Technical Notes and Correspondence

Stability Analysis of Interconnected Systems With "Mixed" Negative-Imaginary and Small-Gain Properties

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***Abstract*—In this letter, an analytical framework is proposed to examine stability of two stable, linear time invariant systems interconnected in positive feedback where the systems have "mixed" properties of negative-imaginary and small-gain. Using the notion of dissipativity, the interconnection of systems is guaranteed to be finite-gain stable under the condition that the dc loop gain is contractive. This work builds on Griggs, *et al.* [14] and exploits a new set of frequency dependent triplets that was introduced in above reference to "mix" two unconditional stability statements, i.e., small-gain and passivity. Unlike the above reference the present work explores the important question of how a conditional stability statement as needed when two negative-imaginary systems are connected in a feedback loop can be "mixed" with an unconditional stability statement as needed when two contractive systems are connected in a feedback loop. The usefulness of the proposed analytical framework is demonstrated via a numerical example.**

***Index Terms*—Bounded-real system, dissipativeness, finite-gain stability, interconnected system, negative-imaginary system, positive feedback.**

I. Introduction

The concept of negative-imaginary systems was introduced in [1], [5] and explored with some practical relevances in recent literatures [2]–[4], [6]–[10]. It is of interest to the engineering community as this class of systems appears physically in for example lightly damped structure with colocated position sensors and force actuators [11]–[13]. By definition, a negative-imaginary system is a stable system with equal number of inputs and outputs, and it has negative-imaginary frequency response as frequency varies in an open interval 0 to ∞ [1]. For instance, lightly damped structures with unmodeled spill-over dynamics is one of the physically motivated examples of this class of systems, where a synthesized negative-imaginary system is connected with a strictly negative-imaginary spill-over dynamics uncertainty via positive feedback as shown in Fig. 1. For stability, this interconnection of systems requires a condition on the dc loop gain to be fulfilled unlike the unconditional stability statement of the small-gain or passivity theorem [16], [17]. This analysis result is captured via a theorem developed in [1] to establish the internal stability of this class of interconnected systems. Under some properness assumptions, the internal stability of such a feedback interconnection is guaranteed if and only if dc loop gain is contractive [1]. This stability analysis theorem is a recently established result parallel to the passivity and small-gain theorems, although the former theorem is a conditional stability result while the latter two theorems are unconditional stability statements. Moreover, this conditional stability result is established for positive feedback interconnection whereas in the passivity theorem, the feedback is essentially negative and in the small-gain theorem, the stability is established irrespective of the positive or negative feedback structure. The positive feedback interconnection with negative-imaginary property is quite often observed in several practical problems, for example, in vibrational control problems, in lightly damped flexible structures [24], [27], in electrical active filter circuits [25], [26] etc.

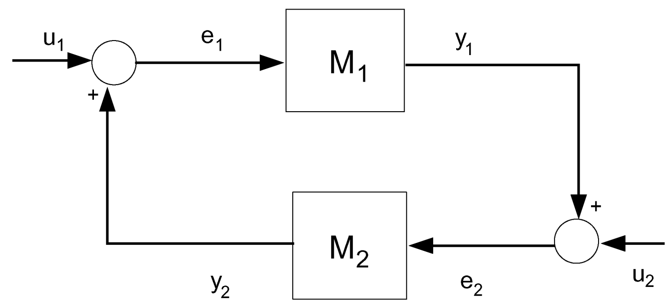

Fig. 1. Interconnection of systems with positive feedback.

In many situations [14], [15], it has been observed that the property of the interconnected systems does not belong to some specific class over the whole frequency range, rather it belongs to some "mixed" classes in different frequency intervals. For example, in low frequency range, the system may lose its small-gain property however the passivity or negative-imaginary properties are retained, and the opposite may happen over high frequencies. In Fig. 2, a Sallen-key low pass filter [25] is cascaded with a gain multiplier circuit and the transfer function from V_i to V_o is the equation shown at the bottom of the page. Considering $R_5 = 1\,\mathrm{K\Omega}, R_4 = 1\,\mathrm{K\Omega}, R_1 = 11\,\mathrm{K\Omega}, R_2 = 110\,\mathrm{K\Omega}, R_3 = 33\,\mathrm{K\Omega}, C_1 = 15\,\mu\mathrm{F}, C_2 = 6.8\,\mu\mathrm{F}$ and $C_3 = 1\,\mu\mathrm{F}$, the Nyquist plot of the above transfer function is shown in Fig. 2. It is apparent that this system shows "mixed" properties, i.e., at low frequencies the gain is high and the negative-imaginary property is retained up to 8.88 rad/s and in high frequencies, the system is with small-gain, however, does not exhibit the negative-imaginary property. In this situation, to establish internal stability for the interconnected systems neither the

