Feedback Stability of Negative Imaginary Systems

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Abstract—This paper extends the robust feedback stability theorem of negative imaginary systems by removing restrictive assumptions on the instantaneous gains of the systems that were imposed in the earlier literature, and it further generalizes this robust analysis result into the case that allows negative imaginary systems to have poles at the origin. In doing so, we extend the class of negative imaginary systems for which this robust stability theorem is applicable. We also show that this new generalized necessary and sufficient result specializes to the earlier theorems under the same assumptions. We additionally prove that the previously known dc gain condition is not only necessary and sufficient for robust feedback stability under the earlier specified instantaneous gain assumptions, but is also necessary and sufficient for robust feedback stability under new, different and equally simple assumptions. The general robust feedback stability theorem for negative imaginary systems with free body dynamics (i.e., poles at the origin) derived in this paper also specializes to the case that is only applicable for the negative imaginary system without poles at the origin. Since the results for negative imaginary systems with free body dynamics developed in this paper depend on the existence of a matrix Ψ with certain properties, we also propose a systematic construction of this matrix Ψ and show that construction of one such Ψ is sufficient for determining the feedback stability of the closed-loop system. Finally, examples are used to demonstrate the applicability of the results.

Index Terms—Feedback stability, negative imaginary (NI) systems, robust control.

I. INTRODUCTION

NEGATIVE imaginary (NI) systems theory was introduced in [1]. In a single-input single-output (SISO) frequency domain setting, the NI notion broadly means that the positive frequency branch of the Nyquist plot has an imaginary part that is negative. In a SISO time-domain setting, the NI notion broadly means that the output of a system follows but lags behind any sinusoidal input to the system by not more than $180°$. In contrast to positive realness, the definitions of NI systems impose weaker restrictions on the relative degree of the transfer function and do not exclude all unstable zeros [2]. The original development
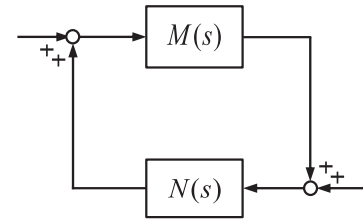
Fig. 1. Positive feedback interconnection.

of NI theory was motivated by inertial systems, all of which can be described by Newton's second law of motion, whose actuation is provided by a force or torque input and whose sensing is obtained through a co-located position or angle output measurement [1], [3]. NI systems theory was found to be suitable for a wide variety of applications including nanopositioning control due to piezoelectric transducers and capacitive sensors (e.g., [4]–[6]) and in multi-agent networked systems (e.g., [7]–[9]). The notion of NI systems also specializes to the subclass of lossless systems [10].

In view of the prevalence of NI systems in different domains, robust stability of the positive feedback interconnection of an NI system $M(s)$ and a strictly negative imaginary (SNI) system $N(s)$ was investigated in [1]. Under the assumptions that the instantaneous gain of the SNI system $N(s)$ is positive semidefinite, i.e., $N(\infty) \geq 0$, and the open-loop transfer function is strictly proper, i.e., $M(\infty)N(\infty) = 0$, Lanzon and Petersen [1] showed that the feedback interconnection in Fig. 1 denoted by $[M(s), N(s)]$ is internally stable if and only if the dc loop gain condition $\bar{\lambda}[M(0)N(0)] < 1$ is satisfied, where $\bar{\lambda}[\cdot]$ denotes the maximum eigenvalue for a matrix with only real eigenvalues. Note that $\bar{\lambda}[\cdot] < 1$ is a one-sided restriction, as $\bar{\lambda}[\cdot]$ can be either positive or negative. This result was also found to hold true in [11] when the definition of an NI system was extended to allow for poles on the imaginary axis but not at the origin. Subsequent modifications in [12] also allow for poles at the origin (under some restrictive assumptions) but this relaxation makes the result significantly more complicated and hence less transparent. These stability conditions are robust in the sense that the internal stability conclusion is invariant to NI perturbations provided that the aforementioned dc and instantaneous gain conditions are preserved (see [2, Th. 6] for NI perturbations that preserve the class).

Despite many developments in NI systems theory including, for example, extensions or connections to infinite-dimensional systems [13], Hamiltonian systems [14], descriptor systems

[15], non-rational systems [2], [16], discrete-time systems [17], finite frequency NI systems [18], mixtures of small-gain and NI properties [19], and NI control synthesis [20]–[22], the aforementioned robust feedback stability theorem has remained unchanged since [1] and that theorem underpins the motivation and application of all the above research. When the assumptions of the feedback stability theorem in [1] and [11] do not hold, such as $N(\infty)$ may be sign-indefinite, the dc loop gain $\bar{\lambda}[M(0)N(0)] < 1$ may not be an appropriate necessary and sufficient test for internal stability of the feedback interconnection. Very recently, Khong *et al.* [23] have attempted to relax these assumptions for NI systems without poles at the origin by invoking integral quadratic constraint (IQC) theory. A pair of complementary IQCs was introduced at dc and infinite frequency to attempt to relax the aforementioned assumptions. However, the methods of Khong *et al.* [23] only yield a conservative sufficient condition as will be illustrated in Section VI. Unlike the sufficient conditions of [23], this paper directly derives necessary and sufficient conditions by building on the work of [1] and [11] and we handle the complete class of NI systems with or without poles at the origin and on the imaginary axis. In doing so, we demonstrate that the mechanism to remove these assumptions is not the IQC theory. As indicated above, the conditions in [12] are derived for NI systems that allow for poles on the imaginary axis and at the origin. However, the conditions obtained in [12] are highly technical, require computation of several matrix factorizations that make the results less intuitive than earlier work and impose a number of additional assumptions for the results to hold amongst which is the restriction that the NI system has to be strictly proper. This paper also generalizes the work in [12] by relaxing the aforementioned assumptions and provides simple necessary and sufficient conditions that invoke directly the steady-state and instantaneous gains of the systems, thereby avoiding cumbersome matrix factorizations and hence making the results significantly more intuitive.

The main contributions of this paper are as follows:
1) removal of the restrictive assumptions in the robust analysis theorem that underpins all NI systems theory;
2) derivation of new necessary and sufficient conditions that reduce to the earlier necessary and sufficient conditions under the same assumptions imposed in the literature;
3) derivation of a new set of simple and easy-to-check assumptions under which the intuitive dc gain condition is both necessary and sufficient;
4) specialization of the results in a SISO setting that illustrates a number of tests not easily explained in a multiple-input multiple-output (MIMO) setting;
5) demonstration that in certain scenarios a mixture of dc and infinite frequency gains are essential to characterize the necessary and sufficient conditions for robust feedback stability (this is counter intuitive);
6) derivation of generalized necessary and sufficient robust feedback stability conditions that can be applied to NI systems having poles at the origin;

7) demonstration that the matrix Ψ, satisfying certain properties, used in the two main stability theorems always exists and we provide a systematic construction for this matrix Ψ;
8) demonstration that the conditions that involve the matrix Ψ are either satisfied for the entire set of matrices Ψ that fulfil certain properties or violated for the entire same set of matrices Ψ. This means one only needs to check the conditions on one Ψ to determine the conclusion (i.e., there is no need for searching across the set of possible matrices Ψ);
9) unification of all robust feedback stability results that appeared in all earlier literature to date into one general theory; and
10) illustration via two numerical examples of either the inapplicability or conservativeness of earlier results and correspondingly the usefulness of the derived general theory.

Notation: $\mathscr{RH}_\infty$ denotes the set of real, rational, stable transfer function matrices. $\mathbb{R}^{m \times n}$ and $\mathbb{C}^{m \times n}$ denote real and complex matrices that have n columns and m rows, respectively. $\lambda_i(A)$ denotes the ith eigenvalue of a square complex matrix A. $\bar{\lambda}(A)$ and $\underline{\lambda}(A)$ denote the largest and smallest eigenvalue of a square complex matrix A that has only real eigenvalues. A^* and A^T denote the complex conjugate transpose and transpose of a complex matrix A, respectively. A^{-*} and $G^\sim(s)$ are shorthand for $(A^{-1})^*$ and $G(-s)^T$, respectively. $\mathfrak{R}(a)$ denotes the real part of a complex number a. Let $\ker(A)$ denote the kernel of matrix A. A square transfer function matrix $G(s)$ is said to have full normal rank if $\exists s \in \mathbb{C} : \det(G(s)) \neq 0$.

II. PRELIMINARY LEMMAS

First, let us recall the definitions of NI systems in Definitions 1 and 2.

Definition 1 ([12]): Let $R : \mathbb{C} \longrightarrow \mathbb{C}^{m \times m}$ be a real, rational, proper transfer function. Then, $R(s)$ is said to be NI if
1) $R(s)$ has no poles in $\mathfrak{R}(s) > 0$;
2) $j[R(j\omega) - R(j\omega)^*] \geq 0$ for all $\omega \in (0, \infty)$ except the values of ω where $j\omega$ is a pole of $R(s)$;
3) if $j\omega_0$ with $\omega_0 > 0$ is a pole of $R(s)$, then it is at most a simple pole and the residue matrix $K_0 = \lim_{s \to j\omega_0} (s - j\omega_0) jR(s)$ is Hermitian and positive semidefinite;
4) if $s = 0$ is a pole of $R(s)$, then $\lim_{s \to 0} s^k R(s) = 0$ for all integer $k \geq 3$ and $\lim_{s \to 0} s^2 R(s)$ is Hermitian and positive semidefinite.

Definition 2 ([1]): Let $R : \mathbb{C} \longrightarrow \mathbb{C}^{m \times m}$ be a real, rational, proper transfer function. Then, $R(s)$ is said to be SNI if
1) $R(s)$ has no poles in $\mathfrak{R}(s) \geq 0$;
2) $j[R(j\omega) - R(j\omega)^*] > 0$ for all $\omega \in (0, \infty)$.

For SISO systems, the NI property ensures that the positive branch of the Nyquist plot lies below the real axis. The definitions of NI systems in [1], [11], and [12] do not require the NI system to be symmetric, i.e., $G(s) = G(s)^T$. It is stated in [16] that there are no examples in the literature that are NI but not symmetric. Here, we give one such non-symmetric example.

Example 1: Let

$$M(s) = \begin{bmatrix} \dfrac{-s}{s+5} & \dfrac{-5}{s+5} \\[2mm] \dfrac{-(4s+5)}{s^2+6s+5} & \dfrac{-s^2+s+15}{s^2+6s+5} \end{bmatrix}$$

and $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be a minimal realization of $M(s)$ where $A = \left[\begin{smallmatrix} 0 & 1 \\ -5 & -6 \end{smallmatrix}\right]$, $B = \left[\begin{smallmatrix} -1 & -2 \\ 6 & -3 \end{smallmatrix}\right]$, $C = \left[\begin{smallmatrix} 1 & 1 \\ -2 & -1 \end{smallmatrix}\right]$, and $D = \left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$. We demonstrate that $M(s)$ is an SNI system despite it not being symmetric. To do this, we use the NI lemma in [1] and [11] and also check for transmission zeros on the imaginary axis. $M(s)$ is NI because there exists a real matrix $Y = \left[\begin{smallmatrix} 3 & -3 \\ -3 & 4 \end{smallmatrix}\right] > 0$ such that $AY + YA^* = \left[\begin{smallmatrix} -6 & 7 \\ 7 & -18 \end{smallmatrix}\right] \leq 0$ and $B = -AYC^*$. Furthermore, $\det(M(s) - M^\sim(s)) = \frac{59s^2}{s^4 - 26s^2 + 25} \neq 0 \quad \forall s \in \{s = j\omega : \omega \in (0, \infty)\}$. Hence, $M(s)$ is NI and there is no transmission zeros of $M(s) - M^\sim(s)$ when $s = j\omega \ \forall \omega \in (0, \infty)$, which implies that $M(s)$ must be SNI.

We now give a few technical lemmas that will be used to prove the main results. We start by giving alternative representations for $\lambda_i[(I - AB)^{-1}(CD - I)]$.

Lemma 3: Suppose that A, B, C, and D are real symmetric matrices and $\det(I - AB) \neq 0$. Then,

$$\begin{aligned} \lambda_i[(I - AB)^{-1}(CD - I)] &= \lambda_i[(CD - I)(I - AB)^{-1}] \\ &= \lambda_i[(I - BA)^{-1}(DC - I)] \\ &= \lambda_i[(DC - I)(I - BA)^{-1}]. \end{aligned}$$

Proof: The first equality is via a similarity transform with $(I - AB)$. The second equality is via $\lambda_i(X) = \lambda_i(X^T)$. The third equality is via a similarity transform with $(I - BA)$. ∎

Next, we show that certain matrix structures have real eigenvalues.

