bar{\mathcal{B}}^M_{\mathcal{W}}$ and $\|y_\tau\| = N$ for any $\tau \in \mathcal{T}$. This is equivalent to

$$[P_\tau G_1(w_1)] \cap [P_\tau G_2'(w_2)] \cap \partial\mathcal{B}^N_{\mathcal{Y}} = \emptyset$$

for all $\tau \in \mathcal{T}, w \in \bar{\mathcal{B}}^M_{\mathcal{W}}$. This is also equivalent to stating that for any $w \in \bar{\mathcal{B}}^M_{\mathcal{W}}$, the solution $y$ to (2) satisfies $y_\tau \notin \partial\mathcal{B}^N_{\mathcal{Y}}\ \forall\tau \in \mathcal{T}$.

*Step 2:* In this step, we apply Lemma 2 to prove that $w \in \bar{\mathcal{B}}^M_{\mathcal{W}} \implies y_\tau \in \mathcal{B}^N_{\mathcal{Y}}\ \forall\tau \in \mathcal{T}$.

By strong well-posedness of the feedback interconnection, $G_1(w_1) \cap G_2'(w_2)$ is a unique point for a given $w \in \mathcal{W}_e$, and hence we define $\Sigma$ from Lemma 2 as $\Sigma(w) = G_1(w_1) \cap G_2'(w_2)$. Also by strong well-posedness, this mapping is locally continuous as is required by Lemma 2, and the assumption that $G_1(0) \cap G_2'(0) = 0$ gives $\Sigma(0) = 0$ as required. Define $\mathcal{S}$ from Lemma 2 as $\mathcal{S} = \bar{\mathcal{B}}^M_{\mathcal{W}}$, which is star shaped as required, and define $\mathcal{Q}$ from Lemma 2 as $\mathcal{Q} = \mathcal{B}^N_{\mathcal{Y}}$, which is open and star shaped as required. Finally, in Step 1 it was proved that, via suppositions 1–3, $w \in \mathcal{S} \implies y_\tau \notin \partial\mathcal{Q}\ \forall\tau \in \mathcal{T}$ where $y = \Sigma(w)$. Hence, by invoking Lemma 2, it follows that $w \in \mathcal{S} = \bar{\mathcal{B}}^M_{\mathcal{W}} \implies y_\tau \in \mathcal{Q} = \mathcal{B}^N_{\mathcal{Y}}\ \forall\tau \in \mathcal{T}$ where $y = G_1(w_1) \cap G_2'(w_2)$ is the solution to the feedback interconnection with input $w$.

*Step 3:* By one final implication of suppositions 1–3 we have $\|y_\tau\| \leq \gamma(\|w_\tau\|)\ \forall\tau \in \mathcal{T}$ and $w \in \bar{\mathcal{B}}^M_{\mathcal{W}}$. Hence, by definition, the system is input-restricted stable for all inputs in $\bar{\mathcal{B}}^M_{\mathcal{W}}$. ■

### C. Proof of Theorem 2

*Proof:* The proof of Theorem 2 is analogous to the proof of Theorem 1.

*Step 1:* As with Theorem 1, we begin by showing that, for any $w \in S^{-1}[(\bar{\mathcal{B}}^M_{\mathcal{Z}})_p]$, the resulting solution $y$ to the feedback interconnection described by (2) and shown in Fig. 1 that satisfies suppositions 1–3 also satisfies $y_\tau \notin \partial(R^{-1}[(\mathcal{B}^N_{\mathcal{X}})_q]) \cap \mathcal{Y}\ \forall\tau \in \mathcal{T}$.