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$$\frac{V_0(s)}{V_i(s)} = \frac{\left(1+\frac{R_4}{R_5}\right)\frac{1}{R_1R_2R_3C_1C_2C_3}}{s^3 + \left(\frac{1}{R_1C_1}+\frac{1}{R_2C_1}+\frac{1}{R_2C_2}+\frac{1}{R_3C_2}\right)s^2 + \left(\frac{C_3R_3+R_1C_3+C_1R_1+C_3R_2}{R_1R_2R_3C_1C_2C_3}\right)s + \frac{1}{R_1R_2R_3C_1C_2C_3}}.$$

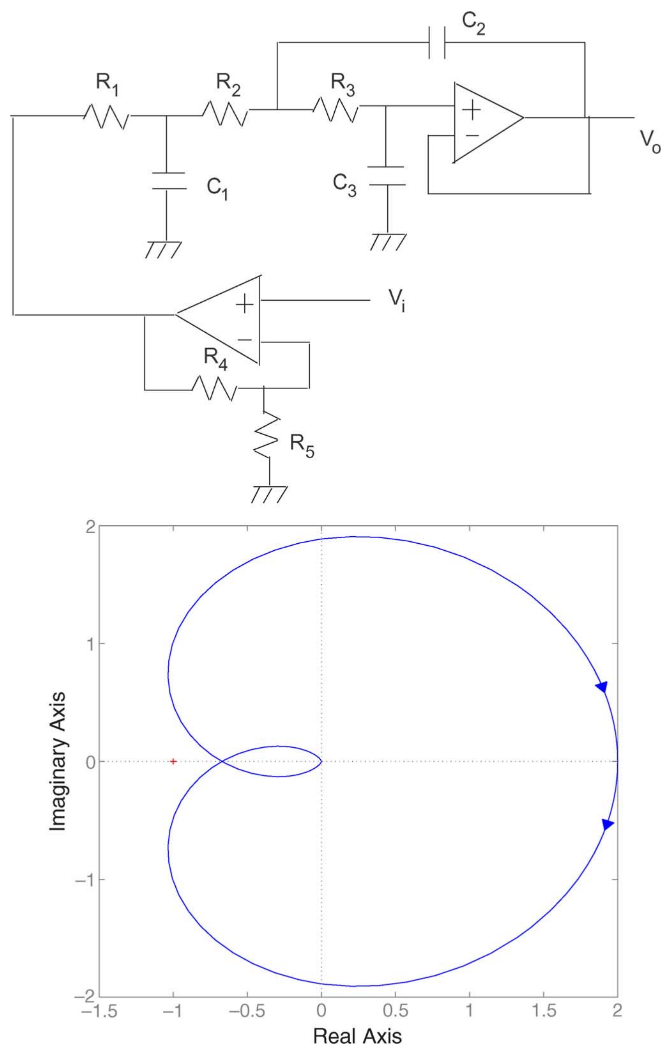

Fig. 2. Sallen-key low-pass filter cascaded with a multiplier circuit and the Nyquist Plot of the transfer function $V_o(s)/V_i(s)$.

small-gain nor passivity, nor negative-imaginary theorem can be applied alone. Keeping this objective in mind, in [14] a novel analytical framework has been established for stability of the interconnected systems by 'combining' the unconditional stability statements of small-gain and passivity theorems (also see [15] for interconnected nonlinear systems). Using the notion of dissipativeness [18]–[21], the interconnection of systems has been shown to be finite-gain stable when the systems are with the "mixed" properties of bounded-real and positive-real systems.

Motivated by this work, an effort has been made in this letter to "merge" the negative-imaginary and small-gain theorems to establish the finite-gain stability of the interconnected systems with positive feedback when the systems are with the "mixed" properties of these two classes. Although the present work builds on [14] and [15], however, it is not a straightforward extension of these previous works. This letter "combines" a conditional stability result with an unconditional stability theorem instead of "mixing" two unconditional stability theorems as it was done in the above two references. The conditional stability result for negative-imaginary system introduces significant technical complexity at dc frequency that makes the technique difficult for combining such two stability theorems. In this letter, to derive stability condition for the interconnected systems the concept of dissipativeness with a new set of frequency dependent triplets has been used. This triplet although it has some resemblance to [14] is significantly different because it provides a new analytical framework for interconnected systems with "mixed" properties where the dc loop gain is a condition on the resulting internal stability and where positive feedback interconnection is considered that normally make the dissipativity based analysis more difficult as addressed in this letter. To tackle these difficulties, the "flag" function involved in the triplet is defined differently from [14] and also, two frequency sets based on the available system properties are defined so that at dc frequency both the systems always exhibit the finite-gain property. Hence, evidently the value of the finite-gain involved in the triplet cannot be selected arbitrarily, however, depends on the maximum singular value of the system matrix evaluated at zero frequency. This proposed analytical framework also captures both the negative-imaginary and small-gain theorems when systems are restricted to specific class and allows multiple shifts from the negative-imaginary class to the small-gain class unlike the work in [14] and [15] which allowed only passivity in low frequency and small-gain in high frequency. Moreover, the proposed triplet establishes stability analysis result for a larger class of systems compared to [1].