Lemma 4: Suppose that A, B, and C are real symmetric matrices satisfying $C \geq B$ and $\det(I - AB) \neq 0$. Then,

$$\lambda_i[(I - AB)^{-1}(AC - I)] \in \mathbb{R} \quad \forall i.$$

Proof: Trivial via rearrangement into a symmetric matrix.

$$\begin{aligned} &\lambda_i[(I - AB)^{-1}(AC - I)] \\ &\quad = \lambda_i[(I - AB)^{-1}A(C - B)] - 1 \\ &\quad = \lambda_i[(C - B)^{1/2}(I - AB)^{-1}A(C - B)^{1/2}] - 1. \end{aligned}$$

∎

The next lemma manipulates scalar inequalities. This will be useful in deriving the main results of Section IV.

Lemma 5: Given four scalars a, b, α, and β with $\alpha \geq 0$ and $\beta \geq 0$. If $ab < 1$ and $(a + \alpha)(b + \beta) < 1$, then $(a + \alpha)b < 1$.

Proof: If $b \leq 0$, then $(a + \alpha)b = ab + \alpha b \leq ab < 1$. If on the other hand $b > 0$, then $b + \beta > 0$ and hence $(a + \alpha)b = (a + \alpha)(b + \beta)\frac{b}{(b+\beta)} < 1$. ∎

We now show that $s = j\omega_0$ with $\omega_0 \in (0, \infty)$ is not a transmission zero of $I - M(s)N(s)$ when $M(s)$ and $N(s)$ are NI systems and $s = j\omega_0$ is a pole of $M(s)$.

Lemma 6: Let $M(s)$ be an NI system and $N(s)$ be an SNI system. Assume $s = j\omega_0$ with $\omega_0 \in (0, \infty)$ is a simple pole

of $M(s)$. Then, $I - M(s)N(s)$ has no transmission zero at $s = j\omega_0$.

Proof: Since $s = j\omega_0$ with $\omega_0 \in (0, \infty)$ is a simple pole of $M(s)$, $M(s)$ can be factored as $M(s) = M_1(s) + \frac{-jA}{s - j\omega_0}$, where $A = A^* \geq 0$ and $M_1(s)$ is analytic in the neighborhood of $s = j\omega_0$.

Let $0 \neq x \in \ker(A)$ and choose a sufficiently small $\delta > 0$. Since $M(s)$ is an NI system,

$$\begin{aligned} &j[M(j\omega) - M(j\omega)^*] \geq 0 \quad \forall \omega \in \{0 < |\omega - \omega_0| < \delta\} \\ \Leftrightarrow{}&[jM(j\omega)] + [jM(j\omega)]^* \geq 0 \quad \forall \omega \in \{0 < |\omega - \omega_0| < \delta\} \\ \Rightarrow{}&x^*[[jM(j\omega)] + [jM(j\omega)]^*]x \geq 0 \quad \forall \omega \in \{0 < |\omega - \omega_0| \\ &\quad < \delta\} \\ \Leftrightarrow{}&x^*[[jM_1(j\omega)] + [jM_1(j\omega)]^*]x \geq 0 \quad \forall \omega \in \{0 < |\omega - \omega_0| \\ &\quad < \delta\} \\ \Rightarrow{}&x^*[[jM_1(j\omega_0)] + [jM_1(j\omega_0)]^*]x \geq 0 \text{ (by continuity of} \\ &\text{real rational functions and because } M_1(s) \text{ is analytic near} \\ &s = j\omega_0). \end{aligned}$$

Also, since $N(s)$ is an SNI system,

$$\begin{aligned} &j[N(j\omega) - N(j\omega)^*] > 0 \quad \forall \omega \in (0, \infty) \\ \Leftrightarrow{}&[jN(j\omega)] + [jN(j\omega)]^* > 0 \quad \forall \omega \in (0, \infty) \\ \Leftrightarrow{}&[jN(j\omega)]^{-1} + [jN(j\omega)]^{-*} > 0 \quad \forall \omega \in (0, \infty) \\ &\text{(since } N(j\omega) \text{ is nonsingular } \forall \omega \in (0, \infty) \text{ via [1])} \\ \Rightarrow{}&[jN(j\omega_0)]^{-1} + [jN(j\omega_0)]^{-*} > 0 \\ \Rightarrow{}&x^*[[jN(j\omega_0)]^{-1} + [jN(j\omega_0)]^{-*}]x > 0. \end{aligned}$$

From the above two conditions, we now have

$$\begin{aligned} &x^*[[jN(j\omega_0)]^{-1} + [jM_1(j\omega_0)]]x \\ &\quad + x^*[[jN(j\omega_0)]^{-1} + [jM_1(j\omega_0)]]^*x > 0. \end{aligned} \tag{1}$$

However, we need to show that $I - M(s)N(s)$ has no transmission zero at $s = j\omega_0$, which is equivalent to $[jN(s)]^{-1} + [jM(s)]$ having no transmission zero at $s = j\omega_0$.

We show this via contradiction. Suppose $s = j\omega_0$ with $\omega_0 > 0$ is a transmission zero of $[jN(s)]^{-1} + [jM(s)]$. Then, $\exists y \in \mathbb{C}^m$ with $y \neq 0$ such that $[[jN(s)]^{-1} + [jM(s)]]y = 0$ at $s = j\omega_0$.

Expanding the above equation, we have that the given y must satisfy $[jN(s)]^{-1}y + [jM_1(s)]y = \frac{-Ay}{s - j\omega_0}$. But in the limit as $s \to j\omega_0$, the left-hand side is finite and the right-hand side is infinite when $y \notin \ker(A)$. Hence, $y \in \ker(A)$. Then, $[[jN(s)]^{-1} + [jM(s)]]y = 0$ at $s = j\omega_0$ implies $[jN(j\omega_0)]^{-1}y + [jM_1(j\omega_0)]y = 0$ which in turn implies that (1) cannot be fulfilled for the choice $x = y$ as x was an arbitrary non-zero vector in $\ker(A)$. Then, by contradiction, $I - M(s)N(s)$ has no transmission zero at $s = j\omega_0$. ∎

The following lemma provides some simple manipulations of NI systems.

Lemma 7: Let $M(s)$ be a square, real, rational and proper transfer function matrix. Then,

1) $M(s)$ is NI if and only if $M(s) - M(\infty)$ is NI and $M(\infty) = M(\infty)^T$;
2) $M(s) - M(\infty)$ is NI if and only if $s[M(s) - M(\infty)]$ is positive real.

Proof: 1) Trivial. 2) See [24, Lemma 2]. ∎

The next lemma extends [1, Lemma 2] to NI systems that can possibly have poles on the $j\omega$-axis excluding the origin.

Lemma 8: Let $M(s)$ be an NI system without poles at the origin. Then, $M(0) = M(0)^T$, $M(\infty) = M(\infty)^T$, and $M(0) \geq M(\infty)$.

Proof: The results are trivial when $M(s)$ is static. When $M(s)$ is dynamic, let it have a minimal realization $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$. Using [11, Lemma 7], we get $D = D^T$ and $M(0) - M(\infty) = -CA^{-1}B = CA^{-1}AYC^* = CYC^* \geq 0$. The results then follow. ∎

III. MAIN RESULTS: PART 1—NO POLES AT THE ORIGIN

A necessary and sufficient condition has been derived in [1] and [11] for internal stability of the positive feedback interconnection of an NI system without poles at the origin and an SNI system as shown in Fig. 1 under two assumptions at infinite frequency. Here, we consider the same feedback structure but remove these two assumptions imposed in [1] and [11], hence obtaining a generalized stability condition. While the condition in [1] and [11] for determining internal stability of the closed-loop system requires a test only on the dc loop gain, in our generalized setting, the internal stability conditions in Theorems 9 and 14 depend on both the frequencies 0 and ∞. Although the resulting stability conditions involving a mixture of frequencies are surprising, this is indeed correct and will be explained in detail in Section IV when the results will be specialized to the SISO setting and given a Nyquist interpretation.

The matrix $I - N(0)M(\infty)$ is invertible in the third condition of Theorem 9 because the second condition (in conjunction with the first condition) in Theorem 9 guarantees its invertibility. Also, the matrices $[I - M(\infty)N(\infty)]^{-1}(M(\infty)N(0) - I)$ and $[I - N(0)M(\infty)]^{-1}(N(0)M(0) - I)$ in the conditions of Theorem 9 have only real eigenvalues via Lemmas 4 and 8.

Theorem 9: Let $M(s)$ be an NI system without poles at the origin and $N(s)$ be an SNI system. Then, $[M(s), N(s)]$ is internally stable if and only if

$I - M(\infty)N(\infty)$ is nonsingular,

$\bar{\lambda}[[I - M(\infty)N(\infty)]^{-1}(M(\infty)N(0) - I)] < 0$, and

$\bar{\lambda}[[I - N(0)M(\infty)]^{-1}(N(0)M(0) - I)] < 0$.

Proof: This proof closely follows the proof of the corresponding result given in [1] with appropriate modifications in pertinent statements to allow relaxation of the restrictive assumptions at ∞ frequency.

Let $N(s) = \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right]$ be a minimal realization. Then, by the assumptions of this theorem and using [11, Lemma 8], $\bar{A}$ is Hurwitz, $\bar{D} = \bar{D}^*$, and there exists a real matrix $\bar{Y} = \bar{Y}^* > 0$ such that

$$\bar{A}\bar{Y} + \bar{Y}\bar{A}^* \leq 0 \text{ and } \bar{B} = -\bar{A}\bar{Y}\bar{C}^*.$$

The proof is brief when $M(s)$ is a static matrix K, i.e., $M(s) = K$, as the first condition in the theorem statement is equivalent to well-posedness, the third condi-

tion is trivially fulfilled and the second condition is equivalent to $(I - KN(s))^{-1} \in \mathcal{RH}_\infty$ via the following argument: $(I - KN(s))^{-1} \in \mathcal{RH}_\infty$ is equivalent to $\hat{A} = \bar{A} + \bar{B}(I - K\bar{D})^{-1}K\bar{C} = \mathcal{PQ}$ is Hurwitz, where $\mathcal{P} = \bar{A}\bar{Y}$ and $\mathcal{Q} = \bar{Y}^{-1} - \bar{C}^*(I - K\bar{D})^{-1}K\bar{C}$. Then, $\hat{A}$ is Hurwitz if and only if $\mathcal{Q} > 0$, which is equivalent to the second condition. This concludes the proof when $M(s)$ is a static matrix.

Next, consider the case when $M(s)$ has dynamics. Let $M(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be a minimal realization. Then, by the assumptions of this theorem and using [11, Lemma 7], $\det(A) \neq 0$, $D = D^*$, and there exists a real matrix $Y = Y^* > 0$ such that

$$AY + YA^* \leq 0 \text{ and } B = -AYC^*.$$

Define $U = I - D\bar{D}$, $V = I - \bar{D}D$, $\Phi = \left[\begin{array}{cc} AY & 0 \\ 0 & \bar{A}\bar{Y} \end{array}\right]$ and $T = \left[\begin{array}{cc} Y^{-1} - C^*\bar{D}U^{-1}C & -C^*V^{-1}\bar{C} \\ -\bar{C}^*U^{-1}C & \bar{Y}^{-1} - \bar{C}^*U^{-1}D\bar{C} \end{array}\right]$. Then, $[M(s), N(s)]$ is internally stable

$\Leftrightarrow$ $I - M(\infty)N(\infty)$ is nonsingular and

$$[I - M(s)N(s)]^{-1}$$
$$= \left[\begin{array}{cc|c} \left(\begin{array}{cc} A & B\bar{C} \\ 0 & \bar{A} \end{array}\right) + \left(\begin{array}{c} B\bar{D} \\ \bar{B} \end{array}\right) U^{-1} \left(C \ \ D\bar{C}\right) & \begin{array}{c} B\bar{D}U^{-1} \\ \bar{B}U^{-1} \end{array} \\ \hline \begin{array}{cc} U^{-1}C & U^{-1}D\bar{C} \end{array} & U^{-1} \end{array}\right]$$
$$\in \mathcal{RH}_\infty$$

(this equivalence is via [25, Th. 5.7] due to the facts that $N(s)$ and $M(s)$ have no poles in the open right-half plane and at the origin, and $N(s)$ having no poles nor zeros on $j\mathbb{R}/\{0\}$. Thus, $M(s)N(s)$ has no pole-zero cancellation in the closed right-half plane)

$\Leftrightarrow$ $I - M(\infty)N(\infty)$ is nonsingular and
$\mathcal{A} = \Phi T$ is Hurwitz (as the previous realization is stabilizable and detectable)

$\Leftrightarrow$ $I - M(\infty)N(\infty)$ is nonsingular and
$T > 0$ [necessity and sufficiency are proved as in the proof of [1, Th. 5] and the proof of [11, Th. 1] except that Lemma 6 must be used instead of $\det(I - M(j\omega)N(j\omega)) \neq 0$ $\forall\omega \in (0, \infty)$ to obtain that $I - M(j\omega)N(j\omega)$ has no transmission zeros $\forall\omega \in (0, \infty)$]