First, we notice that $\|R(y_\tau)\|_q \leq N\ \ \forall\tau \in \mathcal{T} \iff y_\tau \in R^{-1}[(\bar{\mathcal{B}}^N_{\mathcal{X}})_q] \cap \mathcal{Y}\ \forall\tau \in \mathcal{T}$ and similarly $\|R(y_\tau)\|_q < N\ \forall\tau \in \mathcal{T} \iff y_\tau \in R^{-1}[(\mathcal{B}^N_{\mathcal{X}})_q] \cap \mathcal{Y}\ \forall\tau \in \mathcal{T}$ (as $y \in \mathcal{Y}_e \implies y_\tau \in \mathcal{Y}\ \forall\tau \in \mathcal{T}$ by definition).

Next, we define the set $\Gamma$ as

$$\Gamma = \left\{ (\tau, w, y) \;\middle|\; \begin{array}{l} (\tau, w, y) \in \mathcal{T} \times S^{-1}[(\bar{\mathcal{B}}^M_{\mathcal{W}})_p] \times \mathcal{Y}_e, \\ y = G_1(w_1) \cap G_2'(w_2), \\ y_\tau \in R^{-1}[(\bar{\mathcal{B}}^N_{\mathcal{X}})_q] \cap \mathcal{Y} \end{array} \right\}$$

which is analogous to the set $\Gamma$ from Theorem 1 with the space of $w$ modified by $S^{-1}$ and the restriction on $y_\tau$ altered to include $R^{-1}$. Following similar reasoning, $(\tau, w, y) \in \Gamma \implies y = G_1(w_1) \cap G_2(w_2)$ and $\|R(y_\tau)\|_q \leq N$. Then, applying suppositions 1 to 3 gives

$$\|R(y_\tau)\|_q < N\ \forall(\tau, w, y) \in \Gamma$$

which implies that $y_\tau \in R^{-1}[(\mathcal{B}^N_{\mathcal{X}})_q] \cap \mathcal{Y}\,\forall(\tau, w, y) \in \Gamma$.

Continuing to follow the reasoning given in the proof of Theorem 1, it can now be seen that if $w \in S^{-1}[(\bar{\mathcal{B}}^M_{\mathcal{Z}})_p]$ there exists no $y$ which is a solution to the system such that $\|R(y_\tau)\|_q = N$ for some $\tau \in \mathcal{T}$ as this would violate the relationship $(\tau, w, y) \in \Gamma \implies \|R(y_\tau)\|_q < N$. Hence, for $w \in S^{-1}[(\bar{\mathcal{B}}^M_{\mathcal{Z}})_p]$, the solution $y$ to the system satisfies $y_\tau \notin \partial(R^{-1}[(\mathcal{B}^N_{\mathcal{X}})_q] \cap \mathcal{Y})\ \forall\tau \in \mathcal{T}$ as $R^{-1}[(\mathcal{B}^N_{\mathcal{X}})_q] \cap \mathcal{Y}$ is open as per Lemma 4.

*Step 2:* As in Step 2 of the proof of Theorem 1, in this step we apply Lemma 2 to prove that $w \in S^{-1}[(\bar{\mathcal{B}}^M_{\mathcal{Z}})_p] \implies y_\tau \in R^{-1}[(\mathcal{B}^N_{\mathcal{X}})_q] \cap \mathcal{Y}\ \forall\tau \in \mathcal{T}$.

Strong well-posedness of the feedback interconnection allows us to define $\Sigma$ from Lemma 2 as $\Sigma(w) = G_1(w_1) \cap G_2'(w_2)$. The addition of local continuity of $R$ from Lemma 3, ensures this mapping is locally continuous as is required by Lemma 2, and $G_1(0) \cap G_2'(0) = 0 \implies \Sigma(0) = 0$ as required. Define $\mathcal{S} = S^{-1}[(\bar{\mathcal{B}}^M_{\mathcal{Z}})_p]$, which is trivially star shaped [similar to the argument near (3)] as required, and $\mathcal{Q} = R^{-1}[(\mathcal{B}^N_{\mathcal{X}})_q] \cap \mathcal{Y}$, which, by Lemma 4, is open and star shaped as required. Then, the result from Step 1 can be rewritten as $w \in \mathcal{S} \implies y_\tau \notin \partial\mathcal{Q}\ \forall\tau \in \mathcal{T}$ where $y = \Sigma(w)$. Hence, by invoking Lemma 2, it is shown that $w \in \mathcal{S} = S^{-1}[(\bar{\mathcal{B}}^M_{\mathcal{Z}})_p] \implies y_\tau \in \mathcal{Q} = R^{-1}[(\mathcal{B}^N_{\mathcal{X}})_q] \cap \mathcal{Y}\,\forall\tau \in \mathcal{T}$ where $y = G_1(w_1) \cap G_2'(w_2)$ is the solution to the feedback interconnection with input $w$.