Note that IQC theory gives an analytical framework for stability of interconnected systems where one of the systems is possibly nonlinear, time-varying, or uncertain and other is necessarily linear [23]. The unconditional stability result of the IQC theory (which for example reduces to small-gain and passivity) cannot capture the conditional stability result of [1]. Hence, for "mixed" small-gain and negative-imaginary properties, the IQC theory would not be useful. In this letter, dissipative theory is invoked instead to establish stability of the interconnected systems. Although this work considers two LTI systems, the authors believe that dissipativity theory may allow generalization of this work to two nonlinear systems. As such this work paves the way for a full nonlinear extension.

The rest of the letter is organized as follows: In Section II, the notations and some preliminary definitions are given. The dissipativeness of each system is shown in Section III when the systems satisfy "mixed" small-gain and negative-imaginary properties in different frequency intervals. Section IV carries the stability analysis result for a positive feedback interconnection of such systems. The numerical example is presented in Section V and Section VI concludes the letter.

II. Notations and Definitions

Let $\mathbb{R}$ be the set of all real numbers. Let $\mathcal{A}^c$ denote the complement set of $\mathcal{A}$. Also, let $\mathcal{R}$ denote the set of all real rational transfer function matrices and $\mathcal{RH}_\infty^{m\times n}$ be the set of all real rational stable transfer function matrices with m number of rows and n number of columns. Let a transfer function matrix $G \in \mathcal{R}$, then the $\mathcal{L}_2$-adjoint system $G^*(s)$ denotes $G(-s)^T$. Let $\mathcal{L}_2(j\mathbb{R})$ be a Hilbert space under the inner product [16] $\langle g, h\rangle := (1/2\pi)\int_{-\infty}^{\infty} h^*(j\omega)g(j\omega)d\omega$ for $g, h \in \mathcal{L}_2(j\mathbb{R})$. A frequency domain signal $f \in \mathcal{L}_2(j\mathbb{R})$ has norm given by $\|f\| := \langle f, f\rangle^{1/2}$.

Now, in the frequency interval (a, b), the pseudo-inner product is defined by: $\langle x, y\rangle_{(a,b)} := (1/2\pi)\int_a^b y^*(j\omega)x(j\omega)d\omega$ where x, y are such that the integral above is finite (e.g., automatically guaranteed if $x, y \in \mathcal{L}_2(j\mathbb{R})$). Also, $\|f\|_{(a,b)}^2 := \langle f, f\rangle_{(a,b)}$. Now we will define negative-imaginary and finite-gain [22] systems as follows.

Definition 1: Let $M \in \mathcal{RH}_\infty^{n\times n}$ be a strictly proper transfer function matrix. M is said to be a negative-imaginary system in the frequency interval (a, b) with $a \geq 0$ if $\exists \epsilon \geq 0$, $\delta \geq 0$ such that

$$\langle j\omega Mx, x\rangle_{(a,\, b)} \geq \epsilon\|x\|_{(a,\, b)}^2 + \delta\|j\omega Mx\|_{(a,\, b)}^2 \;\forall\, x \in \mathcal{L}_2(j\mathbb{R}). \quad (1)$$

Note that, if $x \in \mathcal{L}_2(j\mathbb{R})$ is Fourier transform of a real-valued signal, then $\langle j\omega Mx, x\rangle_{(-b,-a)} \geq \epsilon\|x\|_{(-b,-a)}^2 + \delta\|j\omega Mx\|_{(-b,-a)}^2 \;\forall\, x \in \mathcal{L}_2(j\mathbb{R})$.

Remark 1: Without loss of generality, the strictly proper condition imposed in Definition 1 can easily be removed by using Lemma 1 of [1] which allows us to get rid of $M(\infty)$ as it must necessarily be symmetric $M(\infty) = M(\infty)^T$.

