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Comments on "On the Existence of Stable, Causal Multipliers for Systems With Slope-Restricted Nonlinearities"

Joaquín Carrasco, William P. Heath, Guang Li, and Alexander Lanzon

***Abstract*—The above technical note presents a novel convex search within the Zames–Falb multiplier class. The aim of this correspondence is to correct the misuse of a relaxation on one condition in the search. This relaxation leads to nontrivial numerical errors in all six examples discussed in the technical note. Correct application of the conditions still gives an improvement over absolute stability criteria in the literature for at least one example, but some of the claims for lack of conservativeness in the above technical note should be moderated.**

***Index Terms*—Integral quadratic constraints (IQC), linear matrix inequalities (LMIs), Zames–Falb multiplier.**

I. INTRODUCTION

In [1], absolute stability of a feedback interconnection between a linear system and slope-restricted and odd nonlinearity is studied using an integral quadratic constraints (IQC) analysis. Given a linear system, this nonlinear problem is solved by finding a multiplier M such that [1, eq. 2] and [1, eq. 4] [1] are satisfied. In [2], it is shown that all elements of a class of multiplier, nowadays referred to as the Zames-Falb multipliers, satisfy [1, eq. 2] for any slope-restricted and odd nonlinearity. "Proposition 2" in [1] provides a quasi-convex search within a subclass of Zames-Falb multipliers to find the maximum slope of these nonlinearities by studying feasibility of linear matrix inequalities (LMIs) [1,

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[1]Throughout this note, [1, eq. n] means equation (n) in [1].

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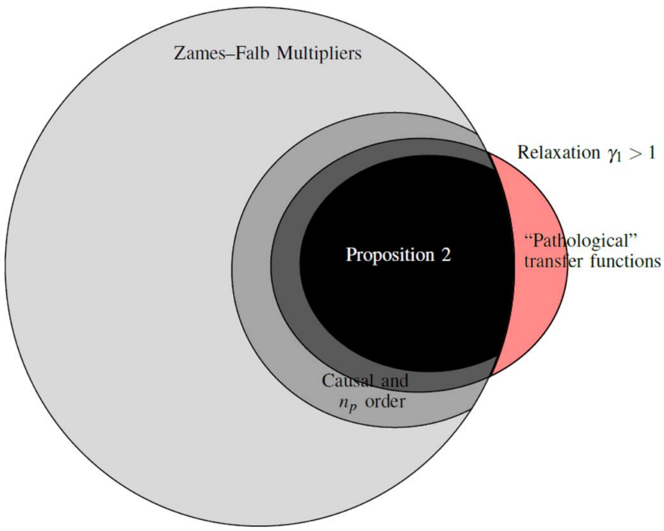

Fig. 1. Schematic representation of various transfer function sets: Proposition 2 only searches within the Zames-Falb class. However relaxation $\gamma_1 > 1$ not only extends the boundaries of the multipliers by using Proposition 2, but also includes "pathological" transfer functions which are not Zames-Falb multipliers. Therefore, "*relaxed* Proposition 2" can lead to incorrect results.

eq. 9], [1, eq. 10], and [1, eq. 11]. The subclass of Zames-Falb multipliers constitutes those which are causal and rational transfer functions of the same order as the linear system. Feasibility of these LMIs implies that there exists a multiplier M satisfying [1, eq. 4] and M is a Zames-Falb multiplier, i.e., M satisfies [1, eq. 2].

In [1, Remark 2.2], a relaxation is proposed by replacing LMI [1, eq. 11] with LMI [1, eq. 27] where $\gamma_1 > 1$, in order to reduce the conservatism of LMIs [1, eq. 7] and [1, eq. 8]. This relaxation is carried out by choosing γ_1 slightly greater than unity, and absolute stability is claimed using this relaxation. It is stated that "normally when the multipliers are reconstructed," they are Zames-Falb multipliers; however, the authors note that multiplier reconstruction could be ill-conditioned. No further details are given in the discussion of examples. The authors label results using this relaxation "Proposition 2" (e.g., [1, Table II]). Since our aim is to show that the relaxation is a source of problems, we will distinguish results using LMIs [1, eq. 9, 10, and 11], from results using LMIs [1, eq. 9], [1, eq. 10], and [1, eq. 27] with $\gamma_1 > 1$ by labeling them "Proposition 2" and "*relaxed* Proposition 2," respectively.

To the best of our knowledge, "Proposition 2" is correct. However the relaxation can lead to incorrect results if the original condition is not checked when the algorithm finishes. This is because the corresponding class of transfer functions includes transfer functions, henceforward "pathological" transfer function, that are not Zames-Falb multipliers, for any relaxation $\gamma_1 > 1$ irrespective of how small; see Fig. 1. Absolute stability cannot hence be claimed because there is no guarantee that the transfer function satisfying [1, eq. 4] is a Zames-Falb multiplier, i.e., the transfer function may not satisfy [1, eq. 2]. In fact, all six transfer functions given in [1, Table III] are not Zames-Falb multipliers, and hence the examples are all incorrect. We argue that this phenomenon is not related to an ill-conditioned reconstruction, since the size of the relaxation is irrelevant in the final result: once $\gamma_1 > 1$ the main constraint is [1, eq. 4], represented by LMI [1, eq. 9].