$\Leftrightarrow$ $I - M(\infty)N(\infty)$ is nonsingular,
$\bar{Y}^{-1} - \bar{C}^*U^{-1}D\bar{C} > 0$, and
$(Y^{-1} - C^*\bar{D}U^{-1}C) - C^*V^{-1}\bar{C}(\bar{Y}^{-1} - \bar{C}^*U^{-1}D\bar{C})^{-1}\bar{C}^*U^{-1}C > 0$

$\Leftrightarrow$ $I - M(\infty)N(\infty)$ is nonsingular,
$\bar{Y}^{-1} - \bar{C}^*U^{-1}D\bar{C} > 0$, and
$Y^{-1} - C^*\bar{D}U^{-1}C - C^*V^{-1}(I - \bar{C}\bar{Y}\bar{C}^*U^{-1}D)^{-1}\bar{C}\bar{Y}\bar{C}^*U^{-1}C > 0$

$\Leftrightarrow$ $I - M(\infty)N(\infty)$ is nonsingular,
$\bar{Y}^{-1} - \bar{C}^*U^{-1}D\bar{C} > 0$, and
$Y^{-1} - C^*\bar{D}U^{-1}C - C^*V^{-1}(I - (N(0) - \bar{D})U^{-1}D)^{-1}(N(0) - \bar{D})U^{-1}C > 0$
(since $\bar{C}\bar{Y}\bar{C}^* = N(0) - \bar{D}$ via [1, Lemma 2])

$\Leftrightarrow$ $I - M(\infty)N(\infty)$ is nonsingular,
$\bar{Y}^{-1} - \bar{C}^*U^{-1}D\bar{C} > 0$, and

$$Y^{-1} - C^* \bar{D} U^{-1} C - C^* V^{-1} [[V - (N(0) - \bar{D}) D] V^{-1}]^{-1} (N(0) - \bar{D}) U^{-1} C > 0$$

$\Leftrightarrow$ $I - M(\infty)N(\infty)$ is nonsingular,
$\bar{Y}^{-1} - \bar{C}^* U^{-1} D \bar{C} > 0$, and
$$Y^{-1} - C^* \bar{D} U^{-1} C - C^* (I - N(0)D)^{-1} (N(0) - \bar{D}) U^{-1} C > 0$$

$\Leftrightarrow$ $I - M(\infty)N(\infty)$ is nonsingular,
$\bar{Y}^{-1} - \bar{C}^* U^{-1} D \bar{C} > 0$, and
$$Y^{-1} - C^* (I - N(0)D)^{-1} [(I - N(0)D)\bar{D} + (N(0) - \bar{D})] U^{-1} C > 0$$

$\Leftrightarrow$ $I - M(\infty)N(\infty)$ is nonsingular,
$\bar{Y}^{-1} - \bar{C}^* U^{-1} D \bar{C} > 0$, and
$$Y^{-1} - C^* (I - N(0)D)^{-1} N(0) C > 0$$

$\Leftrightarrow$ $I - M(\infty)N(\infty)$ is nonsingular,
$\bar{\lambda}[\bar{Y}^{1/2} \bar{C}^* U^{-1} D \bar{C} \bar{Y}^{1/2}] < 1$, and
$\bar{\lambda}[Y^{1/2} C^* (I - N(0)D)^{-1} N(0) C Y^{1/2}] < 1$

$\Leftrightarrow$ $I - M(\infty)N(\infty)$ is nonsingular,
$\bar{\lambda}[U^{-1} D \bar{C} \bar{Y} \bar{C}^*] < 1$, and
$\bar{\lambda}[(I - N(0)D)^{-1} N(0) C Y C^*] < 1$

$\Leftrightarrow$ $I - M(\infty)N(\infty)$ is nonsingular,
$\bar{\lambda}[U^{-1} D(N(0) - \bar{D}) - I] < 0$, and
$\bar{\lambda}[(I - N(0)D)^{-1} N(0)(M(0) - D) - I] < 0$
(since $\bar{C} \bar{Y} \bar{C}^* = N(0) - \bar{D}$ and $C Y C^* = M(0) - D$ via [1, Lemma 2] and Lemma 8)

$\Leftrightarrow$ $I - M(\infty)N(\infty)$ is nonsingular,
$\bar{\lambda}[[I - M(\infty)N(\infty)]^{-1}(M(\infty)N(0) - I)] < 0$, and
$\bar{\lambda}[[I - N(0)M(\infty)]^{-1}(N(0)M(0) - I)] < 0$. ∎

Remark 10: The feedback stability conditions in Theorem 9 involve a mixture of two frequencies, i.e., 0 and ∞. This appears, on the surface, to be counter intuitive as if one considers the Nyquist stability criterion, one might reasonably ponder why a mixture of two frequencies is needed given that the Nyquist plot is a pointwise-in-frequency plot of $M(j\omega)N(j\omega)$. However, note that the feedback stability condition in the Nyquist stability criterion is a winding number condition (or equivalently, an encirclements condition), which is certainly not a pointwise-in-frequency condition. The fact that for certain system classes this encirclements condition can be equivalently rewritten as simpler conditions that do not require full knowledge of the transfer functions is not trivial. It is hence legitimate that the feedback stability conditions in Theorem 9 invoke more than a single frequency. Note that in the sequel, we shall also show how these conditions specialize to a single-frequency condition under some additional assumptions.

The next lemma states that the eigenvalues of the product of certain matrices are always real.

Lemma 11: Let A and B be real symmetric matrices such that either A or B is sign semidefinite (i.e., ≥ 0 or ≤ 0). Then, $\lambda_i(AB) \in \mathbb{R} \; \forall i$.

Proof: Trivial. ∎

If we now impose the same two restrictive assumptions $M(\infty)N(\infty) = 0$ and $N(\infty) \geq 0$ as in [1] and [11], the internal stability conditions in Theorem 9 reduce to the condition in [1] and [11].

Corollary 12: Let $M(s)$ be an NI system without poles at the origin and $N(s)$ be an SNI system. Let $M(\infty)N(\infty) = 0$ and $N(\infty) \geq 0$. Then,
$[M(s), N(s)]$ is internally stable $\Leftrightarrow \bar{\lambda}[M(0)N(0)] < 1$.
Proof: This is a direct consequence of Theorem 9 because
1) $M(\infty)N(\infty) = 0 \Rightarrow I - M(\infty)N(\infty)$ is nonsingular;
2) $\bar{\lambda}[[I - M(\infty)N(\infty)]^{-1}(M(\infty)N(0) - I)] < 0$
   $\Leftrightarrow \bar{\lambda}[M(\infty)N(0)] < 1$
   $\Leftrightarrow N(0)^{1/2} M(\infty) N(0)^{1/2} < I$
    (since $N(0) > N(\infty) \geq 0$ via [1, Lemma 2])
   $\Leftrightarrow M(\infty) < N(0)^{-1}$;
3) $\bar{\lambda}[[I - N(0)M(\infty)]^{-1}(N(0)M(0) - I)] < 0$
   $\Leftrightarrow \bar{\lambda}[[N(0)^{-1} - M(\infty)]^{-1}(M(0) - N(0)^{-1})] < 0$
    (since $N(0) > N(\infty) \geq 0$ via [1, Lemma 2])
   $\Leftrightarrow [N(0)^{-1} - M(\infty)]^{-1/2}(M(0) - N(0)^{-1})[N(0)^{-1} - M(\infty)]^{-1/2} < 0$
    (since $N(0)^{-1} > M(\infty)$ via above)
   $\Leftrightarrow M(0) - N(0)^{-1} < 0$
   $\Leftrightarrow \bar{\lambda}[M(0)N(0)] < 1$.
But $\bar{\lambda}[M(0)N(0)] < 1 \Leftrightarrow M(0) < N(0)^{-1} \Rightarrow M(\infty) < N(0)^{-1}$ (since $M(\infty) \leq M(0)$ via Lemma 8).
This concludes the proof. ∎

Another way of simplifying the stability conditions given in Theorem 9 is to assume that $M(s)$ is a strictly proper transfer function, and under this assumption the internal stability condition obtained here is the same as the one in the previous corollary. Note that the statement in Corollary 13 was not known in the previous literature. In other words, it was not known that the dc loop gain is a necessary and sufficient condition for internal stability also under a simple assumption of a strictly proper $M(s)$. Unlike [12, Remark 1], which requires $N(0)$ to be either a positive or a negative definite matrix, Corollary 13 has no restriction of the sign of $N(0)$, i.e., $N(0)$ can be either sign-indefinite or singular.

Corollary 13: Let $M(s)$ be an NI system without poles at the origin satisfying $M(\infty) = 0$ and $N(s)$ be an SNI system. Then,
$[M(s), N(s)]$ is internally stable $\Leftrightarrow \bar{\lambda}[M(0)N(0)] < 1$.
Proof: This is a direct consequence of Theorem 9 because:
1) $M(\infty) = 0 \Rightarrow I - M(\infty)N(\infty)$ is nonsingular;
2) $\bar{\lambda}[[I - M(\infty)N(\infty)]^{-1}(M(\infty)N(0) - I)] = -1$, which means $\bar{\lambda}[[I - M(\infty)N(\infty)]^{-1}(M(\infty)N(0) - I)] < 0$ is trivially fulfilled;
3) $\bar{\lambda}[[I - N(0)M(\infty)]^{-1}(N(0)M(0) - I)] < 0$ if and only if $\bar{\lambda}[M(0)N(0)] < 1$. ∎

If in the proof of Theorem 9, we take the Schur complement around the other block of $T > 0$, different necessary and sufficient conditions for the internal stability of a positive feedback interconnection of NI systems without poles at the origin can be derived as shown in Theorem 14. The matrices $(M(0)N(\infty) - I)[I - M(\infty)N(\infty)]^{-1}$ and $(N(0)M(0) - I)[I - N(\infty)M(0)]^{-1}$ in the conditions of Theorem 14 have only real eigenvalues due to Lemmas 3, 4, and 8.

Theorem 14: Let $M(s)$ be an NI system without poles at the origin and $N(s)$ be an SNI system. Then, $[M(s), N(s)]$ is

internally stable if and only if

$I - M(\infty)N(\infty)$ is nonsingular,

$\bar{\lambda}[(M(0)N(\infty) - I)[I - M(\infty)N(\infty)]^{-1}] < 0,$ and

$\bar{\lambda}[(N(0)M(0) - I)[I - N(\infty)M(0)]^{-1}] < 0.$

Proof: The proof is identical to that in Theorem 9 until $T > 0$. Then, it proceeds in a similar way to the proof in Theorem 9 after $T > 0$ by taking Schur complements about the $(1,1)$ element of $T > 0$ instead of the $(2,2)$ element of $T > 0$. Note that the conditions in Theorem 14 cannot be obtained directly via trivial algebraic manipulations of the conditions in Theorem 9 as shown in Lemma 3. ∎

Example 2 demonstrates that if the closed-loop system is well-posed (i.e., $I - M(\infty)N(\infty)$ is nonsingular), each of the remaining conditions in Theorem 14 is not simply implied by either one of the remaining conditions in Theorem 9 independently.

Example 2: Choose

$$M(s) = \begin{bmatrix} \dfrac{-s}{s+5} & \dfrac{-5}{s+5} \\ \dfrac{-(4s+5)}{s^2+6s+5} & \dfrac{-s^2+s+15}{s^2+6s+5} \end{bmatrix}$$

and

$$N(s) = \begin{bmatrix} \dfrac{5}{s+5} & \dfrac{-(2s+15)}{s+5} \\ \dfrac{-(2s^2+16s+15)}{s^2+6s+5} & \dfrac{-s^2+s+15}{s^2+6s+5} \end{bmatrix}.$$

Both $M(s)$ and $N(s)$ are SNI systems since $M(s)$ is an SNI system from Example 1 and $N(s) = M(s) + \begin{bmatrix} 1 & -2 \\ -2 & 0 \end{bmatrix}$. Also, the closed-loop system is well-posed, since $I - M(\infty)N(\infty) = \begin{bmatrix} 1 & -2 \\ -2 & 0 \end{bmatrix}$ is nonsingular. In this case, $\bar{\lambda}[[I - M(\infty)N(\infty)]^{-1}(M(\infty)N(0) - I)] = -0.1340 < 0$ and $\bar{\lambda}[[I - N(0)M(\infty)]^{-1}(N(0)M(\infty) - I)] = 8 \not< 0$, whereas $\bar{\lambda}[(M(0)N(\infty) - I)[I - M(\infty)N(\infty)]^{-1}] = 3.3028 \not< 0$ and $\bar{\lambda}[(N(0)M(0) - I)[I - N(\infty)M(0)]^{-1}] = 1.5447 \not< 0$.

A different set of assumptions at 0 and ∞ frequencies can also be imposed to specialize the conditions of Theorem 14, thereby obtaining another intuitive result that is different from Corollaries 12 and 13. Again note that the statement in Corollary 15 is also new and was not known in the previous literature. In other words, it was not known that the dc loop gain condition is a necessary and sufficient condition for internal stability also under the simple assumption of $M(0) > 0$ but no restriction of the sign of $N(\infty)$.