Definition 2: $M \in \mathcal{RH}_\infty^{m\times n}$ is said to be a finite-gain system bounded by a gain $k \geq 0$ in the frequency interval (a, b) with $a \geq 0$ if k satisfies

$$\|Mx\|_{(a,\,b)} \leq k\|x\|_{(a,\,b)}\ \forall\, x \in \mathcal{L}_2(j\mathbb{R}). \tag{2}$$

Similar to above, if $x \in \mathcal{L}_2(j\mathbb{R})$ is Fourier transform of a real-valued signal, then $\|Mx\|_{(-b,-a)} \leq k\|x\|_{(-b,-a)}\ \forall\, x \in \mathcal{L}_2(j\mathbb{R})$.

Now we define two frequency sets as follows:

(i) Let $\mathcal{I}_i, i = 1, \ldots, p$ be frequency intervals where the system is negative-imaginary (according to Definition 1), and define $\mathcal{I} = (\cup_{i=1}^p \mathcal{I}_i) \cup (\cup_{i=-p}^{-1} \mathcal{I}_i)$ where $\mathcal{I}_{-i}$ indicates the reflected negative region of $\mathcal{I}_i$ (i.e. if $\mathcal{I}_i = (a, b)$ with $a \geq 0$, then $\mathcal{I}_{-i} = (-b, -a)$). Let ϵ and δ as used in Definition 1 satisfy inequality (1) over all intervals in $\mathcal{I}$ (i.e. choose ϵ and δ as the smallest ϵ_i and δ_i obtained for each $\mathcal{I}_i$).

(ii) Let $\mathcal{B}_i, i = 1, \ldots, r$ be frequency intervals where the system is finite-gain bounded by a gain $k > 0$ (as given in Definition 2) where $k > \bar{\sigma}(M(0))$, and $\mathcal{B} = (\cup_{i=1}^r \mathcal{B}_i) \cup \{0\} \cup (\cup_{i=-r}^{-1} \mathcal{B}_i)$. Similar to above, $\mathcal{B}_{-i}$ denotes the reflected negative region of $\mathcal{B}_i$ (i.e. if $\mathcal{B}_i = (a, b)$ with $a \geq 0$, then $\mathcal{B}_{-i} = (-b, -a)$).

It is important to notice that zero frequency and its neighborhood are always included in the finite-gain set $\mathcal{B}$ and the gain k is chosen to be greater than $\bar{\sigma}(M(0))$, i.e. the dc gain of the system. It is also important to realise that zero frequency is always excluded from set $\mathcal{I}$.

Now we define a dissipative system.

Definition 3: [14] Given $M \in \mathcal{RH}_\infty^{m\times n}$ with input and output signals, respectively, $e \in \mathcal{L}_2(j\mathbb{R})$ and $y \in \mathcal{L}_2(j\mathbb{R})$. The system is said to be dissipative in the sense of the frequency dependent triplets $(Q(\omega), S(\omega), R(\omega))$ if

$$\langle y, Q(\omega)y\rangle + 2\,\langle y, S(\omega)e\rangle + \langle e, R(\omega)e\rangle \geq 0 \tag{3}$$

$\forall e \in \mathcal{L}_2(j\mathbb{R})$, where $Q(\omega)^T = Q(\omega) \leq 0$ and $R(\omega)^T = R(\omega)\ \forall\, \omega$. Unlike [18]–[21], the dissipativeness of the system has been defined above with respect to frequency dependent triplets $(Q(\omega), S(\omega), R(\omega))$. In the following section, using this definition of dissipativeness we will show that, a system that satisfies different properties in different frequency intervals i.e. either only negative-imaginary, or both negative-imaginary and finite-gain system bounded by a gain k, or only finite-gain bounded by a gain k property, is also dissipative for a special choice of frequency dependent triplets.

III. Dissipativeness of the System

We say that a system has "mixed" negative-imaginary and finite-gain bounded by a gain k properties in $\mathcal{I} \cup \mathcal{B} = \mathbb{R}$. To capture the "mixed" property of the system in different frequency intervals, we need to define a frequency dependent "flag" function. The value of the "flag" function $\alpha(\omega)$ defined below will indicate which system property is active at a particular frequency. Let the frequency dependent continuous "flag" function $\alpha(\omega)$ be defined as follows:

$$\alpha(\omega) \in \begin{cases} \{1\} & \text{when } \omega \in \mathcal{I} \setminus \mathcal{B}; \\ [0, 1] & \text{when } \omega \in \mathcal{I} \cap \mathcal{B}; \\ \{0\} & \text{when } \mathcal{B} \setminus \mathcal{I}. \end{cases} \tag{4}$$