In [1], the claim is made that "the results obtained using this potentially conservative form of multiplier are, in the examples considered, not very conservative at all." The numerical results obtained for six examples using "*relaxed* Proposition 2" are used to support this claim. The main contribution of this correspondence is to correct the claims for the lack of conservativeness of this method, showing that the results obtained by competing methods are better for four out of six examples. This note is structured as follows. Section II shows some falsification and sufficiency tests for Zames-Falb multipliers. Using these tests, it is concluded that all six transfer functions given in [1, Table III] are not Zames-Falb multipliers. Section III analyzes why this is the case. It is shown that any stable transfer function can be scaled such that it can be included in the extended class of transfer functions produced by the relaxation. Corrected results for the examples given in [1] are presented, showing nontrivial numerical errors in [1]. Finally, a well-conditioned reconstruction is proposed and used to reconstruct correct multipliers for the examples. In Section IV, the conclusions of this correspondence are presented.

II. Falsification and Sufficiency Tests for Zames-Falb Multipliers

The typical approach to construct a Zames-Falb multiplier is to take $H_0 = 1$ and find an adequate transfer function $H_1(s)$ such that $\|H_1\|_{\hat{A}} < 1$ (see below for a definition of the $\hat{\mathbf{A}}$-norm). Then, $M(s) = H_0 - H_1(s)$ is a Zames-Falb multiplier by definition (see Definition 2.7). The approach of this section is different; our aim is to find conditions for a transfer function to be a Zames-Falb multiplier. Before approaching the problem, some preliminaries results are given.

A. Notation and Preliminaries

As in [1], the standard notation $\mathbf{RL}_\infty$ is used for the space of all proper real rational transfer functions bounded in the imaginary axis, and $\mathbf{RH}_\infty$ is used for the space of all proper real rational transfer functions such that all their poles have strictly negative real parts. However, we would like to avoid the acknowledged abuse of notation in [1] with respect to the signal and system norms. We will follow [3], [4] and define the following normed spaces.

Definition 2.1: The symbol $\mathbf{A}$ denotes the set of generalized functions (distributions) $f(\cdot)$ such that $f(t) = 0$ when $t < 0$, and

$$f(t) = \sum_{i=0}^{\infty} f_i \delta(t - t_i) + f_a(t), \qquad \text{when } t \geq 0 \tag{1}$$

where $\boldsymbol{\delta}(\cdot)$ denotes the unit delta distribution, $0 \leq t_0 < t_1 < \ldots$ are constants, $f_a \in \mathcal{L}_1$, and in addition

$$\sum_{i=0}^{\infty} |f_i| < \infty. \tag{2}$$

In addition, the norm $\|f\|_A$ of a distribution in $\mathbf{A}$ is defined by

$$\|f\|_A = \sum_{i=0}^{\infty} |f_i| + \|f_a\|_1. \tag{3}$$

Remark 2.2: Note that $\mathcal{L}_1$ is a subset of $\mathbf{A}$; further, if $f \in \mathcal{L}_1$, then

$$\|f\|_A = \|f\|_1. \tag{4}$$

Loosely speaking, the $\mathbf{A}$-norm is the natural generalization of the $\mathcal{L}_1$ norm to the space that includes the impulse responses of biproper stable transfer function. Its use also justifies allowing D_u to be nonzero in, for example, [1, eq. 5]. The $\mathbf{A}$-norm provides an adequate framework for the impulse response of the systems under consideration in this correspondence.

Definition 2.3: The symbol $\hat{\mathbf{A}}$ denotes the set of all transfer functions $H(s)$ that are Laplace transforms of the elements of $\mathbf{A}$. Moreover, the $\hat{\mathbf{A}}$-norm is defined by

$$\|H\|_{\hat{A}} = \|h\|_A \tag{5}$$

where $h(\cdot)$ is the impulse response of $H(s)$.

Remark 2.4: Using "Lemma 27" ([3, p. 297]), this norm is properly defined for any SISO stable proper rational transfer function: given a SISO $H(s) \in \mathbf{RH}_\infty$, let (A, B, C, D) be a minimal state space realization of the transfer function $H(s) = D + C(sI - A)^{-1}B$ then

$$\|H\|_{\hat{A}} = \|h\|_A = |D| + \int_0^\infty |Ce^{At}B|dt \tag{6}$$

where $h(\cdot)$ is the impulse response of $H(s)$.

The following two lemmas give useful properties of the $\hat{\mathbf{A}}$-norm of a transfer function.

Lemma 2.5: Given a SISO transfer function $R(s) \in \mathbf{RH}_\infty$, and any two constants $\beta, \zeta \in \mathbb{R}$, and let us define $Q(s) = \beta + \zeta R(s)$. Then

$$\|Q\|_{\hat{A}} \leq |\beta| + |\zeta| \|R\|_{\hat{A}}. \tag{7}$$

In addition, if $R(s)$ is also strictly proper, then

$$\|Q\|_{\hat{A}} = |\beta| + |\zeta| \|R\|_{\hat{A}}. \tag{8}$$

For the space $\mathbf{RH}_\infty$, we also can define another norm. The following lemma relates both norms.

Lemma 2.6 ([3]): Let $H(s) \in \mathbf{RH}_\infty$ be a SISO transfer function, then

$$\|H\|_{\hat{A}} \geq \sup_{\omega \in \mathbf{R}} |H(j\omega)| = \|H\|_\infty. \tag{9}$$

In the spirit of [1] only causal Zames-Falb multipliers, in particular stable rational transfer function, are considered in this correspondence. Under this restriction, the original class of Zames-Falb multipliers is defined as follows:

Definition 2.7 (Zames-Falb Multiplier [2]): A rational transfer function, $M(s) \in \mathbf{RH}_\infty$, is said to be a Zames-Falb multiplier if it is given by $M(s) = H_0 - H_1(s)$, where $\|H_1\|_{\hat{A}} < H_0$.

Using IQC analysis the definition can be extended to [1, Definition 1], where the strict inequality is replaced by a nonstrict inequality. However, it has been shown in [5] that original definition can be used without loss of generality.