Corollary 15: Let $M(s)$ be an NI system without poles at the origin and $N(s)$ be an SNI system. Suppose $M(\infty)N(\infty) = 0$ and $M(0) > 0$. Then,

$[M(s), N(s)]$ is internally stable $\Leftrightarrow \bar{\lambda}[M(0)N(0)] < 1.$

Proof: This is a direct consequence of Theorem 14 because:
1) $M(\infty)N(\infty) = 0$ implies $I - M(\infty)N(\infty)$ is nonsingular;
2) $\bar{\lambda}[(M(0)N(\infty) - I)[I - M(\infty)N(\infty)]^{-1}] < 0$
   $\Leftrightarrow \bar{\lambda}[M(0)N(\infty)] < 1$

   $\Leftrightarrow M(0)^{1/2}N(\infty)M(0)^{1/2} < I$
   $\Leftrightarrow N(\infty) < M(0)^{-1};$
3) $\bar{\lambda}[(N(0)M(0) - I)[I - N(\infty)M(0)]^{-1}] < 0$
   $\Leftrightarrow \bar{\lambda}[(N(0) - M(0)^{-1})[M(0)^{-1} - N(\infty)]^{-1}] < 0$
   $\Leftrightarrow \bar{\lambda}[[M(0)^{-1} - N(\infty)]^{-1/2}(N(0) - M(0)^{-1})[M(0)^{-1} - N(\infty)]^{-1/2}] < 0$
   (since $M(0)^{-1} > N(\infty)$ via above)
   $\Leftrightarrow N(0) - M(0)^{-1} < 0$
   $\Leftrightarrow \bar{\lambda}[M(0)N(0)] < 1.$

But $\bar{\lambda}[M(0)N(0)] < 1 \Leftrightarrow N(0) < M(0)^{-1} \Rightarrow N(\infty) < M(0)^{-1}$ (since $N(\infty) < N(0)$ via [1, Lemma 2]). This concludes the proof. ∎

IV. SISO SPECIALIZATION

In Theorems 9 and 14, two different internal stability conditions are given. For the SISO case, these two stability conditions can be further reduced as shown in Lemma 16.

Lemma 16: Let $M(s)$ be a scalar NI system without poles at the origin and $N(s)$ be a scalar SNI system. Then, the following three statements are equivalent:
1) $[M(s), N(s)]$ is internally stable;
2) either condition a) or condition b) holds:
   a) $M(0)N(0) < 1$, $M(\infty)N(\infty) < 1$ and $M(\infty)N(0) < 1;$
   b) $M(0)N(0) > 1$, $M(\infty)N(\infty) > 1$ and $M(\infty)N(0) > 1;$
3) either condition a) or condition b) holds:
   a) $M(0)N(0) < 1$, $M(\infty)N(\infty) < 1$ and $M(0)N(\infty) < 1;$
   b) $M(0)N(0) > 1$, $M(\infty)N(\infty) > 1$ and $M(0)N(\infty) > 1.$

Proof: Via Theorem 9, $[M(s), N(s)]$ is internally stable if and only if $M(\infty)N(\infty) \neq 1$, $\frac{1 - M(\infty)N(0)}{M(\infty)N(\infty) - 1} < 0$, and $\frac{M(0)N(0) - 1}{1 - M(\infty)N(0)} < 0$. These three conditions yield condition 2a) or condition 2b). Theorem 14 gives condition 3a) or condition 3b). ∎

The internal stability conditions in Lemma 16 can be further simplified as shown in Theorem 17.

Theorem 17: Let $M(s)$ be a scalar NI system without poles at the origin and $N(s)$ be a scalar SNI system. Then, $[M(s), N(s)]$ is internally stable if and only if either one of the following three conditions holds:
   i) $M(0)N(0) < 1$ and $M(\infty)N(\infty) < 1;$
   ii) $N(\infty) > 0$ and $M(\infty)N(\infty) > 1;$
   iii) $N(0) < 0$ and $M(0)N(0) > 1.$

Proof: Both conditions 2a) and 3a) in Lemma 16 reduce to condition i) in this theorem statement via Lemma 5. This then implies that conditions 2b) and 3b) in Lemma 16 are equivalent.

Next recall that $N(0) > N(\infty)$ via [1, Lemma 2] and $M(0) \geq M(\infty)$ via Lemma 8. Condition ii) and these two NI properties imply $N(0) > N(\infty) > 0$ and $M(0) \geq M(\infty) > 0$. Then, $M(0)N(0) > M(0)N(\infty)$, $M(0)N(0) \geq M(\infty)N(0)$, $M(\infty)N(0) > M(\infty)N(\infty) > 1$, and $M(0)N(\infty) \geq M(\infty)N(\infty) > 1$. These inequalities then imply conditions 2b) and 3b) in Lemma 16. Similarly, condition iii) and

the same two NI properties imply $0 > N(0) > N(\infty)$ and $0 > M(0) \geq M(\infty)$. Then, $1 < M(0)N(0) < M(0)N(\infty)$, $1 < M(0)N(0) \leq M(\infty)N(0)$, $M(\infty)N(0) < M(\infty)N(\infty)$ and $M(0)N(\infty) \leq M(\infty)N(\infty)$. These inequalities then imply conditions 2b) and 3b) in Lemma 16.

To show the converse, consider the following five complimentary cases: $0 < N(\infty) < N(0)$, $0 = N(\infty) < N(0)$, $N(\infty) < 0 < N(0)$, $N(\infty) < N(0) = 0$, and $N(\infty) < N(0) < 0$. The three middle cases violate conditions 2b) and 3b) in Lemma 16. Hence, only two valid complimentary cases are permitted by conditions 2b) and 3b) in Lemma 16: $0 < N(\infty)$ and $N(0) < 0$. Hence, condition 2b) [respectively, condition 3b)] in Lemma 16 implies either condition ii) or condition iii) of this theorem statement. ■

For the internal stability of the closed-loop system to be guaranteed via condition i) in Theorem 17, it is necessary and sufficient that the Nyquist plot of $M(s)N(s)$ starts and ends on the left-hand side of 1. Recall that the Nyquist point is $+1 + j0$ since we are considering a positive feedback interconnection. Similarly, for the internal stability to be guaranteed by either conditions ii) or iii) in Theorem 17, it is necessary and sufficient that the Nyquist plot of $M(s)N(s)$ begins and ends on the right-hand side of 1, and also that $M(0)$, $M(\infty)$, $N(0)$, and $N(\infty)$ have the same sign [positive in condition ii) and negative in condition iii)].

The following example demonstrates use of Lemma 16 and Theorem 17.

Example 3: Choose $M(s) = \frac{3-s}{s+1}$ and $N(s) = \frac{1-2s}{s+1}$. Both $M(s)$ and $N(s)$ are SNI systems. Hence, they fulfill the assumptions of Lemma 16 and Theorem 17. In this case, $M(0)N(0) = (3)(1) > 1$ and $M(\infty)N(\infty) = (-1)(-2) > 1$, but $M(\infty)N(0) = (-1)(1) \not> 1$, $M(0)N(\infty) = (3)(-2) \not> 1$, $N(\infty) = -2 \not> 0$ and $N(0) = 1 \not< 0$. Lemma 16 and Theorem 17 correctly infer that $[M(s), N(s)]$ is not internally stable since $[I - M(s)N(s)]^{-1} = \frac{(s+1)^2}{-2+9s-s^2} = \frac{-(s+1)^2}{s^2-9s+2} \notin \mathcal{RH}_\infty$.

Example 3 illustrates that although the positive frequency branch of the Nyquist plot of $M(s)N(s)$ begins and ends on the right-hand side of the Nyquist point $+1 + j0$, $[M(s), N(s)]$ is unstable since $M(0)$, $M(\infty)$, $N(0)$, and $N(\infty)$ have different signs. Indeed, $M(s)N(s)$ has no open-loop poles in the closed right-half plane but the Nyquist plot of $M(s)N(s)$ encircles the Nyquist point $+1 + j0$ twice. Hence, there will be two unstable closed-loop poles that results in the instability of the closed-loop system according to the Nyquist stability criterion.

V. MAIN RESULTS: PART 2—ALLOWING POLES AT THE ORIGIN

In the previous sections, we have discussed necessary and sufficient conditions for internal stability of the positive feedback interconnection shown in Fig. 1 involving an SNI system and an NI system without poles at the origin. Here, we derive more general results by allowing the NI system to have poles at the origin. To do this, we need to build on the results of the previous sections.

In the following sections, we first provide several technical lemmas that underpin the subsequent theorems and results.

Then, the general feedback stability results are derived. We also show how these general results specialize to the results in the earlier sections through simple algebraic manipulation and reduce to the results of [12] under the same assumptions as imposed in [12]. Finally, we demonstrate that a matrix Ψ with specific properties needed in the results always exists and propose a systematic way to construct a matrix Ψ. We also show that one such Ψ is sufficient to conclude closed-loop stability, thereby we avoid having to search over the set of matrices Ψ with certain properties.

A. Technical Lemmas

We introduce the following three technical lemmas as a basis for deriving the internal stability conditions in Section V-B. We first show that an original feedback system and a transformed system via a linear shift transformation have the same stability properties. Note that the loop transformation technique also holds even for the nonlinear systems [14], [26].

Lemma 18: Let $M(s)$ be an NI system and $N(s)$ be an SNI system. Let $\Psi < 0$ be such that $I - M(\infty)\Psi$ is nonsingular. Define $M_1(s) = [I - M(s)\Psi]^{-1}M(s)$ and $N_1(s) = N(s) - \Psi$. Then,

1) $I - M(\infty)N(\infty)$ is nonsingular if and only if $I - M_1(\infty)N_1(\infty)$ is nonsingular;
2) $[M(s), N(s)]$ is internally stable if and only if $[M_1(s), N_1(s)]$ is internally stable.

Proof:

1) $I - M(\infty)N(\infty)$ is nonsingular
   $\Leftrightarrow I - M_1(\infty)N_1(\infty)$
   $\quad = I - [I - M(\infty)\Psi]^{-1}M(\infty)(N(\infty) - \Psi)$
   $\quad = [I - M(\infty)\Psi]^{-1}[I - M(\infty)N(\infty)]$
   is nonsingular.

2) $[M(s), N(s)]$ is internally stable
   $\Leftrightarrow$ $I - M(\infty)N(\infty)$ is nonsingular (due to well-posedness condition) and $M(s)[I - N(s)M(s)]^{-1} \in \mathcal{RH}_\infty$ (since $N(s) \in \mathcal{RH}_\infty$, see [25, Corollary 5.4])
   $\Leftrightarrow$ $I - M_1(\infty)N_1(\infty)$ is nonsingular via 1) and $M_1(s)[I - N_1(s)M_1(s)]^{-1} = M(s)[I - \Psi M(s)]^{-1}[I - (N(s) - \Psi)M(s)[I - \Psi M(s)]^{-1}]^{-1}$
   $\quad = M(s)[I - N(s)M(s)]^{-1} \in \mathcal{RH}_\infty$
   $\Leftrightarrow$ $[M_1(s), N_1(s)]$ is internally stable. ■

In the next lemma, we show that under a full normal rank assumption, a condition for an NI system to have no transmission zeros in the open right-half plane and at the origin is to choose the system gain at the infinity frequency to be positive semidefinite.

Lemma 19: Let $M(s)$ be an NI system with $M(\infty) \geq 0$. Suppose $M(s)$ has full normal rank. Then, $M(s)$ has no transmission zeros in $\mathfrak{R}(s) > 0$ and at the origin.

Proof: In this proof, we use the notion of positive real systems as given by [2, Definition 1] and [2, Lemma 1]. We define the transfer function $\hat{M}(s) = M(s) - M(\infty)$ and $F(s) = s\hat{M}(s)$. Then, $M(s)$ is NI if and only if $\hat{M}(s)$ is NI and $M(\infty) = M(\infty)^T$ via 1) in Lemma 7, and furthermore $\hat{M}(s)$ is NI if and only if $F(s)$ is positive real via 2) in Lemma 7. Then,

$M(\infty) \geq 0$

$\Rightarrow F_1(s) = F(s) + sM(\infty)$ is positive real (since both $F(s)$ and $sM(\infty)$ are positive real and hence their addition must be positive real [27])

$\Leftrightarrow F_1(s) = sM(s)$ is positive real

$\Rightarrow F_1(s)^{-1} = \frac{1}{s}M(s)^{-1}$ is positive real (since $F_1(s)$ is positive real and $\exists s \in \mathbb{C} : \det[F_1(s)] \neq 0$, see [27, Th. 5–8]. So, $F_1(s)^{-1}$ has no poles in $\Re(s) > 0$ and possibly has a simple pole at the origin)

$\Rightarrow M(s)^{-1} = sF_1(s)^{-1}$ has no poles in $\Re(s) > 0$ and at the origin, since either $F_1(s)^{-1}$ has no pole at the origin or if $F_1(s)^{-1}$ has a simple pole at the origin then $sF_1(s)^{-1}$ has a pole-zero cancellation at the origin

$\Leftrightarrow M(s)$ has no transmission zeros in $\Re(s) > 0$ and at the origin since $\exists s \in \mathbb{C} : \det[M(s)] \neq 0$. ∎

The following technical lemma provides a condition for two transfer matrices to be marginally stable and have no poles at the origin.