Here $\mathcal{I} \setminus \mathcal{B}$ denotes $\mathcal{I} \cap \mathcal{B}^c$ and $\mathcal{B} \setminus \mathcal{I}$ denotes $\mathcal{B} \cap \mathcal{I}^c$. Since zero frequency belongs to the frequency region $\mathcal{B} \setminus \mathcal{I}$ only, then $\alpha(0) = 0$ always. Also, $k > \bar{\sigma}(M(0))$ implies that there always a neighborhood of zero frequency where $\alpha(\omega) \in [0, 1]$. To show dissipativeness of a system which has "mixed" negative-imaginary and finite-gain bounded by a k properties in different frequency intervals, we define frequency dependent triplets $(Q(\omega), S(\omega), R(\omega))$ as follows:

$$\begin{aligned} Q(\omega) &= -\,2\alpha(\omega)\delta\omega^2 I - \frac{1}{k}\,(1 - \alpha(\omega))\,I \\ S(\omega) &= -\,j\omega\alpha(\omega)I \\ R(\omega) &= k\,(1 - \alpha(\omega))\,I - 2\epsilon\alpha(\omega)I \end{aligned} \tag{5}$$

where ϵ and δ satisfy inequality (1) over all intervals in $\mathcal{I}$, $k > 0$ also fulfills $k > \bar{\sigma}(M(0))$ and was used in defining $\mathcal{B}$, and I is the identity matrix with compatible dimension. Now we will show that a system that has "mixed" properties in different frequency intervals as indicated by the "flag" function $\alpha(\omega)$ is dissipative in the sense of the triplets $(Q(\omega), S(\omega), R(\omega))$ as defined above.

To show dissipativeness of $M(s)$ using Definition 3, we have to show that $\forall\omega,\ \forall e \in \mathcal{L}_2(j(R)\ \langle y, Q(\omega)y\rangle + 2\langle y, S(\omega)e\rangle + \langle e, R(\omega)e\rangle \geq 0$ where $y = Me$. Now using the definition of the inner product and assuming $\mathcal{I} \cup \mathcal{B} = \mathbb{R}$ (i.e. the system is assumed to have "mixed" negative-imaginary property with parameters $\epsilon \geq 0$ and $\delta \geq 0$ and finite-gain property bounded by a gain $k > 0$ and $k > \bar{\sigma}(M(0))$, we have

$$\begin{aligned} &\langle y, Q(\omega)y\rangle + 2\,\langle y, S(\omega)e\rangle + \langle e, R(\omega)e\rangle \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \big(y(j\omega)^* Q(\omega)^* y(j\omega) + e(j\omega)^* S(\omega)^* y(j\omega) \\ &\qquad + y(j\omega)^* S(\omega)e(j\omega) + e(j\omega)^* R(w)^* e(j\omega)\big)\, d\omega \\ &= \frac{1}{2\pi}\int_{\mathcal{I}\setminus\mathcal{B}} e(j\omega)^* \big(-2\delta\omega^2 M(j\omega)^* M(j\omega) + j\omega M(j\omega) \\ &\qquad - j\omega M(j\omega)^* - 2\epsilon I\big)\, e(j\omega) d\omega \\ &\quad + \frac{1}{2\pi}\int_{\mathcal{I}\cap\mathcal{B}} e(j\omega)^* \\ &\qquad \times \big(\alpha(\omega)[-2\delta\omega^2 M(j\omega)^* M(j\omega) + j\omega M(j\omega) \\ &\qquad - j\omega M(j\omega)^* - 2\epsilon I] + (1 - \alpha(\omega)) \\ &\qquad \times \left[-\frac{1}{k} M(j\omega)^* M(j\omega) + kI\right]\big)\, e(j\omega) d\omega \\ &\quad + \frac{1}{2\pi}\int_{\mathcal{B}\setminus\mathcal{I}} e(j\omega)^* \left(-\frac{1}{k} M(j\omega)^* M(j\omega) + kI\right) \\ &\qquad \times e(j\omega) d\omega. \end{aligned} \tag{6}$$

Since $0 \leq \alpha(\omega) \leq 1$, using the Definitions 1 and 2, it is straightforward to show that (6) is positive semi-definite. Hence, $\langle y, Q(\omega)y\rangle + 2\langle y, S(\omega)e\rangle + \langle e, R(\omega)e\rangle \geq 0\ \forall e \in \mathcal{L}_2(j\mathbb{R}), \forall\omega \in \mathbb{R}$ and $y = Me$. It implies, the system $M(s)$ that satisfies negative-imaginary property in $\mathcal{I}\setminus\mathcal{B}$, both the negative-imaginary and finite-gain bounded by a gain k properties in $\mathcal{I} \cap \mathcal{B}$, and finite-gain bounded by a gain k property in $\mathcal{B} \setminus \mathcal{I}$ is dissipative in the sense of $(Q(\omega), S(\omega), R(\omega))$.

