Generalizing Negative Imaginary Systems Theory to Include Free Body Dynamics: Control of Highly Resonant Structures With Free Body Motion

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***Abstract*—Negative imaginary (NI) systems play an important role in the robust control of highly resonant flexible structures. In this paper, a generalized NI system framework is presented. A new NI system definition is given, which allows for flexible structure systems with colocated force actuators and position sensors, and with free body motion. This definition extends the existing definitions of NI systems. Also, necessary and sufficient conditions are provided for the stability of positive feedback control systems where the plant is NI according to the new definition and the controller is strictly negative imaginary. Furthermore, the stability conditions given are independent of the plant and controller system order. As an application of these results, a case study involving the control of a flexible robotic arm with a piezo-electric actuator and sensor is presented.**

***Index Terms*—Flexible structures, free body motion, negative imaginary systems.**

I. Introduction

FLEXIBLE structure dynamics arise in many areas such as flexible robot manipulators [1], ground and aerospace vehicles [2], atomic force microscopes (AFMs) [3], [4] and other nano-positioning systems [5]–[8]. Flexible structures can be modeled as infinite dimensional distributed parameter systems [9]–[11]. However, finite dimensional models are often used for the purpose of designing controllers [9], [11]–[13]. In designing controllers for these flexible systems, it is important to consider the effect of highly resonant modes. Such resonant modes are known to adversely affect the stability and performance of flexible structure feedback control systems [9], [14], [15], and are often very sensitive to changes in environmental variables. In addition, highly resonant modes lead to vibrational effects which limit the ability of control systems to achieve desired levels of performance in many applications such as precision instrumentation, optical systems, precision machine tools, wafer steppers, telescopes, and atomic force microscopes [9]. These issues arising from the presence of highly resonant modes in flexible structures motivate the need for tools to guarantee robust stability and performance in flexible structure control systems.

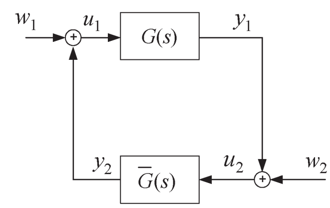

Fig. 1. Negative-imaginary feedback control system. If the plant transfer function matrix $G(s)$ is NI and the controller transfer function matrix $\bar{G}(s)$ is SNI, then the positive-feedback interconnection is internally stable if and only if the dc gain condition, $\lambda_{\max}(G(0)\bar{G}(0)) < 1$, is satisfied.

Lanzon and Petersen introduced a notion of negative imaginary (NI) systems in [15], [16] for the robust control of flexible structures with force actuators combined with position or acceleration sensors. The NI property arises in many practical systems. For example, such systems arise when considering the transfer function from a force actuator to a corresponding colocated position sensor (for instance, a piezoelectric sensor) in a lightly damped structure [3], [14], [15], [17], [18]. Another area where the underlying system dynamics are NI, in the area of nano-positioning systems; see e.g., [3], [5]–[8, 19], [20]–[23].

A negative imaginary definition for transfer functions which not necessarily rational is given in [24]. Also, results for checking and enforcing the NI property are presented in [25], [26].

The stability robustness of interconnected NI systems has been studied in [15], [16]. In these papers, it is shown that a necessary and sufficient condition for the internal stability of a positive-feedback control system (see Fig. 1) consisting of an NI plant with transfer function matrix $G(s)$ and a strictly negative imaginary (SNI) controller with transfer function matrix $\bar{G}(s)$ is given by the dc gain condition

$$\lambda_{\max}\left(G(0)\bar{G}(0)\right) < 1 \tag{1}$$

where the notation $\lambda_{\max}(\cdot)$ denotes the maximum eigenvalue of a matrix with only real eigenvalues. This stability result has been used in a number of practical applications [3], [4], [8], [18], [27], [28].

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The NI systems framework was generalized in [29] to allow for poles on the imaginary axis except the origin. Therefore, an important class of systems, that cannot be captured by the existing NI systems framework, corresponds to flexible systems with free body motion. These systems arise in areas such as rotating flexible spacecraft [30], rotary cranes [31], robotics and flexible link manipulators [32]–[34], and dual-stage hard disk drives [35]–[38]. Flexible structures with free body motion lead to dynamical models including poles at the origin, which is not covered in earlier work on NI systems theory. In particular, the stability condition (1) is not well defined in the case of flexible structures with free body motion which results in poles at the origin.

Thus we are motivated to extend the NI robust stability theory developed in [15], [16], [29] so that it can be applied to control systems involving highly resonant flexible structures with free body motion.

In this paper, we present a new generalized definition of NI systems which allows for flexible structures with colocated force actuators and position sensors and with free body motion. This definition extends the previous definitions of NI systems presented in [15], [16], [29] to allow for up to two poles at the origin. We also derive new generalized stability conditions for positive-feedback control systems involving an NI plant and an SNI controller.

Preliminary conference versions of the stability results presented in this paper were presented in [39], [40]. However, in this paper, much more general versions of these stability results are presented in Theorem 1 and Corollaries 1, 3, 4, which allow for the existence of free body motion in some but not all input-output channels. Also, this paper includes a case study involving the control of a flexible robotic arm, which has not been considered in the previous conference versions of the paper. An expanded archive version of this paper can be found in [41].

This paper is further organised as follows: Section II contains the main results presented in this paper. In this section, we introduce the new generalized definition for NI systems, which allows for systems with free body dynamics. Also, we present new stability results for interconnected NI systems. Section III presents a case study, which involves a flexible robotic arm, as an application of the NI theory presented in this paper. In Appendix A, we present and prove stability results in a particular state space realization. These stability results are used to prove the results presented in Section II. All proofs of the presented theorems and corollaries are given in the Appendices A and B. Also, an extended archive version of the paper is available in [41].

II. Main Results

The main contribution of this paper is a generalization of the framework for NI systems presented in [29]. We introduce a new definition of NI systems that will allow for systems with free body dynamics. This generalized definition will be used in a new set of stability conditions that will allow for NI systems with free body motion to be included into the framework of NI systems theory. Henceforth, when a system is said to be NI, we will mean NI as defined below, not NI as defined in earlier papers.

Definition 1: A square transfer function matrix $G(s)$ is NI if the following conditions are satisfied:

1) $G(s)$ has no pole in $Re[s] > 0$.
2) For all $\omega > 0$ such that $j\omega$ is not a pole of $G(s)$, $j(G(j\omega) - G(j\omega)^*) \geq 0$.
3) If $s = j\omega_0$ with $\omega_0 > 0$ is a pole of $G(s)$, then it is a simple pole and the residue matrix $K = \lim_{s\longrightarrow j\omega_0} (s - j\omega_0)jG(s)$ is Hermitian and positive semidefinite.
4) If $s = 0$ is a pole of $G(s)$, then $\lim_{s\longrightarrow 0} s^k G(s) = 0$ for all $k \geq 3$ and $\lim_{s\longrightarrow 0} s^2 G(s)$ is Hermitian and positive semidefinite.

Here, $G(j\omega)$ is the frequency response corresponding to the transfer function $G(s)$.

Consider the LTI system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{2}$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m},\ C \in \mathbb{R}^{m\times n},\ D \in \mathbb{R}^{m\times m}$, and with the square transfer function matrix $G(s) = C(sI - A)^{-1}B + D$. The transfer function matrix $G(s)$ is said to be strictly proper if $G(\infty) = D = 0$. We will use the notation $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ to denote the state space realization (2).

Now, consider a positive feedback interconnection between an NI system with transfer function matrix $G(s)$ and an SNI system with transfer function matrix $\bar{G}(s)$ as shown in Fig. 1. Also, suppose that the transfer function matrix $G(s)$ has a minimal state space realization $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, and $\bar{G}(s)$ has a minimal state space realization $\left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right]$. Furthermore, it is assumed that the matrix $I - D\bar{D}$ is nonsingular. Then the closed-loop system has a system matrix given by

$$\breve{A} = \begin{bmatrix} A + B\bar{D}(I - D\bar{D})^{-1}C & B\bar{C} + B\bar{D}(I - D\bar{D})^{-1}D\bar{C} \\ \bar{B}(I - D\bar{D})^{-1}C & \bar{A} + \bar{B}(I - D\bar{D})^{-1}D\bar{C} \end{bmatrix}. \tag{3}$$

Moreover, the positive feedback interconnection between $G(s)$ and $\bar{G}(s)$ as shown in Fig. 1 is said to be internally stable if the closed-loop system matrix $\breve{A}$ in (3) is Hurwitz; e.g., see [42].

In order to derive a set of stability conditions that allow for NI systems with free body motion, we define the following constant matrices for a given $m \times m$ NI transfer function matrix $G(s)$:

$$\begin{aligned} G_2 &= \lim_{s\longrightarrow 0} s^2 G(s), \quad G_1 = \lim_{s\longrightarrow 0} s\left(G(s) - \frac{G_2}{s^2}\right), \\ G_0 &= \lim_{s\longrightarrow 0} \left(G(s) - \frac{G_2}{s^2} - \frac{G_1}{s}\right). \end{aligned} \tag{4}$$

These matrices are the first three coefficients in the Laurent series expansion of the transfer function $G(s)$ around the zero. These matrices carry information about properties of the free body motion of the system under consideration and will be used in stability conditions for the positive feedback interconnection of NI and SNI systems. Note that the dc gain condition (1) cannot be defined for an NI system with transfer function matrix $G(s)$ unless $G_2 = G_1 = 0$, which reduces to the case where the dynamical system has no free body motion. From Condition 4) in Definition 1, the matrix G_2 is required to be Hermitian and positive semidefinite. Hence, it follows (e.g., see [43]) that if $G_2 \neq 0$, it can be decomposed in the form

$$G_2 = JJ^T \tag{5}$$

where J is a full column rank matrix.

We now present conditions for the stability of a positive feedback control system involving an NI plant with free body motion. These conditions are stated using the quantities defined in (4). First, we define the $2m \times 2m$ Hankel matrix Γ as

$$\Gamma = \begin{bmatrix} G_1 & G_2 \\ G_2 & 0 \end{bmatrix}. \tag{6}$$

Suppose that $\Gamma \neq 0$. Using the singular value decomposition (SVD), we can decompose the Hankel matrix Γ as

$$\begin{aligned} \Gamma &= [\, H_1 \quad H_2 \,] \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} \\ &= H_1 S V_1^T = U V_1^T = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} V_1^T \end{aligned} \tag{7}$$

where $[H_1\ H_2]$, $\left[\begin{smallmatrix} V_1^T \\ V_2^T \end{smallmatrix}\right]$ are unitary matrices, $S > 0, U = H_1 S \in \mathbb{R}^{2m\times\tilde{n}}$, $U_1 \in \mathbb{R}^{m\times\tilde{n}}$, $U_2 \in \mathbb{R}^{m\times\tilde{n}}$ and the matrices U and V_1 each have orthogonal columns. Furthermore, we can decompose the $\tilde{n} \times \tilde{n}$ matrix $U_1^T U_2$ using the SVD as

$$U_1^T U_2 = \hat{U}\hat{S}\hat{V}^T = \hat{U} \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{V_1}^T \\ \hat{V_2}^T \end{bmatrix} \tag{8}$$

where $\hat{U} \in \mathbb{R}^{\tilde{n}\times\tilde{n}}$ and $\hat{V} \in \mathbb{R}^{\tilde{n}\times\tilde{n}}$ are orthogonal matrices, $\hat{V}_2 \in \mathbb{R}^{\tilde{n}\times\check{n}}$ and $S_1 > 0$.

We now introduce some notation which will be used throughout the paper. Given matrices $X \in \mathbb{R}^{m\times m}$ and $Y \in \mathbb{R}^{m\times\tilde{n}}$ such that $\det(Y^T XY) \neq 0$, then the matrix valued function $\mathcal{P}$ (X, Y) is defined by $\mathcal{P}(X, Y) \triangleq X - XY(Y^T XY)^{-1} Y^T X$. Using this notation, we define the matrix

$$N_f = \mathcal{P}\left(\bar{G}(0), F\right) \tag{9}$$

where the $m \times \check{n}$ matrix F is given by

$$F = U_1 \hat{V}_2 \tag{10}$$

and we will assume that $\det(F^T \bar{G}(0)F) \neq 0$.