Lemma 20: Let $M(s)$ be an NI system. Let $\Psi < 0$ be such that $\bar{\lambda}[M(\infty)\Psi] < 1$. Then,
  1) $[I - \Psi M(s)]^{-1}$ has no poles in $\Re(s) > 0$ and at the origin;
  2) $M_1(s) = M(s)[I - \Psi M(s)]^{-1}$ is an NI system without poles at the origin.

Proof:

$\bar{\lambda}[M(\infty)\Psi] < 1$

$\Leftrightarrow \bar{\lambda}[(-\Psi)^{1/2}(-M(\infty))(-\Psi)^{1/2}] < 1$

$\Leftrightarrow M(\infty) - \Psi^{-1} > 0$

$\Rightarrow \det[M(\infty) - \Psi^{-1}] \neq 0$ and $M(s) - \Psi^{-1}$ has no transmission zeros in $\Re(s) > 0$ and at $s = 0$ via Lemma 19 since $M(s) - \Psi^{-1}$ is NI with $M(\infty) - \Psi^{-1} > 0$ and $M(s) - \Psi^{-1}$ has full normal rank

$\Leftrightarrow [I - \Psi M(s)]^{-1}$ has no poles in $\Re(s) > 0$ and at $s = 0$ and $\det [I - \Psi M(\infty)] \neq 0$

$\Leftrightarrow -I + [I - \Psi M(s)]^{-1}$ has no poles in $\Re(s) > 0$ and at $s = 0$ and $\det [I - \Psi M(\infty)] \neq 0$

$\Leftrightarrow \Psi M(s)[I - \Psi M(s)]^{-1}$ has no poles in $\Re(s) > 0$ and at $s = 0$ and $\det [I - \Psi M(\infty)] \neq 0$

$\Leftrightarrow M_1(s) = M(s)[I - \Psi M(s)]^{-1}$ has no poles in $\Re(s) > 0$ and at $s = 0$ and $\det[I - \Psi M(\infty)] \neq 0$

$\Rightarrow M_1(s) = M(s)[I - \Psi M(s)]^{-1}$ is an NI system without poles at the origin via [2, Th. 6] and since $M_1(s)$ has no poles at the origin. Note that the internal stability assumption in [2, Th. 6] can be relaxed to analyticity in $\Re(s) > 0$ in the non-strict NI setting. ∎

The next lemma provides an equivalent condition for the maximum eigenvalue of the product of two symmetric matrices being less than unity.

Lemma 21: Let $A = A^* \in \mathbb{R}^{m \times m}$, $B = B^* \in \mathbb{R}^{m \times m}$, and $C = C^* \in \mathbb{R}^{m \times m}$ with $C < 0$ and $A - B \geq 0$. Then, $\bar{\lambda}[BC] < 1$ if and only if $A - C^{-1} > 0$ and $I - (A - B)^{1/2}(A - C^{-1})^{-1}(A - B)^{1/2} > 0$.

Proof:

$\bar{\lambda}[BC] < 1$

$\Leftrightarrow \bar{\lambda}[(-C)^{1/2}(-B)(-C)^{1/2}] < 1$ (via $C < 0$)

$\Leftrightarrow -C^{-1} + B > 0$

$\Leftrightarrow (A - C^{-1}) - (A - B) > 0$

$\Leftrightarrow \begin{bmatrix} A - C^{-1} & (A - B)^{1/2} \\ (A - B)^{1/2} & I \end{bmatrix} > 0$ (via $A - B \geq 0$)

$\Leftrightarrow A - C^{-1} > 0$ and
$I - (A - B)^{1/2}(A - C^{-1})^{-1}(A - B)^{1/2} > 0$. ∎

The next two highly technical lemmas below are used to underpin the results in Section V-D. We provide a rewritten expression for $\lambda_i[(-I + AC)(I - BC)]$ in the following lemma.

Lemma 22: Let A, B, and C be real symmetric matrices with $C \geq 0$. Then, $\lambda_i[(-I + AC)(I - BC)] = \lambda_i[(-I + C^{1/2}AC^{1/2})(I - C^{1/2}BC^{1/2})]$ for all i.

Proof:

$\lambda_i[(-I + AC)(I - BC)]$

$= \lambda_i[C^{1/2}(A + B - ACB)C^{1/2}] - 1$ (via $C \geq 0$)

$= \lambda_i[(-I + C^{1/2}AC^{1/2})(I - C^{1/2}BC^{1/2})]$. ∎

The positiveness and positive semidefiniteness of the transfer matrices are shown in the following lemma via algebraic manipulation with specific choices of the Ψ.

Lemma 23: Let $M(s)$ be an NI system. Let $\Psi < 0$ be such that $\bar{\lambda}[M(\infty)\Psi] < 1$. Define $Q = [M(\infty) - \Psi^{-1}]^{1/2}(I - [M(\infty) - \Psi^{-1}]^{1/2} \lim_{s \to 0}[M(s) - \Psi^{-1}]^{-1}[M(\infty) - \Psi^{-1}]^{1/2})[M(\infty) - \Psi^{-1}]^{1/2}$. Then,
  1) $M(\infty) - \Psi^{-1} > 0$;
  2) $\lim_{s \to 0}[M(s) - \Psi^{-1}]^{-1} \geq 0$;
  3) $Q \geq 0$;
  4) $I - Q^{1/2}[M(\infty) - \Psi^{-1}]^{-1}Q^{1/2} \geq 0$.
Furthermore, let $\Psi_1 < 0$ be such that $\bar{\lambda}[M(\infty)\Psi_1] < 1$ and define $M_1(s) = M(s)[I - \Psi M(s)]^{-1}$. Then,
  5) $\lim_{s \to 0}[[M(s) - \Psi^{-1}][M(s) - \Psi_1^{-1}]^{-1}]$ is nonsingular;
  6) $I - [M_1(0) - M_1(\infty)]^{1/2}[I + (\Psi - \Psi_1)M_1(0)]^{-1}(\Psi - \Psi_1)[M_1(0) - M_1(\infty)]^{1/2} > 0$.

Proof:
  1) $\bar{\lambda}[M(\infty)\Psi] < 1$
     $\Leftrightarrow \bar{\lambda}[(-\Psi)^{1/2}[-M(\infty)](-\Psi)^{1/2}] < 1$
     $\Leftrightarrow M(\infty) - \Psi^{-1} > 0$.
  2) $\lim_{s \to 0}[M(s) - \Psi^{-1}]^{-1}$
     $= \lim_{s \to 0}[[M(s) - M(\infty)] + [M(\infty) - \Psi^{-1}]]^{-1}$
     $= \lim_{s \to 0} s[s[M(s) - M(\infty)] + s[M(\infty) - \Psi^{-1}]]^{-1} \geq 0$
     (since both $s[M(s) - M(\infty)]$ and $s[M(\infty) - \Psi^{-1}]$ are positive real via Lemma 7 and 1), hence their addition must be positive real [27] and since $s[M(s) - M(\infty)] + s[M(\infty) - \Psi^{-1}]$ is positive real, its inverse is also positive real [27, Th. 5–8]. If $[s[M(s) - M(\infty)] + s[M(\infty) - \Psi^{-1}]]^{-1}$ has a simple pole at the origin, then its residue is Hermitian and

positive semidefinite. Otherwise, $\lim_{s \to 0} s[s[M(s) - M(\infty)] + s[M(\infty) - \Psi^{-1}]]^{-1} = 0)$.

3) $M(s)$ is an NI system
$\Rightarrow M(s)[I - \Psi M(s)]^{-1}$ is an NI system without poles at the origin, via 2) in Lemma 20
$\Rightarrow \lim_{s \to 0} M(s)[I - \Psi M(s)]^{-1} - M(\infty)[I - \Psi M(\infty)]^{-1} \geq 0$ (via Lemma 8)
$\Leftrightarrow \underline{\lambda}[-\lim_{s \to 0}[I - \Psi M(s)]^{-1} + [I - \Psi M(\infty)]^{-1}] \geq 0$
$\Leftrightarrow I - [M(\infty) - \Psi^{-1}]^{1/2} \lim_{s \to 0}[M(s) - \Psi^{-1}]^{-1}[M(\infty) - \Psi^{-1}]^{1/2} \geq 0$ [since $M(\infty) - \Psi^{-1} > 0$ via 1)]
$\Leftrightarrow Q \geq 0$.

4) $\lim_{s \to 0}[M(s) - \Psi^{-1}]^{-1} \geq 0$ [via 2)]
$\Leftrightarrow \underline{\lambda}[\lim_{s \to 0}[M(s) - \Psi^{-1}]^{-1}[M(\infty) - \Psi^{-1}]] \geq 0$
$\Leftrightarrow \underline{\lambda}[I - [M(\infty) - \Psi^{-1}]^{-1}Q] \geq 0$
$\Leftrightarrow I - Q^{1/2}[M(\infty) - \Psi^{-1}]^{-1}Q^{1/2} \geq 0$ [since $Q \geq 0$ via 3)].

5) For convenience, define $\Psi_2 < 0$ such that $\Psi_2^{-1} < \Psi^{-1}$ and $\Psi_2^{-1} < \Psi_1^{-1}$. Then,
$\lim_{s \to 0}[[M(s) - \Psi^{-1}][M(s) - \Psi_1^{-1}]^{-1}]$
$= \lim_{s \to 0}[[M(s) - \Psi^{-1}][M(s) - \Psi_2^{-1}]^{-1}[M(s) - \Psi_2^{-1}][M(s) - \Psi_1^{-1}]^{-1}]$
$= [(\Psi^{-1} - \Psi_2^{-1})^{-1} + \lim_{s \to 0}[M(s) - \Psi^{-1}]^{-1}]^{-1}(\Psi^{-1} - \Psi_2^{-1})^{-1}(\Psi_1^{-1} - \Psi_2^{-1})[(\Psi_1^{-1} - \Psi_2^{-1})^{-1} + \lim_{s \to 0}[M(s) - \Psi_1^{-1}]^{-1}]$ is nonsingular [since $\Psi^{-1} - \Psi_2^{-1} > 0$, $\Psi_1^{-1} - \Psi_2^{-1} > 0$, $\lim_{s \to 0}[M(s) - \Psi^{-1}]^{-1} \geq 0$, and $\lim_{s \to 0}[M(s) - \Psi_1^{-1}]^{-1} \geq 0$ via 2)].

6) For convenience, define $Q_1 = [M(\infty) - \Psi_1^{-1}]^{1/2}(I - [M(\infty) - \Psi_1^{-1}]^{1/2} \lim_{s \to 0}[M(s) - \Psi_1^{-1}]^{-1}[M(\infty) - \Psi_1^{-1}]^{1/2})[M(\infty) - \Psi_1^{-1}]^{1/2}$.
Then,
$I - Q_1^{1/2}[M(\infty) - \Psi_1^{-1}]^{-1}Q_1^{1/2} + Q_1^{1/2}[M(\infty) - \Psi^{-1}]^{-1}Q_1^{1/2} \geq 0$ [since $[M(\infty) - \Psi^{-1}]^{-1} > 0$ and $Q_1 \geq 0$ via 1) and 3) and $I - Q_1^{1/2}[M(\infty) - \Psi_1^{-1}]^{-1}Q_1^{1/2} \geq 0$ via 4)]
$\Leftrightarrow \underline{\lambda}[\lim_{s \to 0}[I + [[M(\infty) - \Psi^{-1}]^{-1} - [M(\infty) - \Psi_1^{-1}]^{-1}][M(s) - M(\infty)][I + [M(\infty) - \Psi_1^{-1}]^{-1}[M(s) - M(\infty)]]^{-1}]] \geq 0$
$\Leftrightarrow \underline{\lambda}[\lim_{s \to 0}[[I + [M(\infty) - \Psi^{-1}]^{-1}[M(s) - M(\infty)]][I + [M(\infty) - \Psi_1^{-1}]^{-1}[M(s) - M(\infty)]]^{-1}]] \geq 0$
$\Leftrightarrow \underline{\lambda}[\lim_{s \to 0}[[M(\infty) - \Psi^{-1}]^{-1}[M(s) - \Psi^{-1}][M(s) - \Psi_1^{-1}]^{-1}[M(\infty) - \Psi_1^{-1}]]] > 0$ [via 1) and 5)]
$\Leftrightarrow I - [M_1(0) - M_1(\infty)]^{1/2}[I + (\Psi - \Psi_1)M_1(0)]^{-1}(\Psi - \Psi_1)[M_1(0) - M_1(\infty)]^{1/2} > 0$. ∎

B. Generalized Internal Stability Results for NI Systems With Possible Poles at the Origin

In this section, necessary and sufficient conditions are derived for internal stability of a positive feedback interconnection composed of an NI system (which may or may not have poles at the origin) and an SNI system. The limits in the stability conditions will be shown to be all finite. We obtain the general result in

Theorem 24 by building on Theorem 9, which was only applicable for NI systems without poles at the origin, by using an appropriate linear shift transformation to remove the undesirable restriction of no poles at the origin. This then enables Theorem 24 to be also applicable for NI systems which may or may not have free body dynamics (i.e., poles at the origin).