Given a transfer function $M(s)$, the symbol $\widetilde{M}(s)$ means its associated strictly proper transfer function, i.e., $\widetilde{M}(s) = M(s) - M(\infty)$.

B. *Falsification Test*

This section presents two falsification tests. In [6], "Remark 3" can be rewritten as a first test.

Lemma 2.8 ([6]): Let $M(s)$ be a Zames-Falb multiplier, then $\mathrm{Re}(M(j\omega)) > 0$ for all $\omega \in [-\infty, \infty]$.

Proof: (No proof is given in [6].) For any transfer function given by $M(s) = H_0 - H_1(s)$, where $\|H_1\|_{\hat{A}} < H_0$, using Lemma 2.6, then $\|H_1\|_\infty < H_0$, thus the real part of $H_1(j\omega)$ lies within $(-H_0, H_0)$ for all $\omega \in \mathbb{R}$. As a result

$$\mathrm{Re}\left(M(j\omega)\right) \geq H_0 - \mathrm{Re}\left(H_1(j\omega)\right) > 0. \tag{10}$$

Since the definition of $\|H_1\|_\infty$ contains the supremum for all ω, then (10) is also satisfied when $\omega = \infty$. ■

If a SISO transfer function is strictly positive in the sense of Lemma 2.8 and causal, then it is said to be strictly positive real [7]. Hence our first falsification test for causal multipliers.

Falsification Test ZF1: If $M(s) \in \mathbf{RH}_\infty$ is a Zames-Falb multiplier then it must be a strictly positive real (SPR) transfer function. This can be checked with standard commercial software. Graphically, the Nyquist plot of $M(s)$ must lie in the positive real half plane. □

The following lemma gives a necessary and sufficient condition on a transfer function to be a Zames-Falb multiplier.

Lemma 2.9: Let $M(s) \in \mathbf{RH}_\infty$ be a transfer function with $M(s) = M(\infty) + \widetilde{M}(s)$. Then, $M(s)$ is a Zames-Falb multiplier if and only if $\|\widetilde{M}\|_{\hat{A}} < M(\infty)$.

Proof: Sufficiency: If $\|\widetilde{M}\|_{\hat{A}} < M(\infty)$, then taking $H_0 = M(\infty)$ and $H_1(s) = \widetilde{M}(s)$, the result is obtained.

Necessity: Assume that $M(s)$ is a Zames-Falb multiplier, i.e., there exists $H_0 > 0$ such that $M(s) = H_0 - H_1(s)$, where $\|H_1\|_{\hat{A}} < H_0$. If $H_1(s)$ is rewritten as $H_1(s) = H_1(\infty) + \widetilde{H}_1(s)$, then

$$M(\infty) + \widetilde{M}(s) = H_0 - H_1(\infty) - \widetilde{H}_1(s) \tag{11}$$

and hence

$$M(\infty) = H_0 - H_1(\infty) \tag{12}$$

$$\widetilde{M}(s) = -\widetilde{H}_1(s). \tag{13}$$

Hence, using (8), $\|H_1\|_{\hat{A}}$ is related to $\|\widetilde{M}\|_{\hat{A}}$ as follows:

$$\|H_1\|_{\hat{A}} = |H_1(\infty)| + \|\widetilde{M}\|_{\hat{A}} < H_0 \quad \text{and hence } \|\widetilde{M}\|_{\hat{A}} H_0 \quad \forall H_0 > 0. \tag{14}$$

To cope with the absolute value obtained when norms are taken, two cases can be considered:

- If $H_1(\infty) \geq 0$, the left equality in (14) implies that $\|\widetilde{M}\|_{\hat{A}} < H_0 - H_1(\infty) = M(\infty)$;
- if $H_1(\infty) < 0$, the right inequality in (14) implies that $\|\widetilde{M}\|_{\hat{A}} < H_0 < H_0 - H_1(\infty) = M(\infty)$.

In summary, if $M(s)$ is a Zames-Falb multiplier, then $\|\widetilde{M}\|_{\hat{A}} < M(\infty)$. ■

The following falsification test follows from Lemmas 2.6 and 2.9. It does not require the numerical computation of the $\hat{\mathbf{A}}$-norm.

Falsification Test ZF2: If $M(s) \in \mathbf{RH}_\infty$ is a Zames-Falb multiplier then it must satisfy $M(\infty) > \|\widetilde{M}\|_\infty$. This condition can also be checked with standard commercial software. Graphically, the circle of center $(M(\infty), 0)$ and radius $\|\widetilde{M}\|_\infty$ must not cross the imaginary axis. □

C. *Sufficiency Test*

Note that the $\hat{\mathbf{A}}$-norm can be bounded using [1, eq. 7] and [1, eq. 8] with an LMI solver. If these LMIs are feasible in such a way that the given bound of $\|\widetilde{M}\|_{\hat{A}}$ is less than $M(\infty)$, then $M(s)$ is a Zames-Falb multiplier, i.e., a slight modification of LMIs [1, eq. 7] and [1, eq. 8] can be used as a sufficiency test. However, if they are not feasible, no conclusion can be obtained—i.e., they cannot be used as part of a falsification test.