*Step 3:* By one final implication of suppositions 1–3 we have $\|y_\tau\| \leq \gamma(\|w_\tau\|)\ \forall \tau \in \mathcal{T}$ and $w \in S^{-1}[(\bar{\mathcal{B}}^M_{\mathcal{Z}})_p]$. Hence, the system is input-restricted stable for all inputs in $S^{-1}[(\bar{\mathcal{B}}^M_{\mathcal{Z}})_p]$. ■

## D. Proof of Theorem 3

*Proof:* Let $\hat{R}_i$ and $\hat{S}_i$ be the associated matrices for the static linear mappings $R_i$ and $S_i$, respectively. Once again, the proof of Theorem 3 follows a similar trajectory as the proofs of the previous stability theorems.

First, define the set $\Gamma$ as

$$\Gamma = \left\{ (\tau, w, u) \left| \begin{array}{l} (\tau, w, u) \in \mathcal{T} \times S^{-1}[(\bar{\mathcal{B}}^{(M_1,M_2)}_{\mathcal{Z}})_{p_1,p_2}] \times \mathcal{W}_e, \\ (u_1, u_2 - w_2) \in G_1, \\ (u_1 - w_1, u_2) \in G'_2, \\ u_\tau \in R^{-1}[(\bar{\mathcal{B}}^{(N_1,N_2)}_{\mathcal{X}})_{q_1,q_2}] \cap \mathcal{W} \end{array} \right. \right\}.$$

Note that the requirements $(u_1, u_2 - w_2) \in G_1$ and $(u_1 - w_1, u_2) \in G'_2$ are precisely the restrictions imposed by (4), and hence these requirements in $\Gamma$ taken together ensure that, by well-posedness, for any $(\tau, w, u) \in \Gamma$, $u$ is the unique solution to the feedback interconnection given by (4) to the input $w$. Hence, $(\tau, w, u) \in \Gamma$, $u_1 = w_1 + y_2$ and $u_2 = w_2 + y_1 \implies (u_1, y_1) \in G_1$ and $(u_2, y_2) \in G_2$.

Analogously to Step 1 of the proof of Theorem 2, applying suppositions 1 through 3 now gives $\|R_1(u_{1\tau})\|_{q_1} < N_1, \|R_2(u_{2\tau})\|_{q_2} < N_2\ \forall (\tau, w, u) \in \Gamma$.

By following a similar argument as posed in Step 1 of the proof of Theorem 1, it is clear that given $w \in S^{-1}[(\bar{\mathcal{B}}^{(M_1,M_2)}_{\mathcal{Z}})_{p_1,p_2}]$ there exists no $\tau \in \mathcal{T}$ such that $\|R_1(u_{1\tau})\|_{q_1} = N_1$ or $\|R_2(u_{2\tau})\|_{q_2} = N_2$ where $u \in \mathcal{W}_e$ is the solution to the feedback (4), as this would violate the relationship $(\tau, w, u) \in \Gamma \implies \|R_1(u_{1\tau})\|_{q_1} < N_1, \|R_2(u_{2\tau})\|_{q_2} < N_2$. Hence, as with Theorem 1, we have $u_{1\tau} \notin \partial(R_1^{-1}[(\mathcal{B}^{N_1}_{\mathcal{X}_1})_{q_1}] \cap \mathcal{W}_1)$ and $u_{2\tau} \notin \partial(R_2^{-1}[(\mathcal{B}^{N_2}_{\mathcal{X}_2})_{q_2}] \cap \mathcal{W}_2)\ \forall \tau \in \mathcal{T}$, where $u$ is the solution to (4) for a given input $w \in S^{-1}[(\bar{\mathcal{B}}^{(M_1,M_2)}_{\mathcal{Z}})_{p_1,p_2}]$.