We will use the following condition in the theorem which follows:

$$F^T \bar{G}(0)F < 0. \tag{11}$$

Also, for the case in which N_f is positive semidefinite, we will use the condition

$$I - N_f^{\frac{1}{2}} G_0 N_f^{\frac{1}{2}} - N_f^{\frac{1}{2}} G_1 J(J^T J)^{-2} J^T G_1^T N_f^{\frac{1}{2}} > 0. \tag{12}$$

Moreover, for the case in which N_f is negative semidefinite, we will use the condition

$$\det(I + \tilde{N}_f G_0 \tilde{N}_f + \tilde{N}_f G_1 J(J^T J)^{-2} J^T G_1^T \tilde{N}_f) \neq 0. \tag{13}$$

Here, $\tilde{N}_f = (-N_f)^{1/2}$ and matrices G_1, G_0, J, N_f and F are defined in (4), (5), (9), and (10) respectively. Also, $(\cdot)^{1/2}$ denotes the square root of a positive semidefinite matrix.

The following theorem is our first main stability result for the case in which $G_2 \neq 0$. That is, the system has a double pole at the origin.

Theorem 1: Suppose that the square transfer function matrix $G(s)$ is strictly proper and NI with $G_2 \neq 0$, and the transfer function matrix $\bar{G}(s)$ is SNI. Also, suppose that the matrix $F^T \bar{G}(0)F$ is non-singular. If N_f is positive semidefinite, then the closed-loop positive-feedback interconnection between $G(s)$ and $\bar{G}(s)$ as shown in Fig. 1 is internally stable if and only if conditions (11) and (12) are satisfied. Furthermore, if N_f is negative semidefinite, then the closed-loop positive-feedback interconnection between $G(s)$ and $\bar{G}(s)$ is internally stable if and only if conditions (11) and (13) are satisfied.

The proof of this and subsequent theorems and corollaries are presented in Appendix B.

We now present a corollary to this theorem which considers the special case in which none of the free body modes of the plant have frictional force present; i.e., $G_1 = 0$. In order to present this corollary, we define the matrix N_2 as follows:

$$N_2 = \mathcal{P}\left(\bar{G}(0), J\right) \tag{14}$$

where we assume that the matrix $J^T \bar{G}(0)J$ is non-singular.

We will use the following condition in the next corollary, which corresponds to condition (11) in Theorem 1:

$$J^T \bar{G}(0)J < 0. \tag{15}$$

Also, for the case in which N_2 is positive semidefinite, we will use the following condition which corresponds to condition (12) in Theorem 1:

$$I - N_2^{\frac{1}{2}} G_0 N_2^{\frac{1}{2}} > 0. \tag{16}$$

Moreover, for the case in which N_2 is negative semidefinite, we will use the following condition which corresponds to condition (13) in Theorem 1:

$$\det(I + \tilde{N}_2 G_0 \tilde{N}_2) \neq 0 \tag{17}$$

where $\tilde{N}_2 = (-N_2)^{1/2}$.

Corollary 1: Suppose that the transfer function matrix $\bar{G}(s)$ is SNI and the strictly proper transfer function matrix $G(s)$ is NI with $G_1 = 0$ and $G_2 \neq 0$. Also, suppose that the matrix $J^T \bar{G}(0)J$ is non-singular. If N_2 is positive semidefinite, then the closed-loop positive-feedback interconnection between $G(s)$ and $\bar{G}(s)$ is internally stable if and only if conditions (15) and (16) are satisfied. Furthermore, if N_2 is negative semidefinite, then the closed-loop positive-feedback interconnection between $G(s)$ and $\bar{G}(s)$ is internally stable if and only if conditions (15) and (17) are satisfied.

The proof of this corollary is presented in Appendix B.

The following theorem imposes some extra conditions on the matrix G_2 which enables us to relax the sign definiteness condition on the matrix N_2. This then leads to a simplified stability condition.

Theorem 2: Suppose that the transfer function matrix $\bar{G}(s)$ is SNI and the strictly proper transfer function matrix $G(s)$ is NI with $G_1 = 0$ and $G_2 \neq 0$. Also, suppose that $\mathcal{N}(G_2) \subseteq \mathcal{N}(G_0^T)$, where $\mathcal{N}(\cdot)$ denotes the null space of a matrix. Then the closed-loop positive-feedback interconnection between $G(s)$ and $\bar{G}(s)$ is internally stable if and only if condition (15) is satisfied.

The proof of this theorem is presented in Appendix B.

In Theorem 3, Theorem 4 and Corollary 2, we consider cases which correspond to free body motion with frictional force present. As in Theorem 1, these cases allow for the fact that the free body motion may not be present in all input-output channels.

In order to present Theorem 3 and Theorem 4, suppose that $G_1 \neq 0$ and $G_2 = 0$. This corresponds to the case when the system has a single pole at the origin. Then we consider the following SVD decomposition of the matrix G_1 defined in (4):

$$G_1 = [\,\tilde{F}_1 \quad \tilde{F}_2\,] \begin{bmatrix} S_2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} = F_1 V_1^T \tag{18}$$

where $S_2 > 0$, and the matrices $F_1 = \tilde{F}_1 S_2$ and V_1 each have orthogonal columns. Also, we define the matrix N_1 as follows:

$$N_1 = \mathcal{P}\left(\bar{G}(0), F_1\right) \tag{19}$$

where the matrix $F_1^T \bar{G}(0) F_1$ is assumed to be non-singular.

We will use the following condition in Theorem 3 and Corollary 2 which corresponds to condition (11) in Theorem 1:

$$F_1^T \bar{G}(0) F_1 < 0. \tag{20}$$

For the case in which N_1 is positive semidefinite, we also will use the following condition which corresponds to condition (12) in Theorem 1:

$$I - N_1^{\frac{1}{2}} G_0 N_1^{\frac{1}{2}} > 0. \tag{21}$$

Moreover, for the case in which N_1 is negative semidefinite, we will use the following condition which corresponds to condition (13) in Theorem 1:

$$\det(I + \tilde{N}_1 G_0 \tilde{N}_1) \neq 0 \tag{22}$$

where $\tilde{N}_1 = (-N_1)^{1/2}$.

Theorem 3: Suppose that the transfer function matrix $\bar{G}(s)$ is SNI and the strictly proper transfer function matrix $G(s)$ is NI with $G_2 = 0$ and $G_1 \neq 0$. Also, suppose that the matrix $F_1^T \bar{G}(0) F_1$ non-singular. If N_1 is positive semidefinite, then the closed-loop positive-feedback interconnection between $G(s)$ and $\bar{G}(s)$ is internally stable if and only if conditions (20) and (21) are satisfied. Furthermore, if N_1 is negative semidefinite, then the closed-loop positive-feedback interconnection between $G(s)$ and $\bar{G}(s)$ is internally stable if and only if conditions (20) and (22) are satisfied.

The proof of this theorem is presented in Appendix B.

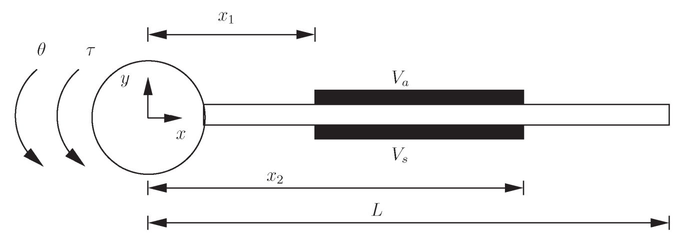

Fig. 2. Schematic diagram of the slewing beam equivalent to the robotic arm.

The following theorem imposes some extra conditions on the matrix G_1 which enables us to relax the sign definiteness condition on the matrix N_1. This then leads to a simplified stability condition.

Theorem 4: Suppose that the transfer function matrix $\bar{G}(s)$ is SNI and the strictly proper transfer function matrix $G(s)$ is NI with $G_2 = 0$ and $G_1 \neq 0$. Also, suppose that $\mathcal{N}(G_1^T) \subseteq \mathcal{N}(G_0^T)$. Then the closed-loop positive-feedback interconnection between $G(s)$ and $\bar{G}(s)$ is internally stable if and only if condition (20) is satisfied.

The proof of this theorem is presented in Appendix B.

The following corollary presents an important special case of Theorems 2 and 4.

Corollary 2: Suppose that the transfer function matrix $\bar{G}(s)$ is SNI and the strictly proper transfer function matrix $G(s)$ is NI with either $G_2 = 0$ and G_1 invertible or $G_1 = 0$ and $G_2 > 0$. Then, the closed-loop positive-feedback interconnection between $G(s)$ and $\bar{G}(s)$ is internally stable if and only if $\bar{G}(0) < 0$.

The proof of this corollary is presented in Appendix B.

Remark 1: The case where $G_2 = 0$ and $G_1 = 0$ corresponds to the existing stability results presented in [15], [29], [44]. In this case, the stability condition reduces to $\lambda_{\max}(\bar{G}(0)G(0)) < 1$. This condition can be obtained from (21) using the fact $N_1 = \bar{G}(0)$ in this case. Also, we require the assumption $\bar{G}(0) > 0$. Hence

$$\begin{aligned} & I - N_1^{\frac{1}{2}} G_0 N_1^{\frac{1}{2}} > 0, \\ \Leftrightarrow & N_1^{-1} - G_0 > 0, \\ \Leftrightarrow & \lambda_{\max}\left(\bar{G}(0) G_0\right) < 1. \end{aligned}$$

Note that using a similar argument to the proof of Theorem 3, we can obtain a similar result under the assumption that $\bar{G}(0) < 0$.

III. Case Study: Control of Flexible Robotic Arm

In this section, we present an application of the stability results presented in this paper to the control of a flexible robotic arm system. The robotic arm is pinned to a motor at one end. For the purposes of modeling the flexible robotic arm, we use an equivalent slewing beam model as depicted in Fig. 2; see [45]. The motor allows the robotic arm to traverse in the vertical plane. Two piezoelectric patches are attached to the arm on either side. Here, one piezoelectric patch acts as an actuator while the other is a sensor. The robotic arm system has two inputs and two outputs: the inputs are the voltage V_a applied to the piezoelectric actuator and the torque τ applied by the motor, whereas the outputs are the voltage V_s produced by the piezoelectric sensor and the motor hub angle θ.

In this case study, the parameter values for the robotic arm are taken from [46]. An infinite dimensional transfer function matrix $G(s)$ model for the robotic arm is given in [45]: $G(s) = \begin{bmatrix} (N_{\tau,\theta}(s)/D(s)) & (N_{V_a,\theta}(s)/D(s)) \\ (N_{\tau,V_s}(s)/D(s)) & (N_{V_a,V_s}(s)/D(s)) \end{bmatrix}$, where $N_{\tau,\theta}(s), N_{V_a,\theta}$, $N_{\tau,V_s}(s)$, $N_{V_a,V_s}(s)$ and $D(s)$ are given in (26)–(28) in [45].

Various methods such as the Maclaurin series expansion presented in [47], the Rayleigh-Ritz method [48], and the assumed modes method [48] are available in literature for the finite dimensional approximation of such an infinite dimensional model. The finite dimensional model can be written as

$$G_f(s) = \begin{bmatrix} G_{f\tau,\theta}(s) & G_{fV_a,\theta}(s) \\ G_{f\tau,V_s}(s) & G_{fV_a,V_s}(s) \end{bmatrix} = \sum_{i=0}^{n} \frac{1}{k} \begin{bmatrix} \frac{a_i}{s^2+p_i^2} & \frac{b_i}{s^2+p_i^2} \\ \frac{c_i}{s^2+p_i^2} & \frac{d_i}{s^2+p_i^2} \end{bmatrix}. \tag{23}$$

Here $D(s)$ in the infinite dimensional model $G(s)$ is approximated by $D_f(s) = k\prod_{i=0}^{n}(s^2 + p_j^2)$, where, $jp_0 \dots jp_n$ are the first n $j\omega$-axis roots of $D(s)$. Also, the coefficient matrices $C_i = \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix}$, are computed using a partial fraction expansion method. We consider the first resonant mode; i.e., $n = 1$ for the controller design. The corresponding coefficient matrices were computed and were found to be $C_0 = \begin{bmatrix} 0.14 & 0 \\ 0 & 0 \end{bmatrix}$; $C_1 = \begin{bmatrix} 3.0907 & 3.5573 \times 10^{-4} \\ 3.5573 \times 10^{-4} & 2.3500 \end{bmatrix}$; and $k = 6.6667 \times 10^{-8}$. Also, the poles were computed to be $p_0 = 0$, $p_1 = 3.4$.