Sufficiency Test ZF: Given a SISO transfer function $M(s) \in \mathbf{RH}_\infty$ with state-space realization $M(s) \sim (A_m, B_m, C_m, D_m)$, i.e., $\widetilde{M}(s) = C_m(sI - A_m)^{-1}B_m$ and $M(\infty) = D_m > 0$, if there exist $Y = Y' > 0$ and scalars $\lambda > 0$ and $\mu > 0$, such that the following inequalities hold:

$$\begin{bmatrix} A_m'Y + YA_m + \lambda Y & YB_m \\ \star & -\mu I \end{bmatrix} < 0 \tag{15}$$

$$\begin{bmatrix} \lambda Y & 0 & C_m' \\ \star & (D_m - \mu)I & 0 \\ \star & \star & D_m \end{bmatrix} \geq 0 \tag{16}$$

then $M(s)$ is a Zames-Falb multiplier. □

Remark 2.10: Although the tests and Lemma 2.9 have been presented for causal multipliers, i.e., $M(s) \in \mathbf{RH}_\infty$, their extensions for noncausal multipliers, i.e., $M(s) \in \mathbf{RL}_\infty$, can be easily obtained by extending the $\hat{\mathbf{A}}$-norm definition to the interval $(-\infty, \infty)$ (see [8, p. 126]) and using the canonical factorization of the Zames-Falb multipliers.

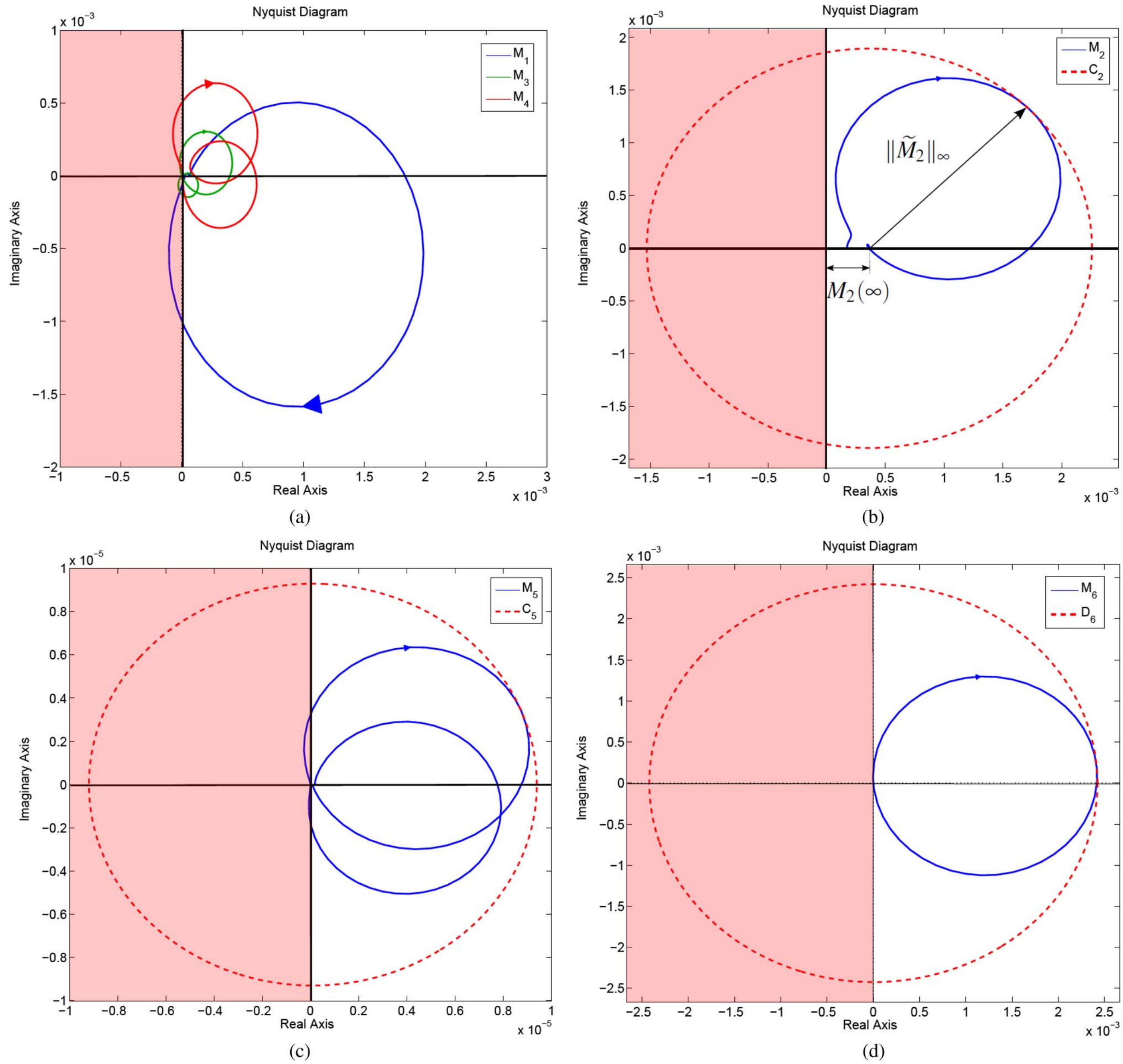

Fig. 2. Transfer functions given in [1, Table III]: (a) Nyquist plot of $M_1(s)$, $M_3(s)$, and $M_4(s)$: they fail test ZF1. (b) Nyquist plot of $M_2(s)$ and circle of center $(M_2(\infty), 0)$ and radius $\|\widetilde{M}_2\|_\infty$: M_2 fails test ZF2. (c) Nyquist plot of $M_5(s)$ and circle of center $(M_5(\infty), 0)$ and radius $\|\widetilde{M}_5\|_\infty$: M_5 fails test ZF1 and ZF2. (d) Nyquist plot of $M_6(s)$ and circle of center $(M_6(\infty), 0)$ and radius $\|\widetilde{M}_6\|_\infty$: M_6 fails test ZF2. It also fails ZF1, but is hard to see graphically without applying a scaling.

D. Application

Using the falsification tests, it can be concluded that all six transfer functions given in [1, Table III] are not Zames-Falb multipliers:

- Transfer functions $M_1(s), M_3(s), M_4(s), M_5(s)$, and $M_6(s)$ fail test ZF1 [see Fig. 2(a), (c), and (d)].
- Transfer function $M_2(s)$ fails ZF2 [see Fig. 2(b)].