Theorem 24: Let $M(s)$ be an NI system and $N(s)$ be an SNI system. Let $\Psi < 0$ be such that $\bar{\lambda}[M(\infty)\Psi] < 1$. Then, $[M(s), N(s)]$ is internally stable if and only if

$I - M(\infty)N(\infty)$ is nonsingular,
$\bar{\lambda}[[I - M(\infty)N(\infty)]^{-1}[M(\infty)N(0) - I]] < 0$, and
$\bar{\lambda}[\lim_{s \to 0}[[I - \Psi M(\infty)][I - N(s)M(\infty)]^{-1}[N(s)M(s) - I][I - \Psi M(s)]^{-1}]] < 0$.

Proof: Define $M_1(s) = [I - M(s)\Psi]^{-1}M(s)$ and $N_1(s) = N(s) - \Psi$. Then,

$[M(s), N(s)]$ is internally stable
$\Leftrightarrow [M_1(s), N_1(s)]$ is internally stable (by Lemma 18)
$\Leftrightarrow I - M_1(\infty)N_1(\infty)$ is nonsingular,
$\bar{\lambda}[[I - M_1(\infty)N_1(\infty)]^{-1}(M_1(\infty)N_1(0) - I)] < 0$, and
$\bar{\lambda}[[I - N_1(0)M_1(\infty)]^{-1}(N_1(0)M_1(0) - I)] < 0$ (since $M_1(s)$ is an NI system without poles at the origin via Lemma 20 and $N_1(s)$ is an SNI system and then direct use of Theorem 9)
$\Leftrightarrow I - M(\infty)N(\infty)$ is nonsingular (via Lemma 18),
$\bar{\lambda}[[I - [I - M(\infty)\Psi]^{-1}M(\infty)(N(\infty) - \Psi)]^{-1}[[I - M(\infty)\Psi]^{-1}M(\infty)(N(0) - \Psi) - I]] < 0$, and
$\bar{\lambda}[\lim_{s \to 0}[[I - (N(s) - \Psi)[I - M(\infty)\Psi]^{-1}M(\infty)]^{-1}[(N(s) - \Psi)[I - M(s)\Psi]^{-1}M(s) - I]]] < 0$
$\Leftrightarrow I - M(\infty)N(\infty)$ is nonsingular,
$\bar{\lambda}[[[I - M(\infty)\Psi]^{-1}[I - M(\infty)\Psi - M(\infty)N(\infty) + M(\infty)\Psi]]^{-1}[[I - M(\infty)\Psi]^{-1}[M(\infty)N(0) - M(\infty)\Psi - I + M(\infty)\Psi]]] < 0$, and
$\bar{\lambda}[\lim_{s \to 0}[[I - (N(s) - \Psi)M(\infty)[I - \Psi M(\infty)]^{-1}]^{-1}[(N(s) - \Psi)M(s)[I - \Psi M(s)]^{-1} - I]]] < 0$
$\Leftrightarrow I - M(\infty)N(\infty)$ is nonsingular,
$\bar{\lambda}[[I - M(\infty)N(\infty)]^{-1}[I - M(\infty)\Psi][I - M(\infty)\Psi]^{-1}[M(\infty)N(0) - I]] < 0$, and
$\bar{\lambda}[\lim_{s \to 0}[[[I - \Psi M(\infty) - N(s)M(\infty) + \Psi M(\infty)][I - \Psi M(\infty)]^{-1}]^{-1}[[N(s)M(s) - \Psi M(s) - I + \Psi M(s)][I - \Psi M(s)]^{-1}]]] < 0$
$\Leftrightarrow I - M(\infty)N(\infty)$ is nonsingular,
$\bar{\lambda}[[I - M(\infty)N(\infty)]^{-1}[M(\infty)N(0) - I]] < 0$, and
$\bar{\lambda}[\lim_{s \to 0}[[I - \Psi M(\infty)][I - N(s)M(\infty)]^{-1}[N(s)M(s) - I][I - \Psi M(s)]^{-1}]] < 0$.

The above algebraic reformulations, together with Lemmas 4 and 8, also demonstrate why the eigenvalues of the matrices in the conditions of the theorem statement are real. ∎

Remark 25: We need to show that $\lim_{s \to 0}[[N(s)M(s) - I][I - \Psi M(s)]^{-1}] = \lim_{s \to 0} N(s)M(s)[I - \Psi M(s)]^{-1} - \lim_{s \to 0}[I - \Psi M(s)]^{-1}$ is finite. Since both $M(s)[I - \Psi M(s)]^{-1}$ and $[I - \Psi M(s)]^{-1}$ have no poles at the origin via Lemma 20, $\lim_{s \to 0}[[N(s)M(s) - I][I - \Psi M(s)]^{-1}]$ is finite.

We now give a different internal stability necessary and sufficient condition in Theorem 26 which is based on Theorem 14 instead of Theorem 9.

Theorem 26: Let $M(s)$ be an NI system and $N(s)$ be an SNI system. Let $\Psi < 0$ be such that $\bar{\lambda}[M(\infty)\Psi] < 1$. Then, $[M(s), N(s)]$ is internally stable if and only if

$I - M(\infty)N(\infty)$ is nonsingular,
$\bar{\lambda}[\lim_{s\to 0}[[I - M(s)\Psi]^{-1}[M(s)N(\infty) - I][I - M(\infty)N(\infty)]^{-1}[I - M(\infty)\Psi]]] < 0$, and
$\bar{\lambda}[\lim_{s\to 0}[[N(s)M(s) - I][I - N(\infty)M(s)]^{-1}]] < 0$.

Proof: The proof is similar to that of Theorem 24 except Theorem 14 is used instead of Theorem 9. ∎

Remark 27: We need to show that $\lim_{s\to 0}[[I - M(s)\Psi]^{-1}[M(s)N(\infty) - I][I - M(\infty)N(\infty)]^{-1}[I - M(\infty)\Psi]]$ is finite. Note that $\lim_{s\to 0}[[I - M(s)\Psi]^{-1}[M(s)N(\infty) - I]]$ is finite by Lemma 20, $(I - M(\infty)N(\infty))^{-1}$ is finite via the first condition in Theorem 26, and $I - M(\infty)\Psi$ is obviously finite.

The following technical lemma will be needed in Remark 29 to show that the limit in the third condition of Theorem 26 is also finite.

Lemma 28: Let all the assumptions in Theorem 26 hold. Furthermore, suppose that $I - M(\infty)N(\infty)$ is nonsingular and $\bar{\lambda}[\lim_{s\to 0}[[I - M(s)\Psi]^{-1}[M(s)N(\infty) - I][I - M(\infty)N(\infty)]^{-1}[I - M(\infty)\Psi]]] < 0$. Then, $\lim_{s\to 0}[[I - \Psi M(s)][I - N(\infty)M(s)]^{-1}]$ is finite and nonsingular.

Proof: First note that $\lim_{s\to 0}[[N(\infty)M(s) - I][I - \Psi M(s)]^{-1}]$ is finite via Lemma 20. Next, note that $\bar{\lambda}[\lim_{s\to 0}[[I - M(s)\Psi]^{-1}[M(s)N(\infty) - I][I - M(\infty)N(\infty)]^{-1}[I - M(\infty)\Psi]]] < 0$

$\Rightarrow \lim_{s\to 0}[[I - M(s)\Psi]^{-1}[M(s)N(\infty) - I][I - M(\infty)N(\infty)]^{-1}[I - M(\infty)\Psi]]$ is nonsingular

$\Rightarrow \lim_{s\to 0}[[I - M(s)\Psi]^{-1}[M(s)N(\infty) - I]]$ is nonsingular

$\Rightarrow \lim_{s\to 0}\frac{\det(M(s)N(\infty) - I)}{\det(I - M(s)\Psi)} \neq 0$

$\Rightarrow \lim_{s\to 0}\frac{\det(N(\infty)M(s) - I)}{\det(I - \Psi M(s))} \neq 0$

$\Rightarrow \lim_{s\to 0}[[N(\infty)M(s) - I][I - \Psi M(s)]^{-1}]$ is nonsingular.

Then,

$\lim_{s\to 0}[[N(\infty)M(s) - I][I - \Psi M(s)]^{-1}]$ is finite and nonsingular

$\Leftrightarrow \lim_{s\to 0}[[I - \Psi M(s)][N(\infty)M(s) - I]^{-1}]$ is finite and nonsingular

$\Leftrightarrow \lim_{s\to 0}[[I - \Psi M(s)][I - N(\infty)M(s)]^{-1}]$ is finite and nonsingular. ∎

Remark 29: We need to show that $\lim_{s\to 0}[[N(s)M(s) - I][I - N(\infty)M(s)]^{-1}] = \lim_{s\to 0}[[N(s)M(s) - I][I - \Psi M(s)]^{-1}[I - \Psi M(s)][I - N(\infty)M(s)]^{-1}]$ is finite. It is clear that $\lim_{s\to 0}[[N(s)M(s) - I][I - \Psi M(s)]^{-1}]$ and $\lim_{s\to 0}[[I - \Psi M(s)][I - N(\infty)M(s)]^{-1}]$ are finite via Lemmas 20 and 28, respectively. Hence, $\lim_{s\to 0}[[N(s)M(s) - I][I - N(\infty)M(s)]^{-1}]$ is finite.

In contrast with the stability results in the literature [1], [11], [12], [23], the conditions presented here are a generalization on those results in at least three aspects:

1) the stability results allow for poles on the imaginary axis and at the origin;
2) the stability results impose no restriction on the instantaneous gains of the systems; and
3) the conditions are significantly easier to check.

Note that several restrictive assumptions are imposed in [12] including strict properness of the NI system. [12] also requires multiple complex matrix factorizations. Furthermore, some of the assumptions in [12] were imposed on the resulting factorizations, thereby destroying intuition. The results in Theorems 24 and 26 impose no such restrictions. Section V-D discusses existence and construction of the matrix Ψ that is needed in Theorems 24 and 26.

C. Specializations of the Generalized Internal Stability Conditions

First, we give a corollary that does not depend on the matrix Ψ. It is a SISO specialization of both Theorems 24 and 26 in the special situation when the NI system necessarily has poles at the origin.

Corollary 30: Let $M(s)$ be a scalar, NI system and $N(s)$ be a scalar, SNI system. Let $s = 0$ be a (single or double) pole of $M(s)$. Then, $[M(s), N(s)]$ is internally stable if and only if either one of the following two conditions holds:

i) $M(\infty)N(\infty) < 1$ and $N(0) < 0$;
ii) $M(\infty)N(\infty) > 1$ and $N(\infty) > 0$.

Proof: Writing $M(s)$ as a Laurent series, the three conditions in Theorem 26 simplify, in the scalar case, to $M(\infty)N(\infty) \neq 1$, $\frac{N(\infty)}{1 - M(\infty)N(\infty)} < 0$ and $\frac{N(0)}{N(\infty)} > 0$ after taking the limit as $s \to 0$. It is easy to see that these three conditions are equivalent to either condition i) or condition ii) in this corollary statement using the NI property $N(0) > N(\infty)$ via [1, Lemma 2].

Alternatively, Theorem 24 can be used instead of Theorem 26 to give the same specializations. Writing $M(s)$ as a Laurent series, the three conditions in Theorem 24 simplify, in the scalar case, to $M(\infty)N(\infty) \neq 1$, $\frac{M(\infty)N(0) - 1}{1 - M(\infty)N(\infty)} < 0$, and $\frac{N(0)}{1 - N(0)M(\infty)} < 0$ after taking the limit as $s \to 0$. These three conditions are equivalent to either one of the following two statements:

1) $M(\infty)N(\infty) < 1$, $M(\infty)N(0) < 1$ and $N(0) < 0$;
2) $M(\infty)N(\infty) > 1$, $M(\infty)N(0) > 1$ and $N(0) > 0$.

Equivalence between condition 1) and condition i) in the corollary statement can be seen on noting that $M(\infty)N(0) = [M(\infty)N(\infty)]\left[\frac{N(0)}{N(\infty)}\right] < 1$ since $N(\infty) < N(0) < 0$ via [1, Lemma 2]. Equivalence between condition 2) and condition ii) in the corollary statement can be seen on noting that $M(\infty)N(0) = [M(\infty)N(\infty)]\left[\frac{N(0)}{N(\infty)}\right] > 1$ since $0 < N(\infty) < N(0)$ via [1, Lemma 2]. ∎

Next, we specialize Theorems 24 and 26 to a MIMO case that is in fact still a generalization of the results in [12]. If we consider the situation that $M(s)$ is a strictly proper transfer function with NI properties, then Theorems 24 and 26 can be specialized to the results in 2) and 3) in Corollary 31, respectively.