The finite dimensional model $G_f(s)$ in (23) is NI, since $j(G_f(j\omega) - G_f(j\omega)^*) = 0$ for all $\omega \geq 0$, where $j\omega$ is not a pole for $G_f(s)$. This follows because in this example, $G_f(j\omega)$ is real and symmetric for all ω such that $j\omega$ is not a pole of $G_f(s)$. Also, the coefficient matrices C_0, C_1 are positive semidefinite which implies that Condition 3) in Definition 1 is satisfied. Moreover, $G_2 = \lim_{s \longrightarrow 0} s^2 G(s) = C_0 \geq 0$, which implies that Condition 4) in Definition 1 is satisfied.

A. Controller Design

According to Theorem 1 if a plant is NI, any SNI controller which satisfies the conditions of Theorem 1 will stabilize the system. The fact that the robotic arm plant involves colocated "force" actuators and "position" sensors indicates that this plant should be NI.

We will now use a finite dimensional model of the form (23) to design a controller for the system. First, we compute the matrices G_2, G_1, and G_0 in (4), for the finite dimension approximate system where $n = 1$ in (23) to obtain

$$G_2 = \begin{bmatrix} 0.14 & 0 \\ 0 & 0 \end{bmatrix} \geqslant 0; \quad G_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix};$$

$$G_0 = \begin{bmatrix} 0.41253083 & 0.0000319 \\ 0.0000319 & 0.15672805 \end{bmatrix}. \tag{24}$$

This implies that we can use Corollary 1 to guarantee the stability of the positive feedback interconnection between the plant and an SNI controller.

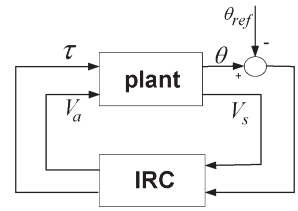

Fig. 3. Block diagram corresponding to a step change in the robotic arm reference position.

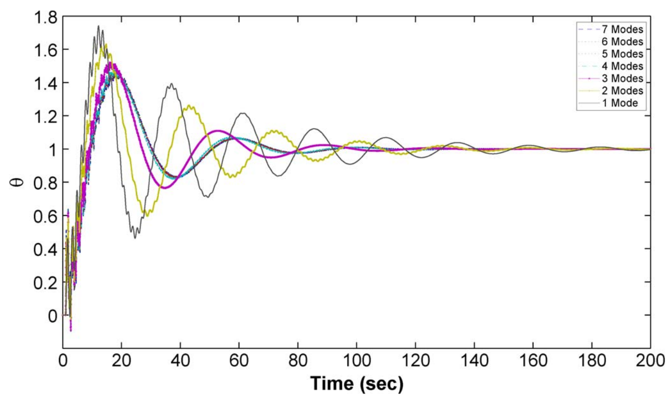

Fig. 4. Position response of the robotic arm system corresponding to a unit step change in reference position. Here, different numbers of modes are used in the plant model.

In this case study, an integral resonant controller (IRC) is chosen to stabilize the system; e.g., see [15]. An IRC is a first order controller which takes the form $\bar{G}(s) = (sI + \Gamma\Phi)^{-1}\Gamma - \Delta$. This controller is SNI if $\Gamma > 0, \Phi > 0$ and Δ is a symmetric matrix [15]. Now, we chose the controller matrices $\Gamma > 0, \Phi > 0$ and Δ such that the conditions of Corollary 1 are satisfied. We choose the controller matrices as follows:

$$\Gamma = \begin{bmatrix} 35 & 15 \\ 15 & 20 \end{bmatrix}; \; \Phi = \begin{bmatrix} 0.745 & 0.521 \\ 0.521 & 1.021 \end{bmatrix}; \; \Delta = \begin{bmatrix} 4.2900 & 0 \\ 0 & 2.22 \end{bmatrix}. \tag{25}$$

This leads to a controller dc gain matrix of $\bar{G}(0) = \begin{bmatrix} -2.2029 & -1.0650 \\ -1.0650 & -0.6971 \end{bmatrix}$. To check the stability conditions in Corollary 1, we first compute the matrix J in (5) using G_2 in (24). This yields $J = \left[\begin{smallmatrix} 0.3751 \\ 0 \end{smallmatrix}\right]$. Also, the matrix N_2 in (14) is calculated as $N_2 = \begin{bmatrix} 0 & 0 \\ 0 & -0.182252 \end{bmatrix}$, which is negative semidefinite. Then we conclude $\det(I + \tilde{N}_2 G_0 \tilde{N}_2) = \det \begin{bmatrix} 1.000000025 & 0.000000003 \\ 0.000000003 & 5.390603 \end{bmatrix} \neq 0$, where $\tilde{N}_2 = (-N_2)^{1/2}$. Also, $J^T \bar{G}(0) J = -0.309908135 < 0$. Thus, the conditions of Corollary 1 are satisfied.

To verify the performance of the closed loop system, we simulate the response of this system corresponding to a step change in the reference position of the robotic arm; see Fig. 3. This step response is shown in Fig. 4. Also, the corresponding response of the piezo sensor output V_s is shown in Fig. 5. Here, the step responses were calculated using finite dimensional plant models defined in (23) for different numbers of modes, $n = 2, 3 \dots 7$.

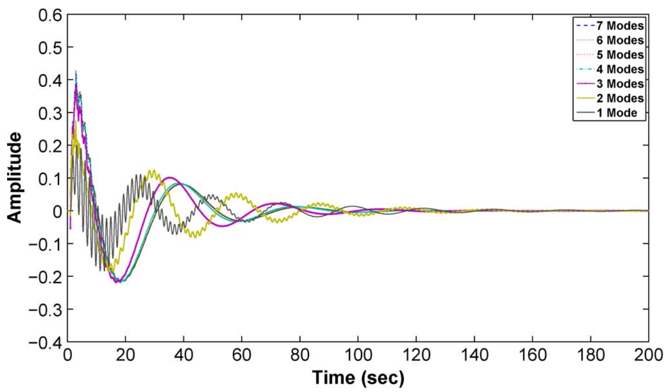

Fig. 5. Piezoelectric response sensor output V_s response corresponding to a unit step change in reference position. Here, different numbers of modes are used in the plant model.

To this end, we have used the proposed controller which is designed for the finite dimensional model with $n=1$ when applied to the plant with finite dimensional model where $n=2,3\ldots 7$ in order to check the performance and robustness of the proposed controller. In fact, the performance of the closed loop system is found to improve by increasing the number of modes; see Fig. 4 and Fig. 5. This is due to the fact that adding more modes to the plant in fact leads to a system which is easier to control. Hence, the controller guarantees a better response when increasing the number of modes in the model being simulated.

Note that the controller parameters in (25) were chosen by process of trial and error to obtain good closed loop performance in the case of the nominal plant model, $n=1$. An alternative approach, which would be useful in the case of a more complicated SNI controller structure, would be to use an optimization procedure to obtain the controller parameters; e.g., see [49].

IV. Conclusion

In this paper, new stability results for the positive-feedback interconnection of negative imaginary systems have been derived. A new NI definition is presented, which allows for systems having free body dynamics to be considered as NI systems. This work can be used in controller design to allow for a broader class of NI systems than considered in previous work. The application of the main results in this paper has been illustrated via a case study involving the control of a flexible robotic arm.

Appendix A

In this appendix, before we present the proofs of the results given in Section II, we present state space results (some of which are of independent interest) using a particular state space representation of the plant transfer function matrix $G(s)$. The first stability result in this appendix is Theorem 5. We will subsequently use Theorem 5 to prove Theorem 1 given in Section II. We also present a number of corollaries which will be used to prove the remaining results presented in Section II of the paper.

We first consider an NI square transfer function matrix $G(s)$ with a minimal state space realization of the form

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) \tag{26}$$

where, $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times m}$, $C\in\mathbb{R}^{m\times n}$, and

$$A=\begin{bmatrix} A_1 & 0 & 0\\ 0 & A_2 & 0\\ 0 & 0 & A_3\end{bmatrix};\ B=\begin{bmatrix} B_1\\ B_2\\ B_3\end{bmatrix};\ C=[C_1 \quad C_2 \quad C_3] \tag{27}$$

$A_1\in\mathbb{R}^{n_1\times n_1}$, $A_2\in\mathbb{R}^{n_2\times n_2}$, $A_3\in\mathbb{R}^{2k\times 2k}$, $B_1\in\mathbb{R}^{n_1\times m}$, $B_2\in\mathbb{R}^{n_2\times m}$, $B_3=\left[\begin{smallmatrix} B_{3a}\\ B_{3b}\end{smallmatrix}\right]$, $B_{3a}\in\mathbb{R}^{k\times m}$, $B_{3b}\in\mathbb{R}^{k\times m}$, $C_1\in\mathbb{R}^{m\times n_1}$, $C_2\in\mathbb{R}^{m\times n_2}$, $C_3=[C_{3a}\ C_{3b}]$, $C_{3a}\in\mathbb{R}^{m\times k}$, $C_{3b}\in\mathbb{R}^{m\times k}$, A_1 is nonsingular, $A_2=0$, and

$$A_3=\begin{bmatrix} 0 & I_{k\times k}\\ 0 & 0\end{bmatrix}. \tag{28}$$

Remark 2: In Appendix B below, it will be shown that any NI system can be transformed to the above particular state space form.

We also consider an SNI transfer function matrix $\bar{G}(s)$ with a minimal state space realization

$$\dot{x}(t) = \bar{A}x(t) + \bar{B}u(t),$$
$$y(t) = \bar{C}x(t) + \bar{D}u(t) \tag{29}$$

where $\bar{A}\in\mathbb{R}^{\bar{n}\times\bar{n}}$, $\bar{B}\in\mathbb{R}^{\bar{n}\times m}$, $\bar{C}\in\mathbb{R}^{m\times\bar{n}}$, and $\bar{D}\in\mathbb{R}^{m\times m}$.

Remark 3: We allow any of the matrices in these models to have zero dimensions. In sequel, any matrix with zero dimension is regarded as being of full rank.

The corresponding transfer function matrix for the state space realization (26), (27) is given as follows:

$$G(s) = C_1(sI-A_1)^{-1}B_1 + C_2(sI-A_2)^{-1}B_2 + C_3(sI-A_3)^{-1}B_3$$
$$= C_1(sI-A_1)^{-1}B_1 + \frac{C_2B_2+C_3B_3}{s} + \frac{C_{3a}B_{3b}}{s^2}. \tag{30}$$

The following theorem provides a necessary and sufficient condition for the stability of the positive-feedback interconnection between the NI transfer function matrix $G(s)$, with state space realization (26), (27), and the SNI transfer function matrix $\bar{G}(s)$, with state space realization (29). In order to present this theorem, we define the following matrix:

$$N = \mathcal{P}\left(\bar{G}(0), [C_2 \quad C_{3a}]\right). \tag{31}$$

Also, the matrix

$$\Xi = \begin{bmatrix} C_2^T\\ C_{3a}^T\end{bmatrix}\bar{G}(0)[C_2 \quad C_{3a}] \tag{32}$$

is assumed to be non-singular.