III. Relaxation

A. Extended Class of Transfer Functions

In this section, we analyze why the reconstructed transfer functions by using "*relaxed* Proposition 2" are not guaranteed to be Zames-Falb multipliers. A Zames-Falb multiplier $M(s)$ can be defined such that $M(\infty) = 1$ without loss of generality, i.e., D_u in [1, eq. 5] can be set to 0. Nevertheless considering D_u as a variable in the corresponding LMI is not illegitimate (see Fig. 3). In this case, the corresponding multiplier satisfies $M(\infty) = 1 - D_u$. But any choice $\gamma_1 > 1$ extends significantly the class of transfer functions under consideration (see Fig. 1). In particular, if D_u is allowed to be a variable, the set of "pathological" transfer functions includes transfer functions where $|C_u B_u|$ is small compared to $|D_u|$.

The extended class of transfer functions given by any relaxation of [1, eq. 11] with [1, eq. 27] where $\gamma_1 > 1$ can be defined as follows.

Definition 3.1: Given $\varepsilon \geq 0$, let $\mathcal{M}_\varepsilon$ be a class of transfer functions given by

$$\mathcal{M}_\epsilon = \left\{ M(s) = 1 - H_1(s) : \|H_1\|_{\hat{A}} < 1 + \epsilon \right\}. \tag{17}$$

Remark 3.2: Note that $\mathcal{M}_0$ is the class of Zames-Falb multipliers. "Proposition 2" produces a search within a subset of the class of Zames-

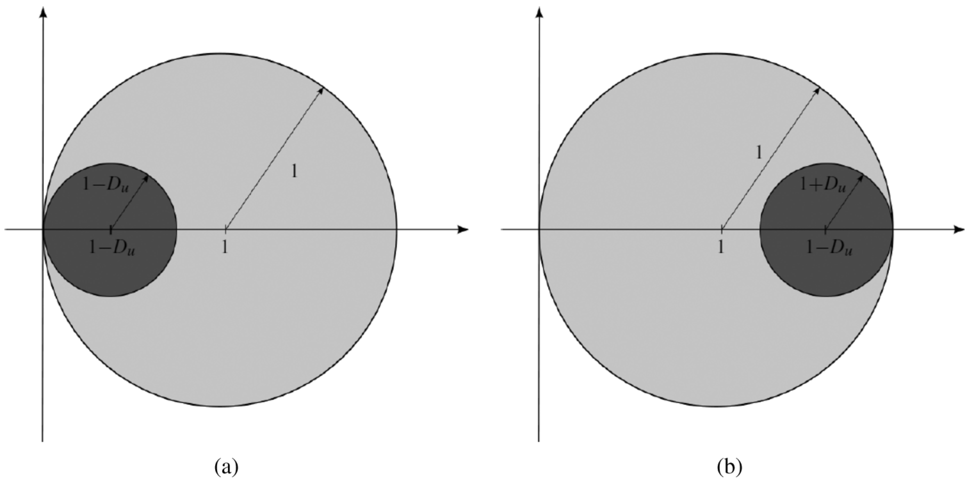

Fig. 3. In light gray, the permissible region for the Nyquist plot of a Zames-Falb multiplier when $D_u = 0$ [6]. In dark gray, the permissible region for the Nyquist plot of a Zames-Falb multiplier when $D_u \neq 0$. In (a) the difference between both regions is a scaling factor; therefore both are equivalent. In (b) the region with $D_u \neq 0$ is more conservative.

Falb multipliers: causal and fixed order transfer functions (see Fig. 1). Since this is not relevant for the argument in this section, we do not consider these restrictions for the sake of simplicity.

The class $\mathcal{M}_\varepsilon$ contains all Zames-Falb multipliers, i.e $\mathcal{M}_0 \subset \mathcal{M}_\varepsilon$. However, the following Lemma shows that it also contains many other transfer functions.

Lemmma 3.3: Given $\varepsilon > 0$ and any SISO transfer function $H(s) \in \mathbf{RH}_\infty$, then there exists $k > 0$ such that $M(s) = kH(s) \in \mathcal{M}_\varepsilon$.

Proof: Given $H(s) \in \mathbf{RH}_\infty$, let us take $k = \varepsilon/(2\|H\|_{\hat{A}})$. Then, $M(s)$ can be rewritten as $M(s) = 1 - (1 - \varepsilon/(2\|H\|_{\hat{A}})H(s))$, hence

$$H_1(s) = 1 - \frac{\epsilon}{2\|H\|_{\hat{A}}}H(s). \tag{18}$$

Taking norms

$$\|H_1\|_{\hat{A}} \leq 1 + \frac{\epsilon}{2\|H\|_{\hat{A}}}\|H\|_{\hat{A}} < 1 + \epsilon \text{ and hence } M(s) \in \mathcal{M}_\epsilon \tag{19}$$

where Lemma 2.5 has been used. ∎

Remark 3.4: Note that (18) shows the importance of D_u as a variable. In this case, LMIs [1, eq. 7] and [1, eq. 8] need not be conservative since $\|U\|_{\hat{A}} \simeq |D_u|$. An adequate scaling in $C_u B_u$ provides full freedom in the selection of a Hurwitz matrix A_u.