Note that zero frequency is contained in set $\mathcal{B} \setminus \mathcal{I}$ only and does not appear in $\mathcal{I} \setminus \mathcal{B}$ or $\mathcal{I} \cap \mathcal{B}$.

IV. Interconnection of Two Dissipative Systems With Positive Feedback

Now we examine stability of two dissipative systems in the sense of their frequency dependent triplets interconnected in positive feedback. Let $M_1(s) \in \mathcal{RH}_\infty^{n\times n}$ and $M_2(s) \in \mathcal{RH}_\infty^{n\times n}$ be two strictly proper systems. Each of them has mixed negative-imaginary and finite-gain properties in different frequency intervals as mentioned in previous section and hence, they are individually dissipative too. Let k_1 and k_2, respectively, be the bounds defined for $M_1(s)$ and $M_2(s)$, and also let ϵ_1, δ_1 and ϵ_2, δ_2 be the negative-imaginary parameter pairs defined for $M_1(s)$ and $M_2(s)$ respectively. To capture the fact that the negative-imaginary and finite-gain properties of both systems have to occur at corresponding frequencies, we use the same frequency dependent "flag" function $\alpha(\omega)$. Notwithstanding a common $\alpha(\omega)$, since each system has different parameters $k_i, \epsilon_i, \delta_i$, $i = 1, 2$ we have to define different frequency dependent triplets $(Q_i(\omega), S_i(\omega), R_i(\omega)), i = 1, 2$. From Fig. 1, we have $e_1 = u_1 + y_2$ and $e_2 = u_2 + y_1$. Since the systems are individually dissipative in the sense of $(Q_i(\omega), S_i(\omega), R_i(\omega)), i = 1, 2$, we have

$$\begin{aligned}
&\langle y_1(j\omega), Q_1(\omega)y_1(j\omega)\rangle + 2\langle y_1(j\omega), S_1(\omega)e_1(j\omega)\rangle \\
&\quad + \langle e_1(j\omega), R_1(w)e_1(j\omega)\rangle + \langle y_2(j\omega), Q_2(\omega)y_2(j\omega)\rangle \\
&\quad + 2\langle y_2(j\omega), S_2(\omega)e_2(j\omega)\rangle + \langle e_2(j\omega), R_2(w)e_2(j\omega)\rangle \geq 0 \\
\Leftrightarrow\ & \frac{1}{2\pi}\int_{-\infty}^{\infty}\left([y_1^*\ y_2^*]\begin{bmatrix} Q_1^* + R_2^* & S_1 + S_2^* \\ S_1^* + S_2 & R_1^* + Q_2^* \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right. \\
&\quad + [u_1^*\ u_2^*]\begin{bmatrix} S_1^* & R_1^* \\ R_2^* & S_2^* \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\
&\quad + [y_1^*\ y_2^*]\begin{bmatrix} S_1 & R_2^* \\ R_1^* & S_2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\
&\quad \left. + [u_1^*\ u_2^*]\begin{bmatrix} R_1^* & 0 \\ 0 & R_2^* \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) d\omega \geq 0 \\
\Leftrightarrow\ & \langle y, \bar{Q}y\rangle + 2\langle u, \bar{S}y\rangle + \langle u, \bar{R}u\rangle \geq 0
\end{aligned}$$

where

$$\bar{Q} = \begin{bmatrix} Q_1^* + R_2^* & S_1 + S_2^* \\ S_1^* + S_2 & R_1^* + Q_2^* \end{bmatrix}, \ \bar{S} = \begin{bmatrix} S_1^* & R_1^* \\ R_2^* & S_2^* \end{bmatrix}$$
$$\text{and} \quad \bar{R} = \begin{bmatrix} R_1^* & 0 \\ 0 & R_2^* \end{bmatrix}. \tag{7}$$

For brevity, the frequency dependence is not shown above. Now $\forall\omega$, $\bar{R} = \bar{R}^T$ and replacing (5) in $\bar{Q}$, we have (8) as shown at the bottom of the page.