In addition, we will use the following condition in the theorem which follows:

$$\begin{bmatrix} C_2^T \\ C_{3a}^T \end{bmatrix} \bar{G}(0)[C_2 \quad C_{3a}] < 0. \tag{33}$$

Also, for the case in which N is positive semidefinite, we will use the condition

$$I + N^{\frac{1}{2}} C_1 A_1^{-1} B_1 N^{\frac{1}{2}} - N^{\frac{1}{2}} C_{3b} P_2^{-1} C_{3b}^T N^{\frac{1}{2}} > 0 \tag{34}$$

where $P_2 = C_{3a}^T B_{3b}^T (B_{3b} B_{3b}^T)^{-1}$, which will be shown to be symmetric and positive definite in Lemma 3. Moreover, for the case in which N is negative semidefinite, we will use the condition

$$\det\left(I - \tilde{N} C_1 A_1^{-1} B_1 \tilde{N} + \tilde{N} C_{3b} P_2^{-1} C_{3b}^T \tilde{N}\right) \neq 0 \tag{35}$$

where $\tilde{N} = (-N)^{1/2}$.

Theorem 5: Suppose that $k \neq 0$ and the matrix Ξ in (32) is non-singular. Also, suppose that the transfer function matrix $G(s)$, with the minimal state space realization (26), is NI and the transfer function matrix $\bar{G}(s)$, with the minimal state space realization (29), is SNI. If N is positive semidefinite, then the closed-loop positive-feedback interconnection between $G(s)$ and $\bar{G}(s)$ is internally stable if and only if conditions (33) and (34) are satisfied. Also, if N is negative semidefinite, then the closed-loop positive-feedback interconnection between $G(s)$ and $\bar{G}(s)$ is internally stable if and only if conditions (33) and (35) are satisfied.

The proof of this theorem is given at the end of this appendix.

Corollary 3: Suppose that the matrix Ξ in (32) is non-singular and the matrix N in (31) satisfies $N[C_1 \ C_{3b}] = 0$. Also suppose that the transfer function matrix $G(s)$, with the minimal state space realization (26), is NI and the transfer function matrix $\bar{G}(s)$, with the minimal state space realization (29), is SNI. Then the closed-loop positive-feedback interconnection between $G(s)$ and $\bar{G}(s)$ is internally stable if and only if condition (33) is satisfied.

The proof of this corollary is given at the end of this appendix.

The following corollary considers the case when $n_2 = 0$ and $k \neq 0$; i.e., the matrix A in (27) has the block diagonal form $A = \begin{bmatrix} A_1 & 0 \\ 0 & A_3 \end{bmatrix}$. In the case when $n_2 = 0$, the matrix N in (31) will be given by

$$N = \mathcal{P}\left(\bar{G}(0), C_{3a}\right) \tag{36}$$

where we assume that $C_{3a}^T \bar{G}(0) C_{3a}$ is non-singular.

We will use the following conditions in the next corollary which correspond to conditions (33)–(35) in Theorem 5. The first condition to be considered is

$$C_{3a}^T \bar{G}(0) C_{3a} < 0. \tag{37}$$

Also, for the case in which N is positive semidefinite, we will use the condition

$$I + N^{\frac{1}{2}} C_1 A_1^{-1} B_1 N^{\frac{1}{2}} - N^{\frac{1}{2}} C_{3b} P_2^{-1} C_{3b}^T N^{\frac{1}{2}} > 0 \tag{38}$$

where $P_2 = C_{3a}^T B_{3b}^T (B_{3b} B_{3b}^T)^{-1}$. Moreover, for the case in which N is negative semidefinite, we will use the condition

$$\det\left(I - \tilde{N} C_1 A_1^{-1} B_1 \tilde{N} + \tilde{N} C_{3b} P_2^{-1} C_{3b}^T \tilde{N}\right) \neq 0 \tag{39}$$

where $\tilde{N} = (-N)^{1/2}$.

Corollary 4: Suppose that the matrix $C_{3a}^T \bar{G}(0) C_{3a}$ is non-singular, $k \neq 0$, and $n_2 = 0$. Also, suppose that the transfer function matrix $G(s)$, with the minimal state space realization (26) is NI and the transfer function matrix $\bar{G}(s)$, with the minimal state space realization in (29), is SNI. If N in (36) is positive semidefinite, then the closed-loop positive-feedback interconnection between $G(s)$ and $\bar{G}(s)$ is internally stable if and only if conditions (37) and (38) are satisfied. Also, if N in (36) is negative semidefinite, then the closed-loop positive-feedback interconnection between $G(s)$ and $\bar{G}(s)$ is internally stable if and only if conditions (37) and (39) are satisfied.

The proof of this corollary is given at the end of this appendix.

The next corollary considers the case when $n \neq 0$ and $k = 0$; i.e., the A matrix in the minimal state realization of $G(s)$ (26), (27) has the block diagonal form $A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$. In this case, when $n \neq 0$ and $k = 0$, the matrix N in (31) will be given by

$$N = \mathcal{P}\left(\bar{G}(0), C_2\right) \tag{40}$$

where the matrix $C_2^T \bar{G}(0) C_2$ is assumed to be non-singular.

We will use the following conditions in the next corollary which corresponds to conditions (33)–(35) in Theorem 5. The first condition to be considered is

$$C_2^T \bar{G}(0) C_2 < 0. \tag{41}$$

Also, for the case in which N in (40) is positive semidefinite, we will use the condition

$$I + N^{\frac{1}{2}} C_1 A_1^{-1} B_1 N^{\frac{1}{2}} > 0. \tag{42}$$

Moreover, for the case in which N in (40) is negative semidefinite, we will use the condition

$$\det\left(I - \tilde{N} C_1 A_1^{-1} B_1 \tilde{N}\right) \neq 0 \tag{43}$$

where $\tilde{N} = (-N)^{1/2}$.

Corollary 5: Suppose that $C_2^T \bar{G}(0) C_2$ is non-singular, $n_2 \neq 0$, and $k = 0$. Also, suppose that the transfer function matrix $G(s)$, with the minimal state space realization (26) is NI and the transfer function matrix $\bar{G}(s)$, with the minimal state space realization in (29), is SNI. If N in (40) is positive semidefinite, then the closed-loop positive-feedback interconnection between $G(s)$ and $\bar{G}(s)$ is internally stable if and only if conditions (41) and (42) are satisfied. Also, if N in (40) is negative semidefinite, then the closed-loop positive-feedback interconnection between $G(s)$ and $\bar{G}(s)$ is internally stable if and only if conditions (41) and (43) are satisfied.

The proof of this corollary is given at the end of this appendix.

In order to prove Theorem 5 and Corollaries 3–5, we will use the following lemmas. First, Lemma 1 maps the quantities G_0, G_1 and G_2 given in (4) to expressions in terms of the state space realization (26), (27). These expression are employed in the proof of Theorem 5.

Lemma 1: Suppose that $G(s)$ has a minimal state space realization (26), (27). Then the quantities G_0, G_1 and G_2 defined in (4) are given as follows:

$$G_2 = C_{3a}B_{3b},\ G_1 = C_2B_2 + C_3B_3,\ G_0 = -C_1A_1^{-1}B_1. \quad (44)$$

Proof: This lemma follows immediately from (30). ■

Now, Lemmas 2, 3, 4 give some useful properties of the minimal state space realization (26), (27). These properties will be used in the proof of Theorem 5.

Lemma 2: Suppose that the transfer function matrix $G(s)$ has a minimal state space realization (26), (27). Then, the matrix $[C_2\ C_{3a}]$ is of full column rank, and the matrix $\left[\begin{smallmatrix} B_2 \\ B_{3b}\end{smallmatrix}\right]$ is of full row rank. Also $m \geq k + n_2$ and the subsystem with realization $\left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & 0 \end{array}\right]$ is minimal.

Proof: Since the state space realization $\left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$ is minimal, the pair (A, C) is observable and the pair (A, B) is controllable. Also, the corresponding observability matrix is given by

$$O(A,C) = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} = \begin{bmatrix} C_1 & C_2 & C_{3a} & C_{3b} \\ C_1A_1 & 0 & 0 & C_{3a} \\ C_1A_1^2 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ C_1A_1^{n-1} & 0 & 0 & 0 \end{bmatrix}.$$

Since the pair (A, C) is observable, it follows that the observability matrix $O(A, C)$ is of full rank. This implies that the pair (A_1, C_1) is observable. Also, since the observability matrix $O(A, C)$ is of full rank, it follows that C_2, C_{3a} and $[C_2\ C_{3a}]$ are of full rank. Furthermore, it follows that $m \geq k + n_2$. Similarly, since the pair (A, B) is controllable, it follows that the corresponding controllability matrix

$$\mathcal{C}(A,B) = [B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B] = \begin{bmatrix} B_1 & A_1B_1 & A_1^2B_1 & \cdots & A_1^{n-1}B_1 \\ B_2 & 0 & 0 & \cdots & 0 \\ B_{3a} & B_{3b} & 0 & \cdots & 0 \\ B_{3b} & 0 & 0 & \cdots & 0 \end{bmatrix}$$

is of full rank. Hence, the pair (A_1, B_1) is controllable and the matrices B_2, B_{3b} and $\left[\begin{smallmatrix} B_2 \\ B_{3b}\end{smallmatrix}\right]$ are of full rank. Also, since the pair (A_1, C_1) is observable and the pair (A_1, B_1) is controllable, it follows that $\left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & 0 \end{array}\right]$ is a minimal realization. ■

Lemma 3: Suppose that the transfer function matrix $G(s)$, with the minimal state space realization (26), (27), is NI. Then, there exist symmetric matrices $P_1 > 0, P_2 > 0$, and matrices L_1, W such that

$$P_1A_1 + A_1^TP_1 = -L_1^TL_1, \quad (45)$$

$$P_1B_1 - A_1^TC_1^T = -L_1^TW, \quad (46)$$

$$P_2B_{3b} = C_{3a}^T, \quad (47)$$

$$C_1B_1 + B_1^TC_1^T + C_2B_2 + B_2^TC_2^T + C_3B_3 + B_3^TC_3^T = W^TW. \quad (48)$$

Furthermore

$$P_2 = C_{3a}^TB_{3b}^T\left(B_{3b}B_{3b}^T\right)^{-1} \quad (49)$$

and

$$G_1 + G_1^T = C_2B_2 + B_2^TC_2^T + C_3B_3 + B_3^TC_3^T = \left(W^T + C_1P_1^{-1}L_1^T\right)\left(W + L_1P_1^{-1}C_1^T\right) \geq 0. \quad (50)$$

Proof: Consider the transfer function matrix $G(s)$ with the minimal state space realization (26), (27). Also, define the transfer function matrix $R(s) = sG(s)$. Using (30), it follows that:

$$\begin{aligned} R(s) &= sC_1(sI - A_1)^{-1}B_1 + \frac{C_{3a}B_{3b}}{s} + C_2B_2 + C_3B_3 \\ &= C_1A_1(sI - A_1)^{-1}B_1 + C_1B_1 + \frac{C_{3a}B_{3b}}{s} \\ &\quad + C_2B_2 + C_3B_3. \end{aligned} \quad (51)$$

This implies that $R(s)$ has a state space realization $\left[\begin{array}{c|c} A_r & B_r \\ \hline C_r & D_r \end{array}\right]$ where $A_r = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}$, $B_r = \left[\begin{smallmatrix} B_1 \\ B_{3b}\end{smallmatrix}\right]$, $C_r = [C_1A_1\ C_{3a}]$ and $D_r = C_1B_1 + C_2B_2 + C_3B_3$. Using the same argument as in the proof of Lemma 2, it follows that the rank of the matrix formed from the first and last columns in $O(A_r, C_r)$ is equal to the rank of the matrix formed from the first and third columns in (56), where, A_1 is invertible. This implies that the matrix $O(A_r, C_r)$ is of full rank; i.e., the pair (A_r, C_r) is observable. Similarly, the pair (A_r, B_r) is controllable. This implies that the state space realization $\left[\begin{array}{c|c} A_r & B_r \\ \hline C_r & D_r \end{array}\right]$ is minimal.