Remark 3.5: The multiplier is a device to transfer passivity from one part of the loop to the other. Despite the use of the IQC machinery, one can consider [1, eq. 4] as a classical passivity condition, i.e.

$$\arg\left(M(j\omega)(1 - \alpha G(j\omega))\right) \in (-90^\circ, 90^\circ).$$

In particular the phase of the multiplier gives the shifting property and the magnitude is not relevant. The phase of a Zames-Falb multiplier is restricted to lie within $(-90^\circ, 90^\circ)$, as shown in Lemma 2.8, because the multiplier must preserve the positivity of the nonlinearity. Hence the small dark gray region in Fig. 4 allows full freedom in the selection of the phase. Any transfer function $M(s)$ that shifts adequately the phase of $M(j\omega)(1 - \alpha G(j\omega))$ can be scaled to lie inside the dark gray region. The mandatory condition of preserving the positivity of the nonlinearity, given by [1, eq. 2], is lost and relaxation provides "better," but illegal, results.

B. Results

Numerical results for the Examples of [1] are shown in Table I [10]. The first three rows are taken directly from [1], but we have relabeled

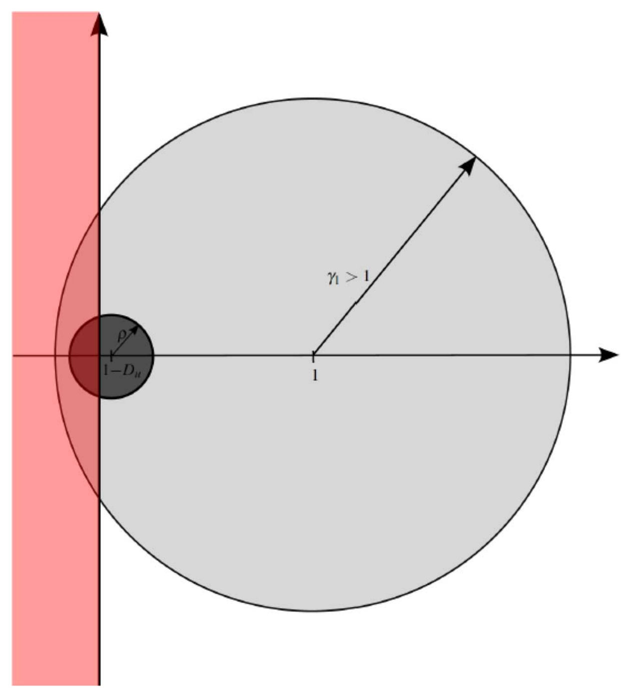

Fig. 4. In light gray, the feasible region for the class of transfer functions when the relaxation is performed. The center of the region is kept in 1, but the radius is increased. The dark gray shows a region which is available for any bounded transfer function by using an adequate scaling factor. Setting $D_u \simeq 1$ and suitably scaling $|C_u B_u|$ allows the selected transfer function to lie in the dark gray region giving near complete freedom in choice of phase.

the third row "*relaxed* Proposition 2." NB our computations suggest that the results given in [1] for the circle and Park's criteria in Examples 3 and 4 should be swapped.

The fourth row shows our own numerical results when Proposition 2 is applied correctly. We have optimized over λ, as suggested in [9], requiring a quasi-convex optimization (results with λ fixed are significantly worse). We have set $H_0 = 1$ and $D_u = 0$. For simplicity we have preferred not to use D_u as an optimization variable, but it should be mentioned that small discrepancies can arise compared to when including D_u as a variable. The likely cause is that LMIs [1, eq. 7 and 8] give "no exact characterization" of the $\hat{\mathbf{A}}$-norm as commented in [9].

TABLE I
SECTOR/SLOPE BOUNDS OBTAINABLE USING VARIOUS STABILITY CRITERIA

Criteria	Ex. 1	Ex. 2	Ex. 3	Ex. 4	Ex. 5	Ex. 6
Circle	1.2431	0.7640	0.3263	0.3081	0.00040	0.00039
Park [10]	4.5894	1.0894	0.7883	0.7083	0.00183	0.00183
"*Relaxed* Proposition 2" ([1])	4.5894	1.0894	10,000+	3.4661	662.5063	1.6599
"Proposition 2"	2.2428	1.0894	0.7049	0.8526	0.00181	0.00095

TABLE II
MULTIPLIER COMPUTED USING [1, eq. 9], [1, eq. 10], AND [1, eq. 11] WITH $H_0 = 1$, $D_u = 0$, AND α GIVEN IN TABLE I

Example	λ	Multiplier
1	0.32	$M_1'(s) = \frac{s^3+0.7336s^2+0.7751s+0.1894}{s^3+1.175s^2+0.7324s+0.139}$
2	3.6	$M_2'(s) = \frac{s^3+9.622s^2+24.8s+11.3}{s^3+12.3s^2+51.64s+77.14}$
3	0.09	$M_3'(s) = \frac{s^4+0.916s^3+6.721s^2+0.8853s+1.309}{s^4+0.9933s^3+6.798s^2+1.347s+1.079}$
4	0.54	$M_4'(s) = \frac{s^4+2.027s^3+7.178s^2+4.356s+0.5378}{s^4+2.488s^3+7.545s^2+6.862s+2.857}$
5	275	$M_5'(s) = \frac{s^4+1107s^3+3.795\ 10^5s^2+4.164\ 10^7s+5.537\ 10^7}{s^4+1374s^3+6.75\ 10^5s^2+1.431\ 10^8s+1.115\ 10^{10}}$
6	0.012	$M_6'(s) = \frac{s^4+0.04971s^3+10.04s^2+0.3503s+9.041}{s^4+0.06182s^3+10s^2+0.3631s+9.003}$

Note that the $\hat{\mathbf{A}}$-norm has the same values as the peak-to-peak norm in [4, eq. (29)].

It can be seen that Proposition 2 indeed beats Park's criterion in Example 4, but gives worse results in Examples 1, 3, 5, and 6.