Now the interconnected system is dissipative in the sense of $(\bar{Q}(\omega), \bar{S}(\omega), \bar{R}(\omega))$ if $\bar{Q}(\omega) = \bar{Q}(\omega)^T \leq 0$ and $\bar{R}(\omega) = \bar{R}(\omega)^T\ \forall\omega$. Furthermore, the interconnected system is dissipative as well as finite-gain stable if $\bar{Q}(\omega) < 0$ [14], [18]–[21]. From above, we have $\bar{Q}(\omega) = \bar{Q}(\omega)^T$ and $\bar{R}(\omega) = \bar{R}(\omega)^T\ \forall\omega$. In the following theorem, we will explore the conditions for which $\bar{Q}(\omega) < 0$ to show dissipativity as well as finite-gain stability (i.e. internal stability) of the interconnected system with positive feedback.

Theorem 1: Let $M_1(s) \in RH_\infty^{n\times n}$ and $M_2(s) \in RH_\infty^{n\times n}$ be two strictly proper, causal and linear time invariant systems interconnected via a positive feedback as shown in Fig. 1. Let $k_1 > \bar{\sigma}(M_1(0))$ and $k_2 > \bar{\sigma}(M_2(0))$. For $j = 1, 2$, let $\mathcal{B}_i^j, i = 1, \ldots, r$ be continuous positive frequency intervals where the system $M_j(s)$ is finite-gain stable bounded by a gain k_j and define $\mathcal{B}^j = (\cup_{i=1}^r \mathcal{B}_i^j) \cup \{0\} \cup (\cup_{i=-r}^{-1}\mathcal{B}_i^j)$ with $\mathcal{B}_{-i}^j$ denoting the reflected negative region of $\mathcal{B}_i^j$. For $j = 1, 2$, let $\mathcal{I}_i^j, i = 1, \ldots, p$ be continuous positive frequency intervals where the system $M_j(s)$ is negative-imaginary with parameters $\epsilon_j \geq 0$ and $\delta_j \geq 0$ and define $\mathcal{I}^j = (\cup_{i=1}^p \mathcal{I}_i^j) \cup (\cup_{i=-p}^{-1}\mathcal{I}_i^j)$ with $\mathcal{I}_{-i}^j$ denoting the reflected negative region of $\mathcal{I}_i^j$. Suppose $\mathcal{B}^j \cup \mathcal{I}^j = \mathbb{R}$ for both $j = 1$ and $j = 2$. Suppose also that there exists a single continuous frequency dependent function $\alpha(\omega)$ to satisfy

$$\alpha(\omega) \in \begin{cases} \{1\} & \text{when } \omega \in \mathcal{I}^j \setminus \mathcal{B}^j \\ [0,1] & \text{when } \omega \in \mathcal{I}^j \cap \mathcal{B}^j \\ \{0\} & \text{when } \mathcal{B}^j \setminus \mathcal{I}^j \end{cases}$$

for both $j = 1$ and $j = 2$. The interconnection in Fig. 1 with positive feedback is finite-gain stable if $k_1k_2 < 1$ and whenever $\mathcal{B}^1 \cap \mathcal{B}^2 \neq \mathbb{R}$, also the following condition holds: (δ_1 or $\epsilon_2 > 0$) and (ϵ_1 or $\delta_2 > 0$).

Proof: First, note that the suppositions of the theorem ensure it is possible to define frequency dependent triplets $(Q_i(\omega), S_i(\omega), R_i(\omega))$ $i = 1, 2$ as follows: $Q_i(\omega) = -2\alpha(\omega)\delta_i\omega^2 I - (1/k_i)(1 - \alpha(\omega))I$, $S_i(\omega) = -j\omega\alpha(\omega)I$ and $R_i(\omega) = k_i(1 - \alpha(\omega))I - 2\epsilon_i\alpha(\omega)I$; and that each separate system $M_i(s)$ $i = 1, 2$ is individually dissipative with respect to $(Q_i(\omega), S_i(\omega), R_i(\omega)), i = 1, 2$. Then, it is possible to derive frequency dependent triplets $(\bar{Q}(\omega), \bar{S}(\omega), \bar{R}(\omega))$ as in (7) for interconnected system in Fig. 1 with positive feedback where $\bar{Q}(\omega) = \bar{Q}(\omega)^T$ is given by (10) and $\bar{R}(\omega) = \bar{R}(\omega)^T\ \forall\omega$. Now there are three possibilities:

(i) When $\alpha(\omega) = 1$, we get $\bar{Q}(\omega) < 0$ if the following condition holds: (δ_1 or $\epsilon_2 > 0$) and (ϵ_1 or $\delta_2 > 0$).

(ii) When $0 < \alpha(\omega) < 1$, we get $\bar{Q}(\omega) < 0$ if $k_1k_2 < 1$ and the following condition holds: (δ_1 or $\epsilon_2 > 0$) and (ϵ_1 or $\delta_2 > 0$).