We now show that $R(s)$ is positive real; e.g., see page 47 in [50] for a definition of positive real transfer function matrices. Since $G(s)$ is NI, it follows that $j(G(j\omega) - G(j\omega)^*) \geq 0$, for all $\omega > 0$ such that $j\omega$ is not a pole of $G(s)$. Then given any such $\omega > 0$, $R(j\omega) + R(j\omega)^* = j\omega(G(j\omega) - G(j\omega)^*) \geq 0$, and $\overline{(R(j\omega) + R(j\omega)^*)} \geq 0$. This implies that $R(-j\omega) + R(-j\omega)^* \geq 0$ for all $\omega > 0$, since $\overline{R(j\omega)} = R(-j\omega)$. Hence, $R(j\omega) + R(j\omega)^* \geq 0$ for all $\omega < 0$ such that $j\omega$ is not a pole of $G(s)$. Therefore, $R(j\omega) + R(j\omega)^* \geq 0$ for all $\omega \in (-\infty, \infty)$ such that $j\omega$ is not a pole of $G(s)$.

Now, consider the case where $j\omega_0$ is a pole of $G(s)$ and $\omega_0 = 0$. In the case where $C_{3a}B_{3b} = 0$, the transfer function matrix $R(s) = C_1A_1(sI - A_1)^{-1}B_1 + C_1B_1 + C_2B_2 + C_3B_3$ will have no pole at the origin. This implies that $R(0)$ is finite. Since $R(j\omega) + R(j\omega)^* \geq 0$ for all $\omega > 0$ such that $j\omega$

is not a pole of $G(s)$ and $R(j\omega)$ is continuous at $\omega = 0$, this implies that $R(0) + R(0)^* \geq 0$. In the case where $C_{3a}B_{3b} \neq 0$, the transfer function matrix $R(s)$ is as given in (51). Since $G(s)$ is NI, then $\lim_{s \longrightarrow 0} s^2 G(s) \geq 0$ which implies that $\lim_{s \longrightarrow 0} sR(s) \geq 0$.

If $j\omega_0$ is a pole of $G(s)$ and $\omega_0 > 0$, then $G(s)$ can be factored as $(1/s^2 + \omega_0^2)F(s)$, which according to the definition for NI systems implies that the residue matrix $K_0 = (1/2\omega_0)F(j\omega_0)$ is positive semidefinite Hermitian. Hence, $F(j\omega_0) = F(j\omega_0)^* \geq 0$. Now, the residue matrix of $R(s)$ at $j\omega_0$ with $\omega_0 > 0$ is given by

$$\lim_{s \longrightarrow j\omega_0} (s - j\omega_0)R(s) = \lim_{s \longrightarrow j\omega_0} (s - j\omega_0)sG(s),$$
$$= \lim_{s \longrightarrow j\omega_0} (s - j\omega_0)s\frac{1}{s^2 + \omega_0^2}F(s) = \frac{1}{2}F(j\omega_0)$$

which is positive semidefinite Hermitian. Hence, we can conclude that $R(s)$ is positive real; see page 47 in [50]. Using the KYP lemma (e.g., see Lemma 3.1 in [50]), it now follows that there exist matrices $P_r > 0, L$ and W such that

$$\begin{aligned} P_r A_r + A_r^T P_r &= -L^T L, \\ P_r B_r - C_r^T &= -L^T W, \\ D_r + D_r^T &= W^T W. \end{aligned} \tag{52}$$

If we write $P_r = \begin{bmatrix} P_1 & P_{12} \\ P_{12}^T & P_2 \end{bmatrix}$ and $L = [L_1 \ L_2]$, it follows from (52) that

$$\begin{bmatrix} P_1 & P_{12} \\ P_{12}^T & P_2 \end{bmatrix}\begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} A_1^T & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} P_1 & P_{12} \\ P_{12}^T & P_2 \end{bmatrix}$$
$$= -\begin{bmatrix} L_1^T \\ L_2^T \end{bmatrix}[L_1 \quad L_2],$$
$$\Leftrightarrow \begin{bmatrix} P_1 A_1 + A_1^T P_1 & A_1^T P_{12} \\ P_{12}^T A_1 & 0 \end{bmatrix} = -\begin{bmatrix} L_1^T L_1 & L_1^T L_2 \\ L_2^T L_1 & L_2^T L_2 \end{bmatrix}. \tag{53}$$

Hence $L_2 = 0$ and since A_1 is a nonsingular matrix, it also follows that $P_{12} = 0$. Also, (53) implies that (45) is satisfied. From (52), it follows that:

$$\begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}\begin{bmatrix} B_1 \\ B_{3b} \end{bmatrix} - \begin{bmatrix} A_1^T C_1^T \\ C_{3a}^T \end{bmatrix} = -\begin{bmatrix} L_1^T \\ 0 \end{bmatrix} W$$

which implies (46) and (47). Lemma 2 implies that B_{3b} is of full rank and hence, (47) implies that (49) is also satisfied. From (52), it follows that (48) holds. Also, using (46), we can write B_1 as

$$B_1 = P_1^{-1}\left(A_1^T C_1^T - L_1^T W\right).$$

Substituting this and (45) into (48), it follows that:

$$C_2B_2 + B_2^T C_2^T + C_3 B_3 + B_3^T C_3^T$$
$$= \left(W^T + C_1 P_1^{-1} L_1^T\right)\left(W + L_1 P_1^{-1} C_1^T\right) \geq 0.$$

Using (44) in Lemma 1, this implies (50). This completes the proof. ■

Lemma 4: Suppose that the transfer function matrix $G(s)$ with the minimal state space realization (26), (27) is NI. Then, there exists an invertible matrix R_d such that $\left[\begin{smallmatrix} B_2 \\ B_{3b} \end{smallmatrix}\right] = R_d \left[\begin{smallmatrix} C_2^T \\ C_{3a}^T \end{smallmatrix}\right]$. Also, if $x \in \mathcal{N}(\left[\begin{smallmatrix} C_2^T \\ C_{3a}^T \end{smallmatrix}\right])$, then $x \in \mathcal{N}(B_{3a} + P_2^{-1}C_{3b}^T)$, where the matrix P_2 is defined as in Lemma 3. Here, $\mathcal{N}(\cdot)$ denotes as the null space of a matrix.

Proof: Suppose that $x \in \mathcal{N}(\left[\begin{smallmatrix} C_2^T \\ C_{3a}^T \end{smallmatrix}\right])$. It follows that $\left[\begin{smallmatrix} C_2^T \\ C_{3a}^T \end{smallmatrix}\right]x = 0$. Hence using (47) in Lemma 3, it follows that there exists a matrix $P_2 > 0$ such that $P_2 B_{3b} x = 0$. Therefore, $x^T C_2 = 0$, $x^T C_{3a} = 0$ and $B_{3b}x = 0$. Hence, using (44) it follows that:

$$x^T\left(G_1 + G_1^T\right)x = x^T G_1 x + x^T G_1^T x.$$

Using the fact that $x^T G_1 x$ is a scalar, this implies

$$x^T(G_1 + G_1^T)x = 2x^T G_1 x$$
$$\Rightarrow 2x^T(C_2B_2 + C_{3a}B_{3a} + C_{3b}B_{3b})x = 0,$$
$$\Rightarrow \left(G_1 + G_1^T\right)x = 0$$

since, $G_1 + G_1^T \geq 0$ using (50) in Lemma 3. Hence

$$\left(C_2B_2 + C_{3a}B_{3a} + B_{3b}^T C_{3b}^T\right)x = 0,$$
$$\Rightarrow \left(C_2B_2 + C_{3a}B_{3a} + C_{3a}P_2^{-1}C_{3b}^T\right)x = 0$$

using (47) in Lemma 3. Therefore

$$[C_2 \quad C_{3a}]\begin{bmatrix} B_2 \\ B_{3a} + P_2^{-1}C_{3b}^T \end{bmatrix}x = 0,$$
$$\Rightarrow \begin{bmatrix} B_2 \\ B_{3a} + P_2^{-1}C_{3b}^T \end{bmatrix}x = 0$$

since $[C_2 \ C_{3a}]$ is full rank using Lemma 2. Therefore

$$B_2 x = 0 \text{ and } \left(B_{3a} + P_2^{-1}C_{3b}^T\right)x = 0.$$

This implies that $x \in \mathcal{N}(B_{3a} + P_2^{-1}C_{3b}^T)$. Thus, we have established the second part of the lemma. Also since $B_2 x = 0$ and $B_{3b}x = 0$, it follows that $\left[\begin{smallmatrix} B_2 \\ B_{3b} \end{smallmatrix}\right]x = 0$. This implies that if $x \in \mathcal{N}(\left[\begin{smallmatrix} C_2^T \\ C_{3a}^T \end{smallmatrix}\right])$, then $x \in \mathcal{N}(\left[\begin{smallmatrix} B_2 \\ B_{3b} \end{smallmatrix}\right])$.

Similarly, suppose that $x \in \mathcal{N}(\left[\begin{smallmatrix} B_2 \\ B_{3b} \end{smallmatrix}\right])$, and hence $\left[\begin{smallmatrix} B_2 \\ B_{3b} \end{smallmatrix}\right]x = 0$. Therefore, $x \in \mathcal{N}(\left[\begin{smallmatrix} C_2^T \\ C_{3a}^T \end{smallmatrix}\right])$. Hence

$$\mathcal{N}\left(\begin{bmatrix} B_2 \\ B_{3b} \end{bmatrix}\right) = \mathcal{N}\left(\begin{bmatrix} C_2^T \\ C_{3a}^T \end{bmatrix}\right).$$

Hence, there exists an invertible matrix R_d such that

$$\begin{bmatrix} B_2 \\ B_{3b} \end{bmatrix} = R_d \begin{bmatrix} C_2^T \\ C_{3a}^T \end{bmatrix}.$$

This completes the proof. ■

The following lemma, which follows directly from the proof of a result presented in [16], [29], gives useful properties of the minimal realization (29) of the SNI transfer function matrix $\bar{G}(s)$.

Lemma 5: (See the proof of Lemma 6 in [29]): Suppose that the transfer function matrix $\bar{G}(s)$, with minimal state space realization (29), is SNI

$$\bar{A}\bar{P}^{-1} + \bar{P}^{-1}\bar{A}^T \le 0 \text{ and } \bar{B} = -\bar{A}\bar{P}^{-1}\bar{C}^T. \quad (54)$$

The following lemma is a simple matrix theory result, which is needed in the proof of Theorem 5.

Lemma 5 (See e.g., [16]): Given $A \in \mathbb{C}^{n\times n}$ with $j(A - A^*) \ge 0$ and $B \in \mathbb{C}^{n\times n}$ with $j(B - B^*) > 0$, then

$$\det(I - AB) \neq 0.$$

Now, we are in a position to present the proof of Theorem 5.

Proof of Theorem 5: The internal stability of the positive-feedback interconnection between $G(s)$ and $\bar{G}(s)$ will be guaranteed by considering the closed loop system matrix defined in (3) which is given by

$$\breve{A} = \begin{bmatrix} A + B\bar{D}C & B\bar{C} \\ \bar{B}C & \bar{A} \end{bmatrix}.$$

Here, A, B and C are defined as in (26), (27) and $\bar{A}, \bar{B}, \bar{C}, \bar{D}$ are defined as in (29). To establish internal stability, we show that the matrix $\breve{A}$ is Hurwitz; i.e., all the eigenvalues of $\breve{A}$ lie in the open left-half of the complex plane.

Consider $T = \begin{bmatrix} P - C^T\bar{D}C & -C^T\bar{C} \\ -\bar{C}^T C & \bar{P} \end{bmatrix}$ to be a candidate Lyapunov matrix, where

$$P = \begin{bmatrix} P_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & P_2 \end{bmatrix} \ge 0 \quad (55)$$

$P_1 > 0, P_2 > 0$ are defined as in Lemma 3 and $\bar{P} > 0$ is defined as in Lemma 5.

Claim 1: In the case when the matrix N in (31) is negative semidefinite, then $T > 0$ if and only if (33) is satisfied. Also, in the case when the matrix N in (31) is positive semidefinite, then $T > 0$ if and only if (33) and (34) are satisfied.