Corollary 31: Let $M(s)$ be a strictly proper, NI system and $N(s)$ be an SNI system. Let $\Psi < 0$. Then, the following three conditions are equivalent:

1) $[M(s), N(s)]$ is internally stable;
2) $\bar{\lambda}[\lim_{s \to 0}[[N(s)M(s) - I][I - \Psi M(s)]^{-1}]] < 0$;
3) $\bar{\lambda}[\lim_{s \to 0}[[I - M(s)\Psi]^{-1}[M(s)N(\infty) - I]]] < 0$ and $\bar{\lambda}[\lim_{s \to 0}[[N(s)M(s) - I][I - N(\infty)M(s)]^{-1}]] < 0$.

Proof: The equivalence between 1) and 2) is omitted since it is the direct consequence of Theorem 24 and the assumption of strictly properness of $M(s)$. The equivalence between 1) and 3) is omitted since it is a direct consequence of Theorem 26 and the assumption of strict properness of $M(s)$. ∎

Note that the limit in 2) of Corollary 31 is finite via Lemma 20 and the limits in 3) of Corollary 31 are finite via Lemmas 20 and 28. Under the same assumption of strictly proper $M(s)$, Mabrok *et al.* [12] proposed necessary and sufficient conditions for internal stability of positive feedback interconnections of NI systems. However, the stability results in [12] require additional assumptions such as the invertibility of $F^T \bar{G}(0)F$ and N_f being either positive or negative definite (see [12, Th. 1]). When these assumptions imposed in [12] do not hold, such as $F^T \bar{G}(0)F$ being singular or N_f being sign-indefinite, the conditions in [12] are no longer applicable. Unlike the results in [12], the conditions in Corollary 31 are easy-to-check, are necessary and sufficient conditions, impose no extra assumptions, and do not require any matrix factorizations. Indeed, the extra assumptions in [12] on objects that are a result of such factorizations unfortunately inhibit intuition into the results of [12].

The internal stability condition in 2) of Corollary 31 only depends on the steady-state (or dc) gains and a negative definite matrix Ψ. If we rearrange the inequality in 2), we end up to have $\bar{\lambda}[M_1(0)N_1(0)] < 1$, where $M_1(s) = [I - M(s)\Psi]^{-1}M(s)$ and $N_1(s) = N(s) - \Psi$, which means that the stability of the positive-feedback interconnection relies on only the dc loop gain of the transformed system. Surprisingly, simplification of the two inequalities in 3) to obtain the inequality in 2) is not directly obvious for the MIMO situation. However, for the SISO case, it is straightforward to realize that the terms of $[M(s)N(\infty) - I]$ and $[I - N(\infty)M(s)]^{-1}$ in first and second inequalities of 3) can be eliminated via multiplying both inequalities in 3) to give the inequality in 2).

We next give a corollary that does not depend of the matrix Ψ. It is a MIMO specialization of Corollary 31 in the situation when the NI system necessarily has single or double poles at the origin in all directions.

Corollary 32: Let $M(s)$ be a strictly proper, NI system and $N(s)$ be an SNI system. Assume one of the following conditions holds:

1) $\lim_{s \to 0} s^2 M(s)$ is nonsingular;
2) $\lim_{s \to 0} s^2 M(s) = 0$ and $\lim_{s \to 0} sM(s)$ is nonsingular.

Then, $[M(s), N(s)]$ is internally stable if and only if $N(0) < 0$.

Proof: Consider each of the two cases of this corollary separately. Writing $M(s)$ as a Laurent series into either condition 2) or condition 3) of Corollary 31 gives the required result after evaluating the limit and simplifying. ∎

In Section V-B, general internal stability conditions of positive feedback interconnections of NI systems were derived. We now show that the main stability theorems of that section (i.e., Theorems 24 and 26) specialize to the prior stability conditions in Theorems 9 and 14, respectively, under the corresponding assumption that $M(s)$ is an NI system without poles at the origin, thereby demonstrating the generality of Theorems 24 and 26.

Since, by inspection, two of the conditions in Theorem 24 (respectively Theorem 26) are trivially equivalent to two of the conditions in Theorem 9 (respectively Theorem 14), we only need to proof equivalence of the remaining inequality in Theorem 24 (respectively Theorem 26) with the remaining inequality in Theorem 9 (respectively Theorem 14). Note that to show these equivalences in Lemmas 33 and 34, we cannot simply assume that $\Psi = 0$ since $\Psi < 0$ is required in the assumptions of both Theorems 24 and 26 and a limiting argument cannot be used as the results in Theorems 24 and 26 are valid for any arbitrary $\Psi < 0$ that satisfies $\bar{\lambda}(M(\infty)\Psi) < 1$.

Lemma 33: Let all the assumptions of Theorem 24 hold and furthermore suppose $M(s)$ has no poles at the origin. Then,

$$\bar{\lambda}[\lim_{s \to 0}[[I - \Psi M(\infty)][I - N(s)M(\infty)]^{-1}[N(s)M(s) - I][I - \Psi M(s)]^{-1}]] < 0$$
$$\Leftrightarrow \bar{\lambda}[[I - N(0)M(\infty)]^{-1}(N(0)M(0) - I)] < 0.$$

Proof: Let $R = [M(0) - M(\infty)]^{1/2}$. See Lemma 8 for proof that $M(0) - M(\infty) \geq 0$. Then,

$\bar{\lambda}[\lim_{s \to 0}[[I - \Psi M(\infty)][I - N(s)M(\infty)]^{-1}[N(s)M(s) - I][I - \Psi M(s)]^{-1}]] < 0$

$\Leftrightarrow \bar{\lambda}[[I - \Psi M(\infty)][I - N(0)M(\infty)]^{-1}[N(0)M(0) - I][I - \Psi M(0)]^{-1}] < 0$ (since $M(s)$ is an NI system without poles at the origin and $N(s)$ is an SNI system, $M(0)$ and $N(0)$ exist)

$\Leftrightarrow \bar{\lambda}[[I - \Psi M(0)]^{-1}[I - \Psi M(\infty)][I - N(0)M(\infty)]^{-1}[N(0)M(0) - I]] < 0$

$\Leftrightarrow \bar{\lambda}[[I - \Psi M(0)]^{-1}[I - \Psi M(0) + \Psi M(0) - \Psi M(\infty)][I - N(0)M(\infty)]^{-1}[N(0)M(0) - N(0)M(\infty) + N(0)M(\infty) - I]] < 0$

$\Leftrightarrow \bar{\lambda}[[I + [I - \Psi M(0)]^{-1}\Psi R^2][[I - N(0)M(\infty)]^{-1}N(0)R^2 - I]] < 0$

$\Leftrightarrow \bar{\lambda}[[[I - N(0)M(\infty)]^{-1}N(0) - [I - \Psi M(0)]^{-1}\Psi + [I - \Psi M(0)]^{-1}\Psi R^2[I - N(0)M(\infty)]^{-1}N(0)]R^2] < 1$

$\Leftrightarrow \bar{\lambda}[R[[I - N(0)M(\infty)]^{-1}N(0) - [I - \Psi M(0)]^{-1}\Psi + [I - \Psi M(0)]^{-1}\Psi R^2[I - N(0)M(\infty)]^{-1}N(0)]R] < 1$

$\Leftrightarrow \bar{\lambda}[[I + R[I - \Psi M(0)]^{-1}\Psi R][R[I - N(0)M(\infty)]^{-1}N(0)R - I]] < 0$

$\Leftrightarrow \bar{\lambda}[[I - R[M(0) - \Psi^{-1}]^{-1}R][R[I - N(0)M(\infty)]^{-1}N(0)R - I]] < 0$

$\Leftrightarrow [I - R[M(0) - \Psi^{-1}]^{-1}R]^{1/2}[R[I - N(0)M(\infty)]^{-1}N(0)R - I][I - R[M(0) - \Psi^{-1}]^{-1}R]^{1/2} < 0$ (since $I - R[M(0) - \Psi^{-1}]^{-1}R > 0$ via Lemma 21)

$\Leftrightarrow \bar{\lambda}[R[I - N(0)M(\infty)]^{-1}N(0)R] < 1$

$\Leftrightarrow \bar{\lambda}[[I - N(0)M(\infty)]^{-1}N(0)R^2 - I] < 0$

$\Leftrightarrow \bar{\lambda}[[I - N(0)M(\infty)]^{-1}[N(0)M(0) - N(0)M(\infty) - I + N(0)M(\infty)]] < 0$

$\Leftrightarrow \bar{\lambda}[[I - N(0)M(\infty)]^{-1}(N(0)M(0) - I)] < 0.$ ∎

In Lemma 34, we state the equivalence between the stability conditions of Theorem 26 and Theorem 14 under the specialization that $M(s)$ has no poles at the origin.

Lemma 34: Let all the assumptions of Theorem 26 hold and furthermore suppose $M(s)$ has no poles at the origin. Then,

$$\bar{\lambda}[\lim_{s\to 0}[[I - M(s)\Psi]^{-1}[M(s)N(\infty) - I][I - M(\infty)N(\infty)]^{-1}[I - M(\infty)\Psi]]] < 0$$
$$\Leftrightarrow \bar{\lambda}[(M(0)N(\infty) - I)[I - M(\infty)N(\infty)]^{-1}] < 0.$$

Proof: The proof is similar to that of Lemma 33 except that the second inequality in Theorem 26 is used instead of the third inequality in Theorem 24. ∎

D. Existence and Construction of a Negative Definite Matrix Ψ

In Sections V-B and V-C, we have derived powerful internal stability conditions that allow for NI systems with free body dynamics. Some of the conditions in Theorems 24 and 26 and Corollary 31 are expressed in terms of a negative definite matrix Ψ that is not a part of the problem data. In Corollary 31, since $M(\infty) = 0$, then any choice of a negative definite Ψ is fine. However, if the plant is not strictly proper, i.e., $M(\infty) \neq 0$, $\Psi < 0$ also needs to satisfy $\bar{\lambda}[M(\infty)\Psi] < 1$ as required by the assumptions of Theorems 24 and 26. We first demonstrate that such a Ψ always exists, even when $M(\infty) \neq 0$, and give a simple constructive procedure to determine one such Ψ.

Since $M(\infty) \neq 0$ is a symmetric matrix, $M(\infty)$ can be factored into $M(\infty) = Q\Lambda Q^T$ where Q and Λ are orthogonal and diagonal matrices, respectively. Now, choose $\Psi = Q\Lambda_1^{-1}Q^T$, where Λ_1 is a diagonal matrix with strictly negative numbers on its diagonal. It is clear that $\Psi < 0$. Then, via 1) in Lemma 23, $\bar{\lambda}[M(\infty)\Psi] < 1$ is equivalent to $M(\infty) - \Psi^{-1} > 0$. Hence, the problem reduces to selecting a strictly negative definite and diagonal matrix Λ_1 that satisfies $\Lambda - \Lambda_1 > 0$. It is obvious that there always exists a matrix Λ_1 fulfilling $\Lambda_1 < \Lambda$. Hence, it is trivial to construct a matrix $\Psi < 0$ such that $\bar{\lambda}[M(\infty)\Psi] < 1$.

It is trivial to observe that there is a large set of matrices $\Psi < 0$ satisfying $\bar{\lambda}[M(\infty)\Psi] < 1$. The statements in Theorems 24 and 26 and Corollary 31 indicate that any arbitrary Ψ in this set could be selected. The following two theorems show that the conditions that depend on Ψ in Theorems 24 and 26 and Corollary 31 are either fulfilled for all $\Psi < 0$ satisfying $\bar{\lambda}[M(\infty)\Psi] < 1$ or violated for all $\Psi < 0$ satisfying $\bar{\lambda}[M(\infty)\Psi] < 1$. Hence, choosing any arbitrary $\Psi < 0$ satisfying $\bar{\lambda}[M(\infty)\Psi] < 1$, for example, via the constructive procedure described earlier, will be all that is required to use the powerful results in Theorems 24 and 26 and Corollary 31.

Theorem 35: Let $M(s)$ be an NI system and $N(s)$ be an SNI system. Define $\underline{X} = \{\Psi < 0 : \bar{\lambda}[M(\infty)\Psi] < 1\}$. Then, $\exists \Psi_1 \in \underline{X} : \bar{\lambda}[\lim_{s\to 0}[[I - \Psi_1 M(\infty)][I - N(s)M(\infty)]^{-1}[N(s)M(s) - I][I - \Psi_1 M(s)]^{-1}]] < 0$ if and only if $\forall \Psi \in \underline{X}$, $\bar{\lambda}[\lim_{s\to 0}[[I - \Psi M(\infty)][I - N(s)M(\infty)]^{-1}[N(s)M(s) - I][I - \Psi M(s)]^{-1}]] < 0$.