Having found a region in $\mathcal{W}$ which is infeasible for $u_\tau$ for all $\tau \in \mathcal{T}$ when $w \in S^{-1}[(\bar{\mathcal{B}}^{(M_1,M_2)}_{\mathcal{Z}})_{p_1,p_2}]$, similarly to the proof for Theorem 1 we can apply Lemma 3 by choosing $\mathcal{S} = S^{-1}[(\bar{\mathcal{B}}^{(M_1,M_2)}_{\mathcal{Z}})_{p_1,p_2}]$, $\mathcal{Q} = R^{-1}[(\mathcal{B}^{(N_1,N_2)}_{\mathcal{X}})_{q_1,q_2}] \cap \mathcal{W}$ and $\Sigma(w) = u$ where $u_1 = \Sigma_2(u_2) + w_1$ and $u_2 = \Sigma_1(u_1) + w_2$ as per (4), i.e. $u$ is the solution to the feedback interconnection with input $w$. These choices ensure that $\mathcal{S}$ is star shaped and that $\mathcal{Q}$ is open and star shaped (by application of Lemma 4). They also ensure that $u = \Sigma(w)$ is unique (due to well-posedness), $\Sigma$ is a locally continuous mapping (due to strong well-posendess and Lemma 3), $\Sigma(0) = 0$ (due to the assumption that $G_1 \cap G'_2 = 0$), and $w \in \mathcal{S} \implies P_\tau(\Sigma(w)) \notin \partial\mathcal{Q}\ \ \forall \tau \in \mathcal{T}$ by the product rule of boundaries.[2] We are then able to apply Lemma 3, which gives $w \in \mathcal{S} = S^{-1}[(\bar{\mathcal{B}}^{(M_1,M_2)}_{\mathcal{Z}})_{p_1,p_2}] \implies u_\tau \in \mathcal{Q} = R^{-1}[(\mathcal{B}^{(N_1,N_2)}_{\mathcal{X}})_{q_1,q_2}] \cap \mathcal{W}\ \ \forall \tau \in \mathcal{T}$. Then, $\forall \tau \in \mathcal{T}, w \in S^{-1}[(\bar{\mathcal{B}}^{(M_1,M_2)}_{\mathcal{Z}})_{p_1,p_2}]$, one last implication of suppositions 1–3 gives $\|u_\tau\| \leq \gamma(\|w_\tau\|)\ \ \forall \tau \in \mathcal{T}$ and $w \in S^{-1}[(\bar{\mathcal{B}}^{(M_1,M_2)}_{\mathcal{Z}})_{p_1,p_2}]$. Hence, the system is input-restricted stable for all inputs in $S^{-1}[(\bar{\mathcal{B}}^{(M_1,M_2)}_{\mathcal{Z}})_{p_1,p_2}]$. ■

[2]For some $\mathcal{S}_1 \subset \mathcal{Y}_1, \mathcal{S}_2 \subset \mathcal{Y}_2$, the product rule of set boundaries states $\partial(\mathcal{S}_1 \times \mathcal{S}_2) = (\partial\mathcal{S}_1 \times \mathcal{S}_2) \cup (\mathcal{S}_1 \times \partial\mathcal{S}_2)$, and hence $y_1 \notin \partial\mathcal{S}_1$ and $y_2 \notin \partial\mathcal{S}_2$ implies $y \notin \partial(\mathcal{S}_1 \times \mathcal{S}_2)$.

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