To establish this claim, we first note that since $\bar{G}(s)$ is SNI, it follows from Lemma 5 that $\bar{P}$ satisfies (54). This implies that the condition $T > 0$ is equivalent to

$$P - C^T\bar{D}C - C^T\bar{C}\bar{P}^{-1}\bar{C}^T C > 0,$$

$$\Leftrightarrow P - C^T(\bar{D} + \bar{C}\bar{P}^{-1}\bar{C}^T)C > 0,$$

$$\Leftrightarrow P - C^T(\bar{D} - \bar{C}\bar{A}^{-1}\bar{B})C > 0 \text{ via (54) in Lemma 5},$$

$$\Leftrightarrow P - C^T\bar{G}(0)C > 0,$$

$$\Leftrightarrow \begin{bmatrix} P_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & P_2 \end{bmatrix} - \begin{bmatrix} C_1^T \\ C_2^T \\ C_{3a}^T \\ C_{3b}^T \end{bmatrix} \bar{G}(0)[C_1 \quad C_2 \quad C_{3a} \quad C_{3b}] > 0. \quad (56)$$

Furthermore, using the Schur complement of the LMI in (56), it is straightforward to verify that the condition $T > 0$ is equivalent to the conditions

$$\begin{bmatrix} -C_2^T\bar{G}(0)C_2 & -C_2^T\bar{G}(0)C_{3a} \\ -C_{3a}^T\bar{G}(0)C_2 & -C_{3a}^T\bar{G}(0)C_{3a} \end{bmatrix} > 0 \quad (57)$$

and

$$\begin{bmatrix} P_1 - C_1^T\bar{G}(0)C_1 & -C_1^T\bar{G}(0)C_{3b} \\ -C_{3b}^T\bar{G}(0)C_1 & P_{3b} - C_{3b}^T\bar{G}(0)C_{3b} \end{bmatrix} - \begin{bmatrix} -C_1^T\bar{G}(0)C_2 & -C_1^T\bar{G}(0)C_{3a} \\ -C_{3b}^T\bar{G}(0)C_2 & -C_{3b}^T\bar{G}(0)C_{3a} \end{bmatrix} \times \begin{bmatrix} -C_2^T\bar{G}(0)C_2 & -C_2^T\bar{G}(0)C_{3a} \\ -C_{3a}^T\bar{G}(0)C_2 & -C_{3a}^T\bar{G}(0)C_{3a} \end{bmatrix}^{-1} \times \begin{bmatrix} -C_2^T\bar{G}(0)C_1 & -C_2^T\bar{G}(0)C_{3b} \\ -C_{3a}^T\bar{G}(0)C_1 & -C_{3a}^T\bar{G}(0)C_{3b} \end{bmatrix} > 0. \quad (58)$$

Moreover, (57) is equivalent to

$$-\begin{bmatrix} C_2^T \\ C_{3a}^T \end{bmatrix}\bar{G}(0)[C_2 \quad C_{3a}] = -\Xi > 0 \quad (59)$$

where Ξ is defined in (32). This is equivalent to condition (33).

Also, the condition (58) is equivalent to

$$\begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} - \begin{bmatrix} C_1^T \\ C_{3b}^T \end{bmatrix}\bar{G}(0)[C_1 \quad C_{3b}] + \begin{bmatrix} C_1^T \\ C_{3b}^T \end{bmatrix}\bar{G}(0)[C_2 \quad C_{3a}]\Xi^{-1}\begin{bmatrix} C_2^T \\ C_{3a}^T \end{bmatrix}\bar{G}(0)[C_1 \quad C_{3b}] > 0,$$
$$\Leftrightarrow \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} - \begin{bmatrix} C_1^T \\ C_{3b}^T \end{bmatrix} N [C_1 \quad C_{3b}] > 0 \text{ using (31).} \quad (60)$$

This condition is always satisfied in the case where N is negative semidefinite. Hence using (59), we can conclude that $T > 0$ if and only if (33) is satisfied in the case when N is negative semidefinite.

Now in the case when N is positive semidefinite, the condition (60) can be rewritten as follows

$$P_f - C_f^T N C_f > 0$$

where $P_f = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} > 0$ and $C_f = [C_1 \ \ C_{3b}]$. However, using the Schur complement, this is equivalent to the condition

$$\begin{bmatrix} I & N^{\frac{1}{2}}C_f \\ C_f^T N^{\frac{1}{2}} & P_f \end{bmatrix} > 0 \Leftrightarrow I - N^{\frac{1}{2}}C_f P_f^{-1} C_f^T N^{\frac{1}{2}} > 0,$$

$$\Leftrightarrow I - N^{\frac{1}{2}}C_1 P_1^{-1} C_1^T N^{\frac{1}{2}} - N^{\frac{1}{2}}C_{3b} P_2^{-1} C_{3b}^T N^{\frac{1}{2}} > 0. \quad (61)$$

Now using (50) in Lemma 3, we can define a matrix M as

$$M = W + L_1 P_1^{-1} C_1^T$$

so that

$$\begin{aligned} M^T M &= C_2 B_2 + B_2^T C_2^T + C_3 B_3 + B_3^T C_3^T \\ &= C_2 B_2 + B_2^T C_2^T + C_{3a} B_{3a} + B_{3a}^T C_{3a}^T \\ &\quad + C_{3b} B_{3b} + B_{3b}^T C_{3b}^T. \end{aligned} \tag{62}$$

Also using (46) in Lemma 3, we can write B_1 as

$$B_1 = P_1^{-1} \left(A_1^T C_1^T - L_1^T W \right).$$

Substituting for W in terms of M into this expression for B_1 gives

$$\begin{aligned} B_1 &= P_1^{-1} \left(A_1^T C_1^T - L_1^T \left(M - L_1 P_1^{-1} C_1^T \right) \right) \\ &= P_1^{-1} A_1^T C_1^T - P_1^{-1} L_1^T M - P_1^{-1} L_1^T L_1 P_1^{-1} C_1^T \\ &= -A_1 P_1^{-1} C_1^T - P_1^{-1} L_1^T M. \end{aligned} \tag{63}$$

Also, from the definition of N in (31), it follows that:

$$N[C_2 \quad C_{3a}] = 0. \tag{64}$$

Therefore, Lemma 4 implies

$$\begin{aligned} & N[B_2^T \quad B_{3b}^T] = 0, \\ & \Rightarrow N \left(C_2 B_2 + B_2^T C_2^T + C_{3a} B_{3a} \right. \\ & \qquad \left. + B_{3a}^T C_{3a}^T + C_{3b} B_{3b} + B_{3b}^T C_{3b}^T \right) N = 0, \\ & \Rightarrow N(M^T M) N = 0, \text{ using (62).} \end{aligned}$$

Hence, $MN = 0$. Therefore,

$$M N^{\frac{1}{2}} = 0. \tag{65}$$

Substituting this into (63) implies

$$\begin{aligned} & B_1 N^{\frac{1}{2}} = -A_1 P_1^{-1} C_1^T N^{\frac{1}{2}} - P_1^{-1} L_1^T M N^{\frac{1}{2}}, \\ & \Rightarrow B_1 N^{\frac{1}{2}} = -A_1 P_1^{-1} C_1^T N^{\frac{1}{2}}, \\ & \Rightarrow N^{\frac{1}{2}} C_1 A_1^{-1} B_1 N^{\frac{1}{2}} = -N^{\frac{1}{2}} C_1 P_1^{-1} C_1^T N^{\frac{1}{2}}. \end{aligned} \tag{66}$$

Substituting (66) into (61) gives the condition

$$I + N^{\frac{1}{2}} C_1 A_1^{-1} B_1 N^{\frac{1}{2}} - N^{\frac{1}{2}} C_{3b} P_2^{-1} C_{3b}^T N^{\frac{1}{2}} > 0.$$

This is equivalent to condition (34). Hence, in the case when N is positive semidefinite, it follows from this and (59) that $T > 0$ if and only if conditions (33) and (34) are satisfied. This completes the proof of Claim 1.

Now, observe that

$$\begin{aligned} & T\breve{A} + \breve{A}^T T \\ & = -\begin{bmatrix} C^T \bar{D} W^T + L^T & C^T \bar{W}^T \\ \bar{C}^T W^T & \bar{L}^T \end{bmatrix} \begin{bmatrix} W \bar{D} C + L & W \bar{C} \\ \bar{W} C & \bar{L} \end{bmatrix} \\ & \leq 0. \end{aligned} \tag{67}$$

Together with Claim 1, this implies that $\breve{A}$ has all its eigenvalues in the closed left-half of the complex plane if and only if conditions (33) and (34) are satisfied in the case when N is positive semidefinite; e.g., see Lemma 3.19 in [42]. Similarly, in the case when N is negative semidefinite $\breve{A}$ has all its eigenvalues in the closed left-half of the complex plane if and only if condition (33) is satisfied.

In order to complete the proof of the sufficiency part of the theorem, we must show that if conditions (33) and (34) are satisfied in the case when N is positive semidefinite, then the matrix $\breve{A}$ can have no eigenvalues on the $j\omega$ axis. Similarly, we must show that if conditions (33) and (35) are satisfied in the case that N is negative semidefinite, then the matrix $\breve{A}$ can have no eigenvalues on the $j\omega$ axis.

Indeed, using Lemma 6, the fact that $G(s)$ is NI and the fact that $\bar{G}(s)$ is SNI, we conclude that $\det(I - G(j\omega)\bar{G}(j\omega)) \neq 0$ for all $\omega > 0$. This implies that $\breve{A}$ has no eigenvalues on the imaginary axis for $\omega > 0$. Thus, to complete the proof, we will show that in the case when N is positive semidefinite, conditions (33) and (34) imply that $\det(\breve{A}) \neq 0$. Similarly, in the case when N is negative semidefinite, we will show that conditions (33) and (35) imply that $\det(\breve{A}) \neq 0$. Indeed

$$\begin{aligned} \det(\breve{A}) &= \det(\bar{A}) \det(A + B\bar{D}C - B\bar{C}\bar{A}^{-1}\bar{B}C), \\ &= \det(\bar{A}) \det\left(A + B\bar{G}(0)C\right). \end{aligned} \tag{68}$$

This implies that $\det(\breve{A}) \neq 0$ if $\det(A + B\bar{G}(0)C) \neq 0$, since $\det(\bar{A}) \neq 0$ using Lemma 5 and the fact that $\bar{G}(s)$ is SNI. Now, define the matrix

$$\Lambda = \begin{bmatrix} B_2 \\ B_{3b} \end{bmatrix} \bar{G}(0)[C_2 \quad C_{3a}]. \tag{69}$$

It follows from Lemma 4 that there exists a non-singular matrix R_d such that:

$$\Lambda = R_d \begin{bmatrix} C_2^T \\ C_{3a}^T \end{bmatrix} \bar{G}(0)[C_2 \quad C_{3a}] = R_d \Xi. \tag{70}$$

Since the matrix Ξ is assumed to be invertible, this implies that the matrix Λ in (69) is invertible.

Now, substituting (27) into (68), it is straightforward to verify that

$$\begin{aligned} & \det\left(A + B\bar{G}(0)C\right) \\ & = -\det\Lambda \det\left(\begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} B_1 \\ B_{3a} \end{bmatrix} N \left[C_1 \quad C_{3b} \right] \right) \qquad (71) \\ & = -\det\Lambda \det\begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix} \det\left(I + \left(C_1 A_1^{-1} B_1 + C_{3b} B_{3a}\right) N\right). \qquad (72) \end{aligned}$$

In the case when N is positive semidefinite, (70) and (72) imply that

$$\begin{aligned} & \det\left(A + B\bar{G}(0)C\right) \\ & \quad = -\det R_d \det \Xi \det \begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix} \\ & \qquad \times \det\left[I + N^{\frac{1}{2}} \left(C_1 A_1^{-1} B_1 + C_{3b} B_{3a} \right) N^{\frac{1}{2}} \right]. \end{aligned} \tag{73}$$

Now using (64), it follows that the columns of the matrix $N^{1/2}$ are contained in the set $\mathcal{N}(\left[\begin{smallmatrix} C_2^T \\ C_{3a}^T \end{smallmatrix}\right])$. Hence, it follows from the second part of Lemma 4 that $(B_{3a} + P_2^{-1} C_{3b}^T) N^{1/2} = 0$. This implies that

$$N^{\frac{1}{2}} C_{3b} B_{3a} N^{\frac{1}{2}} = -N^{\frac{1}{2}} C_{3b} P_2^{-1} C_{3b}^T N^{\frac{1}{2}}.$$

Hence (73) can be written as

$$\begin{aligned} &\det\left(A + B\bar{G}(0)C\right) \\ &= -\det R_d \det \Xi \det \begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix} \\ &\quad \times \det\left[I + N^{\frac{1}{2}} C_1 A_1^{-1} B_1 N^{\frac{1}{2}} - N^{\frac{1}{2}} C_{3b} P_2^{-1} C_{3b}^T N^{\frac{1}{2}}\right]. \end{aligned} \tag{74}$$

Since the matrices $R_d, A_1, \bar{A}$ are invertible and also using (33), (34), (68), it follows that $\det(\breve{A}) \neq 0$ as required.