Proof:
($\Leftarrow$) Trivial.
($\Rightarrow$) Choose an arbitrary $\Psi \in \underline{X}$. Let $M_1(s) = M(s)[I - \Psi M(s)]^{-1}$, $N_1(s) = N(s) - \Psi$, and $R = I - [M_1(0) - M_1(\infty)]^{1/2}[I + (\Psi - \Psi_1)M_1(0)]^{-1}(\Psi - \Psi_1)[M_1(0) - M_1(\infty)]^{1/2}$. See Lemma 8 for proof that $M_1(0) - M_1(\infty) \geq 0$ after noting that $M_1(s)$ is NI without poles at the origin via Lemma 20. Then,

$\bar{\lambda}[\lim_{s\to 0}[[I - \Psi_1 M(\infty)][I - N(s)M(\infty)]^{-1}[N(s)M(s) - I][I - \Psi_1 M(s)]^{-1}]] < 0$

$\Leftrightarrow \bar{\lambda}[\lim_{s\to 0}[[I - \Psi_1 M(\infty)][I - \Psi M(\infty)]^{-1}[I - \Psi M(\infty)][I - N(s)M(\infty)]^{-1}[N(s)M(s) - I][I - \Psi M(s)]^{-1}[I - \Psi M(s)][I - \Psi_1 M(s)]^{-1}]] < 0$

$\Leftrightarrow \bar{\lambda}[[-I + [I - N_1(0)M_1(\infty)]^{-1}N_1(0)[M_1(0) - M_1(\infty)]][I - [I + (\Psi - \Psi_1)M_1(0)]^{-1}(\Psi - \Psi_1)[M_1(0) - M_1(\infty)]]] < 0$

$\Leftrightarrow \bar{\lambda}[[-I + [M_1(0) - M_1(\infty)]^{1/2}[I - N_1(0)M_1(\infty)]^{-1}N_1(0)[M_1(0) - M_1(\infty)]^{1/2}]R] < 0$ (via Lemma 22)

$\Leftrightarrow \bar{\lambda}[-I + [M_1(0) - M_1(\infty)]^{1/2}[I - N_1(0)M_1(\infty)]^{-1}N_1(0)[M_1(0) - M_1(\infty)]^{1/2}] < 0$ (since $R > 0$ via 6) in Lemma 23)

$\Leftrightarrow \bar{\lambda}[\lim_{s\to 0}[[I - \Psi M(\infty)][I - N(s)M(\infty)]^{-1}[N(s)M(s) - I][I - \Psi M(s)]^{-1}]] < 0$.

The result then follows since $\Psi \in \underline{X}$ was arbitrary. ∎

Necessity of Theorem 35 states that if there exists a $\Psi < 0$ satisfying $\bar{\lambda}[M(\infty)\Psi] < 1$ and the third condition in Theorem 24 (i.e., the condition involving the matrix Ψ), then this third condition of Theorem 24 is satisfied for all $\Psi < 0$ satisfying $\bar{\lambda}[M(\infty)\Psi] < 1$. The contrapositive statement of this same necessity of Theorem 35 states that if there exists a $\Psi < 0$ satisfying $\bar{\lambda}[M(\infty)\Psi] < 1$ that violates the third condition of Theorem 24, then this third condition of Theorem 24 is violated for all $\Psi < 0$ satisfying $\bar{\lambda}[M(\infty)\Psi] < 1$.

Theorem 36 is similar to Theorem 35, but tackles the second inequality of Theorem 26 (i.e., the one involving the matrix Ψ) instead of the third inequality of Theorem 24.

Theorem 36: Let $M(s)$ be an NI system and $N(s)$ be an SNI system. Define $\underline{X} = \{\Psi < 0 : \bar{\lambda}[M(\infty)\Psi] < 1\}$. Then, $\exists \Psi_1 \in \underline{X} : \bar{\lambda}[\lim_{s\to 0}[[I - M(s)\Psi_1]^{-1}[M(s)N(\infty) - I][I - M(\infty)N(\infty)]^{-1}[I - M(\infty)\Psi_1]]] < 0$ if and only if $\forall \Psi \in \underline{X}$, $\bar{\lambda}[\lim_{s\to 0}[[I - M(s)\Psi]^{-1}[M(s)N(\infty) - I][I - M(\infty)N(\infty)]^{-1}[I - M(\infty)\Psi]]] < 0$.

Proof: Proof is omitted as it is similar to that of Theorem 35. ∎

Theorem 36 states that the second inequality of Theorem 26 is either satisfied for all $\Psi < 0$ with $\bar{\lambda}[M(\infty)\Psi] < 1$ or violated for all $\Psi < 0$ with $\bar{\lambda}[M(\infty)\Psi] < 1$. Hence, one only needs any arbitrary $\Psi < 0$ with $\bar{\lambda}[M(\infty)\Psi] < 1$ to use the result in Theorem 26.

VI. NUMERICAL EXAMPLES

Two numerical examples are given to illustrate the internal stability results developed in the previous sections.
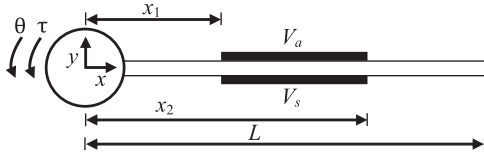
Fig. 2. Schematic diagram of a slewing beam equivalent to a robotic arm.

A. Without Poles at the Origin

In this section, an example is given to illustrate the inapplicability or conservativeness of earlier results and demonstrate the main results developed in this paper. Let us consider a positive feedback interconnection $[M(s), N(s)]$, as shown in Fig. 1. An undamped SISO plant $M(s)$ is given by $M(s) = \frac{1}{s^2+1} + 2$, which is NI without poles at the origin according to [11, Definition 1]. A SISO controller $N(s)$ is given by $N(s) = \frac{1}{s+1} + 2$, which is SNI according to Definition 2. The internal stability of $[M(s), N(s)]$ cannot be determined via [11, Th. 1], since $M(\infty)N(\infty) = 4 \neq 0$. Also, [23, Corollary 4.3] cannot be used to check the internal stability of $[M(s), N(s)]$ since $\bar{\sigma}(M(0)N(0)) = 9 \not< 1$ and $\bar{\sigma}(M(\infty)N(\infty)) = 4 \not< 1$. Next, we attempt to use the IQC-based sufficient conditions of [23, Th. 4.2]. We will demonstrate via this example that the sufficient conditions of [23, Th. 4.2] are conservative. Let $\Pi_0 = \Pi_0^* = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ and $\Pi_\infty = \Pi_\infty^* = \begin{bmatrix} d & e \\ e & f \end{bmatrix}$. [23, Th. 4.2] is only applicable if there exist scalars a, b, c, d, e, and f such that the sufficient conditions in [23, Th. 4.2] are satisfied for all $\tau \in [0, 1]$. However, $\begin{bmatrix} N(0) \\ I \end{bmatrix}^* \Pi_0 \begin{bmatrix} N(0) \\ I \end{bmatrix} = 9a + 6b + c < 0$ and $\begin{bmatrix} I \\ \tau M(0) \end{bmatrix}^* \Pi_0 \begin{bmatrix} I \\ \tau M(0) \end{bmatrix} = 9c\tau^2 + 6b\tau + a \geq 0$ cannot be fulfilled simultaneously when $\tau = 1/9$. Hence, [23, Th. 4.2] cannot be used here.

By applying Theorem 17 in Section IV, we can easily conclude that $[M(s), N(s)]$ is internally stable since either condition 2) or 3) is satisfied, i.e., $M(\infty)N(\infty) = 4 > 1$, $M(0)N(0) = 9 > 1$ and $M(\infty)N(0) = 6 > 1$. The robust stability conditions in this paper are easy to check, are necessary and sufficient, and only depend on the steady-state and instantaneous gains (i.e., the 0 and ∞ frequency gains) of the systems.

B. With Poles at the Origin

In this section, a physically motivated example is presented to demonstrate the application of the generalized feedback stability results derived in this paper. The example is directly taken from [12] for ease of comparison. Let us consider a flexible robotic arm, as shown in Fig. 2, which is driven by a motor mounted at one end and with two piezoelectric patches attached to either side of the arm which are used as an actuator and a sensor, respectively. The control inputs of this flexible structure are the voltage V_a applied to the actuator patch and the torque of the motor τ, while the outputs of the system are the voltage V_s generated by the sensor patch and the motor hub angle θ. Using techniques detailed in [28], this flexible robotic arm can be modeled as an infinite-dimensional transfer function:

$$G(s) = \frac{1}{D(s)} \begin{bmatrix} N_{\tau,\theta}(s) & N_{V_a,\theta}(s) \\ N_{\tau,V_s}(s) & N_{V_a,V_s}(s) \end{bmatrix},$$

where $N_{\tau,\theta}(s), N_{V_a,\theta}(s), N_{\tau,V_s}(s), N_{V_a,V_s}(s),$ and $D(s)$ are given in [28, eq. (26)–(28)]. Through approximation approaches, we can represent the above model $G(s)$ as the finite-dimensional model, and for the sake of simplification, we here only consider the finite-dimensional model $M(s)$ with only the first resonant mode, which can be expressed as

$$M(s) = \sum_{i=0}^{1} \begin{bmatrix} \dfrac{a_i}{s^2+p_i^2} & \dfrac{b_i}{s^2+p_i^2} \\ \dfrac{c_i}{s^2+p_i^2} & \dfrac{d_i}{s^2+p_i^2} \end{bmatrix},$$

where $a_0 = 0.14, b_0 = c_0 = d_0 = p_0 = 0, a_1 = 3.0907, b_1 = c_1 = 3.5573 \times 10^{-4}, d_1 = 2.35,$ and $p_1 = 3.4$ [12]. This finite-dimensional model $M(s)$ is an NI system since all poles of $M(s)$ are located in the closed left-half plane, $j[M(j\omega) - M(j\omega)^*] = 0$ for all $\omega \in (0, \infty) \backslash 3.4$, the residue matrix

$$\lim_{s \to j3.4} (s - j3.4)jM(s)$$

$$= \begin{bmatrix} \dfrac{3.0907}{6.8} & \dfrac{3.5573 \times 10^{-4}}{6.8} \\ \dfrac{3.5573 \times 10^{-4}}{6.8} & \dfrac{2.35}{6.8} \end{bmatrix} > 0,$$

and $\lim_{s \to 0} s^2 M(s) = \begin{bmatrix} 0.14 & 0 \\ 0 & 0 \end{bmatrix} \geq 0$ which imply that the conditions 3) and 4) in Definition 1 are fulfilled. We then apply the SNI controller [3] $N(s) = (sI + \Gamma\Phi)^{-1}\Gamma - \Delta$, where $\Gamma = \begin{bmatrix} 35 & 15 \\ 15 & 20 \end{bmatrix}, \Phi = \begin{bmatrix} 0.745 & 0.521 \\ 0.521 & 1.021 \end{bmatrix}$ and $\Delta = \begin{bmatrix} 4.29 & 0 \\ 0 & 2.22 \end{bmatrix}$, as detailed in [12]. Note that since $M(s)$ is an NI system with two poles at the origin, the internal stability results introduced in [11] and [23] are not applicable. Hence, for ease of calculation, we directly set $\Psi = N(0) < 0$ and use the generalized stability results presented in Section V-C. Using the stability condition in 2) of Corollary 31, we get $\bar{\lambda}[\lim_{s \to 0}[[N(s)M(s) - I][I - \Psi M(s)]^{-1}]] = \bar{\lambda}[\lim_{s \to 0}[[N(0)M(s) - I][I - N(0)M(s)]^{-1}]] = -1 < 0$. Hence, feedback stability of $[M(s), N(s)]$ is guaranteed for the chosen controller via 2) in Corollary 31 without any complicated calculations. In contrast, the results in [12] require computation of several matrix factorizations and also a set of different inequalities need to be checked for different cases that depend on the positive or negative semidefiniteness of a specific matrix which is a result of such factorizations in order to determine whether the positive-feedback interconnection is internally stable or not.

VII. Conclusion

We have removed restrictive assumptions from the robust stability analysis of positive feedback interconnections of NI systems, and generalized results for NI systems that may also have poles at the origin. New necessary and sufficient conditions have thus been derived that reduce to the earlier necessary and sufficient conditions when the assumptions of earlier literature are imposed. This demonstrates that the new results generalize the earlier work in the area. We also show that it is possible to simplify the new necessary and sufficient conditions by imposing alternative new succinct assumptions that were not known

in prior literature. Because the proposed extended internal stability results rely on a negative definite matrix Ψ with specific properties, it was also shown that such a matrix always exists. Moreover, a systematic procedure was given for constructing Ψ to fulfil the required properties. We also prove that no search is needed for an appropriate Ψ as the results are valid for any arbitrarily chosen Ψ. Two numerical examples are given that demonstrate the completeness of the derived results in this paper and illustrate the knowledge gap in the earlier literature.

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