Example 1 is a case where Park's method is less conservative than "Proposition 2." This directly contradicts the statement in [1] that "Park's method is equivalent to choosing IQC's of a particular form whereas our method allows optimization over a larger class of multipliers."

It is worth noting that for Example 2, where "*relaxed* Proposition 2" and "Proposition 2" give equivalent numerical results, the transfer function M_2 does not use the forbidden area although it is still pathological. This can be ascribed to λ being fixed near to zero. In this case, the LMIs [1, eq. 7] and [1, eq. 8] are very conservative, except when $D_u \simeq 1$. As a result, even though there are Zames-Falb multipliers available, the *relaxed* search with $\lambda \simeq 10^{-5}$ leads to a pathological transfer function.

Table II shows the reconstructed multipliers corresponding to row four in Table I. Using Sufficiency Test ZF, it is easy to check that $\|\widetilde{M}_i'\|_{\hat{A}} < M_i'(\infty) = 1$ for all $i = 1, \ldots, 6$. This confirms that all reconstructed transfer functions are Zames-Falb multipliers when "Proposition 2" is used correctly.

It is stated in [1] that "poor numerical conditioning may arise in the reconstruction of multiplier." Letting $P_{22} = I$ as in [1], then [1, eq. 29 and 30] should read

$$P_{12}P_{12}' = P_{11} - S_{11} \tag{20}$$
$$S_{11}Q_{12} = -P_{12} \tag{21}$$

and using a state space similarity transformation, multiplier reconstruction can be carried out as follows:

$$A_u = -(P_{11} - S_{11})^{-1}\mathbf{A_u} \tag{22}$$
$$B_u = -(P_{11} - S_{11})^{-1}\mathbf{B_u} \tag{23}$$
$$C_u = -\mathbf{C_u}. \tag{24}$$

As suggested in [1] this may be ill-conditioned: specifically this occurs when $P_{11} - S_{11}$ is ill-conditioned. But the underlying LMI is also ill-conditioned in this case.

C. Discussion on a Legal Relaxation

The analysis of Section III has shown that the relaxation proposed in [1] can lead to incorrect results. Nevertheless, a relaxation can be performed provided a reliable method is used to compute the $\hat{A}$-norm in order to check that $\|U\|_{\hat{A}} < 1$. A detailed analysis is beyond the scope of this correspondence, but the following should be pointed out:

- Setting $D_u = 0$ is the safest option since the pathological transfer functions such as (18) are then avoided.
- The improvement is usually moderate even with a *significant* relaxation of $\gamma_1 > 1$.
- It is *essential* to reconstruct the multipliers and perform a sufficiency test. Note that a transfer function obtained by relaxing $\gamma > 1$ may fail Sufficiency Test ZF. The $\hat{\mathbf{A}}$-norm of $U(s)$ must be numerically computed or bounded in some other way, which can be a new source of problems. Using the framework of [1], any *optimization* with λ and γ_1 as variables would not even be quasi-convex.

IV. CONCLUSION

"Proposition 2" in [1] provides a convex search within a subclass of Zames-Falb multipliers. "Proposition 2" is mathematically correct; however the numerical results given in [1] are incorrect: Falsification tests proposed in this correspondence show that the transfer functions given in [1, Table III] are not Zames-Falb multipliers. The relaxation proposed in [1] for "Proposition 2" is behind this phenomenon, because "*relaxed* Proposition 2" extends the feasible class of transfer function and allows considerably more freedom in the choice of phase. As a consequence, the use of "*relaxed* Proposition 2" easily leads to incorrect results if the original condition $\|U\|_{\hat{A}} < 1$ is not adequately checked after using "*relaxed* Proposition 2."

We have also proposed both a sufficiency test for Zames-Falb multipliers and a novel reconstruction of the multipliers from the LMIs of [1]. We have thus confirmed that when applied correctly, "Proposition 2" in [1] does indeed find appropriate Zames-Falb multipliers. Numerical results indicate that claims for the lack of conservativeness of the LMIs in [1] should be moderated. Nevertheless, the result for Example 4 gives an improvement over Park's criterion, showing that "Proposition 2" of [1] remains a useful contribution. Similarly our observations in no way invalidate the subsequent results (e.g., [11]) of the same authors.

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Authors Reply to "Comments on 'On the Existence of Stable, Causal Multipliers for Systems With Slope-Restricted Nonlinearities'"

Matthew C. Turner, Murray Kerr, and Ian Postlethwaite

***Abstract*—In this reply, we show that, in certain cases, the analysis approach given by Turner et al. does indeed give "superior" results to that given by Park. We also show, briefly, how the analysis approach given by Turner et al. can be enhanced by considering both Zames-Falb and Popov multipliers (as in the papers by Jonsson and Turner and Kerr).**

***Index Terms*—Linear matrix inequalities, nonlinear control systems.**

I. INTRODUCTION

In [6], we suggested an approach for analysing the stability of feedback interconnections consisting of a linear time-invariant part and a static, slope-restricted nonlinear part. The main results of [6] showed how the existence of a Zames-Falb multiplier guaranteeing the stability of such an interconnection could be verified by solving a set of linear matrix inequalities (LMI's), coupled with a line-search over a scalar parameter, λ. These LMI's consisted of three main parts: a "passivity" type LMI, and two "$\mathcal{L}_1$" inequalities which were constructed using the results of [4]. The purpose of the "$\mathcal{L}_1$" inequalities was to ensure that the multiplier which was guaranteed to exist was indeed a Zames-Falb multiplier; that is it had the form $M(s) = H_0(1 - U(s))$ where $\|U\|_1 \leq 1$. Due to the well-known conservatism of the "$\mathcal{L}_1$" inequalities given in [4], a numerical relaxation was suggested in [6] which, instead of enforcing $\|U\|_1 \leq 1$, enforced $\|U\|_1 \leq 1 + \epsilon$ for some suitably small ϵ. The idea behind this relaxation was to recognize the conservatism of the $\mathcal{L}_1$ inequalities and, hence, to reduce the conservatism in our results. Although not explicitly stated, it was intended that the $\mathcal{L}_1$ bound on $U(s)$ continued to be enforced, which could be checked *a posteriori*. A by-product of this relaxation was that the numerical results given in [6] appeared to be insensitive to the scalar parameter λ, and hence effectively the line search could be dropped and the computation was essentially purely LMI-based.