In the case when N is negative semidefinite, we consider the matrix $\tilde{N} = (-N)^{1/2}$. Then (72) implies that

$$\begin{aligned} &\det\left(A + B\bar{G}(0)C\right) \\ &= -\det R_d \det \Xi \det \begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix} \\ &\quad \times \det\left[I - \tilde{N}\left(C_1 A_1^{-1} B_1 + C_{3b} B_{3a}\right)\tilde{N}\right]. \end{aligned} \tag{75}$$

Using (64), it follows that the columns of the matrix $\tilde{N}$ are contained in the set $\mathcal{N}(\left[\begin{smallmatrix} C_2^T \\ C_{3a}^T \end{smallmatrix}\right])$. Hence, it follows from the second part of Lemma 4 that $(B_{3a} + P_2^{-1} C_{3b}^T)\tilde{N} = 0$. This implies

$$\tilde{N} C_{3b} B_{3a} \tilde{N} = -\tilde{N} C_{3b} P_2^{-1} C_{3b}^T \tilde{N}.$$

Hence, (75) can be written as

$$\begin{aligned} &-\det R_d \det \Xi \det \begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix} \\ &\quad \times \det\left[I - \tilde{N} C_1 A_1^{-1} B_1 \tilde{N} + \tilde{N} C_{3b} P_2^{-1} C_{3b}^T \tilde{N}\right]. \end{aligned} \tag{76}$$

Since the matrices $R_d, A_1, \bar{A}$ are invertible and also using (33)–(35), (68), it follows that $\det(\breve{A}) \neq 0$ as required. This completes the proof of the sufficiency part of the theorem.

To complete the proof of the necessity part of the theorem, suppose that the positive-feedback interconnection between the NI transfer function matrix $G(s)$ and the SNI transfer function matrix $\bar{G}(s)$ is internally stable. This implies that the matrix $\breve{A}$ is Hurwitz and hence has all its eigenvalues are in the open left-half of the complex plane. This together with Claim 1 and (67) implies that conditions (33) and (35) are satisfied in the case when N is negative semidefinite. Similarly, in the case when N is positive semidefinite, Claim 1 and (67) implies that conditions (33) and (34) are satisfied. This completes the proof of the theorem. ■

Proof of Corollary 3: The proof of this corollary will proceeds in an almost identical fashion to the proof of Theorem 5. Indeed, we first state the following claim:

Claim 2: Assume that the matrix N in (31) satisfies $N[C_1\ C_{3b}] = 0$, then $T > 0$ if and only if (33) is satisfied. This claim corresponds to Claim 1 in Theorem 5 when we relax the conditions on the matrix N. The proof of this claim is similar to the proof of Claim 1 in the proof of Theorem 5 since (60) is automatically satisfied in the case when $N[C_1\ C_{3b}] = 0$.

Also, the determinant condition in (68) will be automatically satisfied using the fact $N[C_1\ C_{3b}] = 0$ in (71). The proof of the corollary then follows as in the proof of Theorem 5. ■

Proof of Corollary 4: Note that for the system (26), (27) corresponding to the case of this corollary, the conditions (33)–(35) in Theorem 5 reduce to conditions (37)–(39). Then the proof of the corollary proceeds in an identical fashion to the proof of Theorem 5 for the special case being considered, where the matrix P defined in (55) becomes a matrix of the form $P = \begin{bmatrix} P_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & P_2 \end{bmatrix}$. ■

Proof of Corollary 5: First note that for the system (26), (27) corresponding to the case of this corollary, the conditions (33)–(35) in Theorem 5 reduce to conditions (41)–(43). Then the proof of the corollary proceeds in an identical fashion to the proof of Theorem 5 for the special case being considered, where the matrix P defined in (55) becomes a matrix of the form $P = \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix}$. ■

APPENDIX B

Here, we present the proof of the main results in the paper.

We first show in the following lemma that any NI system can be transformed to the block diagonal form given in (26), (27). This will allow us to use the results presented in Appendix A to prove the main results of the paper presented in Section II.

Lemma 7: Any NI system with transfer function matrix $G(s)$ and minimal state space realization (26), can be transformed to the block diagonal form given in (27).

Proof: Suppose that the transfer function matrix $G(s)$, with a minimal state space realization $\left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$ is NI. It follows from Theorem 2.1.1 in [51] that we can find a nonsingular state space transformation matrix T such that the matrix $T^{-1}AT$ is in real Jordon block diagonal form and the realization $\left[\begin{array}{c|c} T^{-1}AT & T^{-1}B \\ \hline CT & 0 \end{array}\right]$ is minimal. Also, we can choose this transformation so that the Jordon blocks of $T^{-1}AT$ are ordered according to the magnitudes of the corresponding eigenvalues of the matrix A, such that the last blocks correspond to the zero eigenvalues of A if they exist. Furthermore, this transformation can be chosen so that the Jordan blocks corresponding to the zero eigenvalues are ordered according to increasing order of the Jordan blocks. Also, a further transformation can be applied so that the matrix A_3 corresponding to the Jordan blocks of order two is of the form (28).

Now, we claim that if $G(s)$ is NI, then there are no Jordan blocks corresponding to zero eigenvalues of order greater than or equal to three. To prove this claim, suppose that there is a Jordon block of A corresponding to a zero eigenvalue of order greater than or equal to three. This together with the minimality of the realization implies that $G_3 = \lim_{s\longrightarrow 0} s^3 G(s) \neq 0$ which contradicts the NI definition. Thus the zero eigenvalues of A will only have Jordon blocks of order one or two. From this, it now follows that the matrix $T^{-1}AT$ will be of the form (27). This completes the proof of the lemma. ■

The next lemma is a technical lemma, which will be used in order to prove Theorem 1.

Lemma 8: For any full rank matrices A, B, C and D which satisfy $AB = CD$ where $A \in \mathbb{R}^{n\times r}$, $B \in \mathbb{R}^{r\times n}$, $C \in \mathbb{R}^{n\times r}$, $D \in \mathbb{R}^{r\times n}$ and $n \geq r$, there exists an invertible matrix R such that $A = CR$ and $B = R^{-1}D$.

Proof: Since B is of full rank, and $n \geq r$, $AB = CD$ implies

$$ABB^T = CDB^T \Rightarrow A = CDB^T(BB^T)^{-1} \Rightarrow A = CR$$

where $R = DB^T(BB^T)^{-1}$.

To show that R is nonsingular, suppose that R is singular. Then there exists a non-zero $n \times 1$ vector x such that $Rx = 0$. This implies that $Ax = 0$ which contradicts the fact that A is a full rank. Hence, that there exists a nonsingular matrix R such that $A = CR$. Also, since C is of full rank and $n \geq r$, it follows that C has a left inverse, which implies that $RB = D$; i.e., $B = R^{-1}D$. This complete the proof. ■

Proof of Theorem 1: Lemma 7 shows that any strictly proper NI system can be represented in the block diagonal form (26), (27). This implies that we only need to show the equivalence of the assumptions and the conditions (33)–(35) in Theorem 5 and the assumptions and the conditions (11)–(13) in this theorem.

First, it is straightforward to verify that the condition $k \neq 0$ is equivalent to the condition $G_2 \neq 0$. Also, it follows from (44) and (5) that there exists a full rank matrix J such that

$$C_{3a}B_{3b} = JJ^T.$$

Also, it follows from Lemma 2 and Lemma 8 that there exists an invertible matrix X such that $C_{3a} = JX$ and $B_{3b} = X^{-1}J^T$. We let $P_2 = C_{3a}^T B_{3b}^T (B_{3b}B_{3b}^T)^{-1}$ and note that $B_{3b}B_{3b}^T$ is invertible since B_{3b} is of full rank. Then Lemma 3 implies that P_2 is symmetric and also we obtain

$$P_2 = X^T X. \qquad (77)$$

In the case when N is positive semidefinite, the definition of N implies that $N[C_2\ C_{3a}] = 0$, and hence $N^{1/2}[C_2\ C_{3a}] = 0$. Using (44), it follows that $G_1 = [C_2\ \ C_{3a}]\left[\begin{smallmatrix} B_2 \\ B_{3a}\end{smallmatrix}\right] + C_{3b}B_{3b}$, which implies

$$\begin{aligned} &N^{\frac{1}{2}}G_1 = N^{\frac{1}{2}}[C_2 \quad C_{3a}]\begin{bmatrix} B_2 \\ B_{3a}\end{bmatrix} + N^{\frac{1}{2}}C_{3b}B_{3b}, \\ &\Rightarrow N^{\frac{1}{2}}G_1 = N^{\frac{1}{2}}C_{3b}B_{3b} \text{ since } N^{\frac{1}{2}}[C_2 \quad C_{3a}] = 0, \\ &\Rightarrow N^{\frac{1}{2}}G_1 J = N^{\frac{1}{2}}C_{3b}X^{-1}J^T J, \\ &\Rightarrow N^{\frac{1}{2}}G_1 J\left(J^T J\right)^{-1} = N^{\frac{1}{2}}C_{3b}X^{-1}. \end{aligned} \qquad (78)$$

Substituting (44), (77), and (78) into condition (34) in Theorem 5, it follows that this condition can be rewritten as

$$I - \left(N^{\frac{1}{2}}G_0N^{\frac{1}{2}} + N^{\frac{1}{2}}G_1J\left(J^TJ\right)^{-1}\left(J^TJ\right)^{-T}J^TG_1^TN^{\frac{1}{2}}\right) > 0. \qquad (79)$$

Similarly, in the case when N is negative semidefinite, it follows that $\tilde{N}[C_2\ C_{3a}] = 0$. This implies that

$$\begin{aligned} &\tilde{N}G_1 = \tilde{N}[C_2 \quad C_{3a}]\begin{bmatrix} B_2 \\ B_{3a}\end{bmatrix} + \tilde{N}C_{3b}B_{3b}, \\ &\Rightarrow \tilde{N}G_1 = \tilde{N}C_{3b}B_{3b} \text{ since } \tilde{N}\left[C_2 \quad C_{3a}\right] = 0, \\ &\Rightarrow \tilde{N}G_1 J = \tilde{N}C_{3b}X^{-1}J^T J, \\ &\Rightarrow \tilde{N}G_1 J\left(J^T J\right)^{-1} = \tilde{N}C_{3b}X^{-1}. \end{aligned} \qquad (80)$$

Substituting (44), (77), and (80) into condition (35) in Theorem 5, it follows that this condition can be rewritten as

$$\det\left(I + \left(\tilde{N}G_0\tilde{N} + \tilde{N}G_1J\left(J^TJ\right)^{-1}\left(J^TJ\right)^{-T}J^TG_1^T\tilde{N}\right)\right) \neq 0. \qquad (81)$$

Now, using Lemma 1 and substituting for G_1 and G_2 from (4) in the Hankel matrix defined in (6), it follows that

$$\Gamma = \begin{bmatrix} G_1 & G_2 \\ G_2 & 0\end{bmatrix} = \begin{bmatrix} \tilde{C} \\ \tilde{C}\tilde{A}\end{bmatrix}\left[\tilde{B} \quad \tilde{A}\tilde{B}\right]$$

where

$$\tilde{A} = \begin{bmatrix} A_2 & 0 \\ 0 & A_3\end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & A_3\end{bmatrix}; \quad \tilde{B} = \begin{bmatrix} B_2 \\ B_3\end{bmatrix};$$
$$\tilde{C} = \left[C_2 \quad C_3\right].$$