However, it was recently pointed out [1] that such a relaxation was not usually permissible. [1] showed that using the relaxation described above guaranteed the existence of a "multiplier" belonging to a broader class than simply those suggested by Zames and Falb. This then implied that, any "multiplier" reconstructed on the basis of the relaxation proposed in [6] would not necessarily be a Zames-Falb multiplier, and thus the system could not be guaranteed to be stable. This finding of

TABLE I
MAXIMUM SLOPE (α) FOR WHICH STABILITY HOLDS: COMPARISON OF DIFFERENT RESULTS

Example	Maximum slope		
	[3]	[6]	Proposition 1
1	4.5866	2.4243 ($\lambda = 0.32$)	3.5026 ($\lambda = 831.2$)
2	1.0891	1.0891 ($\lambda = 3.6$)	1.0891 ($\lambda = 3.6$)
3	0.78833	0.70606 ($\lambda = 0.091$)	0.77673 ($\lambda = 482.7$)
4	0.70833	0.85261 ($\lambda = 0.54$)	1.0864 ($\lambda = 1.55$)
5	0.0018335	0.0018118 ($\lambda = 276$)	0.0033370 ($\lambda = 0.4$)
6	0.0018325	0.00095969 ($\lambda = 0.025$)	0.0031513 ($\lambda = 0.12$)
7	5.4956	9.1210 ($\lambda = 0.46$)	12.2377 ($\lambda = 0.15$)
8	0.17189	0.22884 ($\lambda = 0.35$)	0.52249 ($\lambda = 0.45$)
9	0.15466	0.25506 ($\lambda = 0.12$)	0.27863 ($\lambda = 0.12$)

TABLE II
ZAMES FALB MULTIPLIERS WHICH PROVE STABILITY USING [6]

Example	Zames-Falb multiplier
1	$\frac{s^3+0.7132s^2+0.7662s+0.1764}{s^3+1.155s^2+0.7147s+0.1294}$
2	$\frac{s^3+9.783s^2+25.58s+11.62}{s^3+12.31s^2+51.9s+78.13}$
3	$\frac{s^4+0.9208s^3+6.725s^2+0.8943s+1.313}{s^4+0.9994s^3+6.804s^2+1.364s+1.081}$
4	$\frac{s^4+2.027s^3+7.178s^2+4.356s+0.5377}{s^4+2.488s^3+7.545s^2+6.863s+2.857}$
5	$\frac{s^4+1271s^3+4.634e005s^2+5.225e007s+6.953e007}{s^4+1535s^3+8.012e005s^2+1.762e008s+1.405e010}$
6	$\frac{s^4+0.09472s^3+10.08s^2+0.6562s+9.086}{s^4+0.1197s^3+10s^2+0.676s+9.011}$
7	$\frac{s^4+6.395s^3+51.85s^2+30.27s+4.288}{s^4+5.757s^3+41.94s^2+27.3s+5.575}$
8	$\frac{s^4+2.394s^3+90.65s^2+53.11s+5.803}{s^4+2.443s^3+84.81s^2+56.37s+11.02}$
9	$\frac{s^6+1712s^5+4581s^4+4997s^3+2500s^2+568.7s+138.3}{s^6+1712s^5+4806s^4+5536s^3+2977s^2+689.6s+105.7}$

TABLE III
ZAMES FALB MULTIPLIERS WHICH PROVE STABILITY USING PROP. 1 ($\mathbf{H_0} = 1$)

Example	Zames-Falb multiplier
1	$\frac{0.02264s^3+38.12s^2+1.604e004s+10.94}{s^3+2515s^2+2.109e006s+5.894e008}$
2	$\frac{0.9991s^3+13.16s^2+44.08s+26.63}{s^3+14.95s^2+67.65s+101.7}$
3	$\frac{0.69s^4+1243s^3+7.156e005s^2+1.338e008s+3368}{s^4+2284s^3+1.907e006s^2+6.947e008s+9.364e010}$
4	$\frac{0.002409s^4+0.01691s^3+0.05407s^2+0.05799s+0.0003819}{s^4+8.646s^3+33.86s^2+60.17s+39.12}$
5	$\frac{0.962s^4+2.233s^3+11.35s^2+14.94s+18.73}{s^4+3.095s^3+13.59s^2+17.33s+15.35}$
6	$\frac{0.9514s^4+0.343s^3+9.553s^2+2.28s+7.912}{s^4+0.4804s^3+10.09s^2+2.41s+9.146}$
7	$\frac{0.8139s^4+9.311s^3+25.31s^2+26.49s+1.762}{s^4+11.83s^3+32s^2+35.22s+5.299}$
8	$\frac{0.005422s^4+0.04082s^3+1.715s^2+1.238s+0.02227}{s^4+8.06s^3+320.4s^2+396.3s+122.3}$
9	$\frac{0.000753s^6+0.0445s^5+0.0608s^4+0.0696s^3+0.0249s^2+0.0064s+0.00117}{s^6+59.2s^5+90.6s^4+104.5s^3+47.2s^2+11.4s+1.4}$

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