Using this and the SVD in (7), it follows that:

$$\begin{bmatrix} \tilde{C} \\ \tilde{C}\tilde{A}\end{bmatrix}[\tilde{B} \quad \tilde{A}\tilde{B}] = \begin{bmatrix} U_1 \\ U_2\end{bmatrix}V^T.$$

Using Lemma 2 and Lemma 8, it follows that there exists a nonsingular matrix R such that

$$U = \begin{bmatrix} U_1 \\ U_2\end{bmatrix} = \begin{bmatrix} \tilde{C} \\ \tilde{C}\tilde{A}\end{bmatrix}R = \begin{bmatrix} \hat{C} \\ \hat{C}\hat{A}\end{bmatrix}, \qquad (82)$$

where

$$\hat{C} = \tilde{C}R, \quad \hat{A} = R^{-1}\tilde{A}R.$$

This implies that $\hat{A}^2 = R^{-1}\tilde{A}^2R = 0$ since $\tilde{A}^2 = 0$. It follows that $U\hat{A} = \left[\begin{smallmatrix} \hat{C}\hat{A} \\ \hat{C}\hat{A}^2\end{smallmatrix}\right] = \left[\begin{smallmatrix} \hat{C}\hat{A} \\ 0\end{smallmatrix}\right] = \left[\begin{smallmatrix} U_2 \\ 0\end{smallmatrix}\right]$, which implies

$$\hat{A} = U^TU\hat{A} = U^T\begin{bmatrix} U_2 \\ 0\end{bmatrix} = U_1^TU_2.$$

Using this and (8), it follows that:

$$\mathcal{N}(\hat{A}) = span\{\hat{V}_2\}. \qquad (83)$$

Also, since $\tilde{A} = \begin{bmatrix} A_2 & 0 \\ 0 & A_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix}$, it follows that:

$$\mathcal{N}(\tilde{A}) = span \left\{ \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix} \right\}. \quad (84)$$

Now observe that we can write the matrix $[C_2 \; C_{3a}]$ as

$$[\, C_2 \quad C_{3a} \,] = \tilde{C} \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix}. \quad (85)$$

Also, observe that $\hat{A}x = 0$ if and only if

$$R^{-1}\tilde{A}Rx = 0 \quad \Leftrightarrow \tilde{A}Rx = 0 \quad \Leftrightarrow Rx \in \mathcal{N}(\tilde{A}).$$

Hence, $\mathcal{N}(\tilde{A}) = R\mathcal{N}(\hat{A})$. Therefore it follows from (83) and (84) that:

$$R\; span\{\hat{V}_2\} = span\{R\hat{V}_2\} = span \left\{ \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix} \right\}.$$

This implies that there exists a nonsingular matrix $\hat{R}$ such that

$$\begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix} = R\hat{V}_2\hat{R}.$$

Substituting this into (85) and using (82) implies

$$[C_2 \quad C_{3a}] = \tilde{C}R\hat{V}_2\hat{R} = \hat{C}\hat{V}_2\hat{R} = U_1\hat{V}_2\hat{R} = F\hat{R}, \quad (86)$$

where $F = U_1\hat{V}_2$ as in (10). Substituting (86) into the matrix (31) and (32) implies that

$$N = \left(\bar{G}(0) - \bar{G}(0)F\hat{R}\left(\hat{R}^T F^T \bar{G}(0) F \hat{R}\right)^{-1} \hat{R}^T F^T \bar{G}(0)\right)$$
$$= \bar{G}(0) - \bar{G}(0)F\left(F^T\bar{G}(0)F\right)^{-1} F^T\bar{G}(0) = N_f, \quad (87)$$

where N_f is defined as into (9). Substituting (87) in (79) and (81) implies that conditions (34) and (35) in Theorem 5 are equivalent to conditions (12) and (13) in the theorem respectively.

Also, (86) implies that

$$\Xi = \begin{bmatrix} C_2^T \\ C_{3a}^T \end{bmatrix} \bar{G}(0)[C_2 \quad C_{3a}] = \hat{R}^T F^T \bar{G}(0) F \hat{R}.$$

It follows that condition (33) in Theorem 5 is equivalent to condition (11) in the theorem since $\hat{R}$ is invertible. This completes the proof of the theorem. ■

Proof of Corollary 1: In order to prove this corollary, we show that the stability conditions and the assumptions in Corollary 4 are equivalent to the stability conditions and the assumptions in this corollary. First, it is straightforward to verify that the conditions $k \neq 0$ and $n_2 = 0$ are equivalent to the conditions $G_2 \neq 0$ and $G_1 = 0$. Also, using (44) and the decomposition in (5), it follows that $C_{3a}B_{3b} = JJ^T$, and hence Lemma 8 implies that there exist an invertible matrix X such that $C_{3a} = JX$. This implies that the matrix N_2 in (14) is equal to the matrix N in (36). Also, since $C_{3a} = JX$ and X is invertible, it follows that condition (15) in Corollary 1 is equivalent to condition (37) in Corollary 4. Since $G_1 = 0$, it follows that:

$$\begin{aligned} &NG_1 = N(C_{3a}B_{3a} + C_{3b}B_{3b}) = 0, \\ &\Rightarrow NC_{3b}B_{3b} = 0, \text{ since } NC_{3a} = 0, \\ &\Rightarrow NC_{3b} = 0, \text{ since } B_{3b} \text{ is of full rank.} \end{aligned} \quad (88)$$

This implies that $N^{1/2}C_{3b} = 0$ in the case when N is positive semidefinite. Using the fact that $G_0 = -C_1A_1^{-1}B_1$ from Lemma 1, it follows that condition (16) in Corollary 1 is equivalent to condition (38) in Corollary 4. Also, in the case when N is negative semidefinite (88) implies that $\tilde{N}C_{3b} = 0$. Using the fact that $G_0 = -C_1A_1^{-1}B_1$ from Lemma 1, it follows that condition (17) in Corollary 1 is equivalent to condition (39) in Corollary 4. This completes the proof of the corollary. ■

Proof of Theorem 2: In order to prove this theorem, we first show that $\mathcal{N}(G_2) \subseteq \mathcal{N}(G_0^T)$ implies the condition $N[C_1 \;\; C_{3b}] = 0$ in Corollary 3, in the case when $G_1 = 0$. Indeed, suppose that $\mathcal{N}(G_2) \subseteq \mathcal{N}(G_0^T)$. This implies that

$$\mathcal{R}(G_2) \supseteq \mathcal{R}(G_0)$$

where $\mathcal{R}(\cdot)$ denotes the range space of a matrix. Since $G_2 = JJ^T$ and J is of full rank, it follows that:

$$\mathcal{R}(JJ^T) \supseteq \mathcal{R}(G_0),$$

which implies that there exist a matrix Q such that $G_0 = JQ$. Then, we consider the matrix N defined as

$$N = \bar{G}(0) - \bar{G}(0)C_{3a}\left(C_{3a}^T\bar{G}(0)C_{3a}\right)^{-1} C_{3a}^T\bar{G}(0),$$

which is the formula for the matrix N in Corollary 3 in the case in which $G_1 = 0$. This implies that

$$NG_0 = NJQ = 0, \text{ since } NJ = 0,$$

and hence from Lemma 1, it follows that:

$$NC_1A_1^{-1}B_1 = 0.$$

Using a similar calculation as in (63) in the proof of Theorem 5, this implies that

$$\begin{aligned} &NC_1A_1^{-1}\left(A_1P_1^{-1}C_1^T - P_1^{-1}L_1^TM\right)N = 0, \\ &\Rightarrow NC_1P_1^{-1}C_1^TN = 0 \text{ using (65)}, \quad \Rightarrow NC_1 = 0. \end{aligned} \quad (89)$$

Also, since $G_1 = C_3B_3 = C_{3a}B_{3a} + C_{3b}B_{3b} = 0$, it follows that:

$$\begin{aligned} &NC_{3a}B_{3a} + NC_{3b}B_{3b} = 0, \\ &\Rightarrow NC_{3b}B_{3b} = 0, \text{ since } NC_{3a} = 0, \\ &\Rightarrow NC_{3b} = 0 \text{ since } B_{3b} \text{ is of full rank.} \end{aligned} \quad (90)$$

Using (89) and (90), it follows that $N[C_1 \;\; C_{3b}] = 0$. This implies the assumptions in Corollary 3 are satisfied in the case when $G_1 = 0$. Also, as in the proof of Theorem 1, the condition (15) reduces to condition (33) in Corollary 3. ■

Proof of Theorem 3: In order to prove this theorem, we show that the stability conditions and the assumptions in this theorem are equivalent to the stability conditions and the assumptions in Corollary 5. First, it is straightforward to verify that the conditions $n_2 \neq 0$ and $k = 0$ are equivalent to the conditions $G_1 \neq 0$ and $G_2 = 0$. Using Lemma 8 and the fact that $G_1 = C_2 B_2$ from Lemma 1, it follows that there exists an invertible matrix R such that $C_2 = F_1 R$, where the matrix F_1 is given in (18). This implies that the matrix N_1 in (19) is equal to the matrix N in (40). Also, since $C_2 = F_1 R$ and R is invertible, it follows that condition (20) in this theorem is equivalent to condition (41) in Corollary 5. Finally, using the fact that $G_0 = -C_1 A_1^{-1} B_1$ from Lemma 1, it follows that conditions (21) and (22) in this theorem are equivalent to conditions (42) and (43) in Corollary 5 respectively. This completes the proof of the theorem. ■

Proof of Theorem 4: In order to prove this theorem, we first show that $\mathcal{N}(G_1^T) \subseteq \mathcal{N}(G_0^T)$ implies the condition $N[C_1 \;\; C_{3b}] = 0$ in Corollary 3, in the case when $G_2 = 0$. Indeed, suppose that $\mathcal{N}(G_1^T) \subseteq \mathcal{N}(G_0^T)$. This implies that

$$\mathcal{R}(G_1) \supseteq \mathcal{R}(G_0). \tag{91}$$

Since $G_1 = C_2 B_2$ from Lemma 1 and B_2 is of full rank using Lemma 2, it follows that $\mathcal{R}(C_2) = \mathcal{R}(G_1)$. Using (91), it follows that:

$$\mathcal{R}(C_2) \supseteq \mathcal{R}(G_0),$$

which implies that there exists a matrix Q such that $G_0 = C_2 Q$. Then, we consider the matrix N defined as

$$N = \bar{G}(0) - \bar{G}(0) C_2 \left(C_2^T \bar{G}(0) C_2 \right)^{-1} C_2^T \bar{G}(0),$$

which is the formula for the matrix N in Corollary 3 for the case in which $G_2 = 0$. This implies that

$$N G_0 = N C_2 Q = 0, \text{ since } N C_2 = 0$$

and hence from Lemma 1 it follows that:

$$N C_1 A_1^{-1} B_1 = 0.$$

Using a similar calculation as in equation (63) in the proof of Theorem 5, this implies

$$N C_1 A_1^{-1} \left(A_1 P_1^{-1} C_1^T - P_1^{-1} L_1^T M \right) N = 0,$$
$$\Rightarrow N C_1 P_1^{-1} C_1^T N = 0, \text{ using (65)} \quad \Rightarrow N C_1 = 0.$$

This implies the assumptions in Corollary 3 are satisfied in the case when $G_2 = 0$. Also, as in the proof of Theorem 1, condition (20) reduces to condition (33) in Corollary 3. ■

Proof of Corollary 2: In the case where G_1 is assumed to be invertible in this corollary, it follows that the matrix F_1 in (18) is invertible. Then, the condition (20) reduces to the condition $\bar{G}(0) < 0$ and the corollary follows immediately from Corollary 4. In the case where G_2 is assumed to be positive definite in this corollary, it follows that the matrix J in (5) is invertible. Then, the condition (20) reduces to the condition $\bar{G}(0) < 0$ and the corollary follows immediately from Corollary 2. ■

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