(iii) When $\alpha(\omega) = 0$, we get $\bar{Q}(\omega) < 0$ if $k_1k_2 < 1$.

Since we need $\bar{Q}(\omega) < 0\ \forall\omega \in \mathbb{R}$ and since $\alpha(0) = 0$, it follows that the condition $k_2k_2 < 1$ is always required for internal stability of the positive feedback loop. Once this is enforced, we have only two remaining situations: $\alpha(\omega) = 1$ and $\alpha(\omega) \in (0, 1)$. Both these correspond to a situation where there is a negative imaginary frequency region and hence the finite-gain frequency regions $\mathcal{B}^i \neq \mathbb{R}\ i = 1, 2$. ■

Remark 2: To capture "mixed" properties of the interconnected systems, the "flag" function $\alpha(\omega)$ is introduced. The value of this function at any frequency is set to be the same for both of the interconnected systems, i.e., there must exist frequency intervals where both of the systems show at least one common system property.

Remark 3: For finite-gain stability, the condition (δ_1 or $\epsilon_2 > 0$) and (ϵ_1 or $\delta_2 > 0$) implies any one of the following combinations holds:

- $M_1(s)$ is input negative-imaginary and output strictly negative-imaginary system and $M_2(s)$ is input negative-imaginary and output strictly negative-imaginary system.
- $M_1(s)$ is input strictly negative-imaginary and output strictly negative-imaginary system and $M_2(s)$ is input negative-imaginary and output negative-imaginary system.
- $M_1(s)$ is input negative-imaginary and output negative-imaginary system and $M_2(s)$ is input strictly negative-imaginary and output strictly negative-imaginary system.
- $M_1(s)$ is input strictly negative-imaginary and output negative-imaginary system and $M_2(s)$ is input strictly negative-imaginary and output negative-imaginary system.

The above combinations of systems connected via positive feedback is similar to passivity and yields a more general analytical framework for finite-gain stability compared to [1] where only the second and third combinations were posed for stability.

Corollary 2: Let the suppositions of Theorem 1 holds but furthermore assume $p = 1$ and $\mathcal{I}_1^j = (0, \infty)$ where $j = 1, 2$. The interconnected system with positive feedback is finite-gain stable whenever

$$\bar{Q} = \begin{bmatrix} \left[-2\epsilon_2\alpha(\omega) - 2\alpha(\omega)\delta_1\omega^2 + \left(k_2 - \frac{1}{k_1}\right)(1 - \alpha(\omega))\right] I & 0 \\ 0 & \left[-2\epsilon_1\alpha(\omega) - 2\alpha(\omega)\delta_2\omega^2 + \left(k_1 - \frac{1}{k_2}\right)(1 - \alpha(\omega))\right] I \end{bmatrix} \tag{8}$$

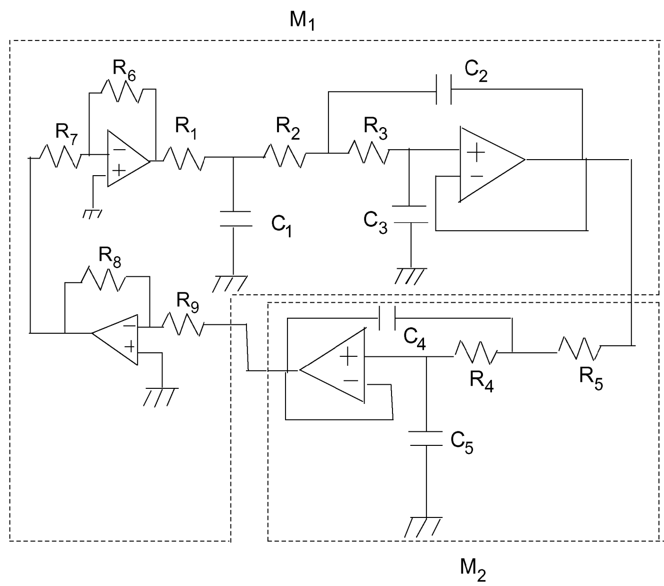

Fig. 3. Interconnected systems.

$k_1k_2 < 1$ and the following condition holds: (δ_1 or $\epsilon_2 > 0$) and (ϵ_1 or $\delta_2 > 0$).

Proof: Trivial reduction of Theorem 1 to the case when systems are negative-imaginary. ■

Remark 4: Corollary 2 captures the internal stability result of analysis (Theorem 5 in [1]) as follows. Let $M_1(s)$ and $M_2(s)$ are, respectively, the negative-imaginary and strictly negative-imaginary system (see case 3 of Remark 2). The $\mathcal{I}^j = (0, \infty)$ for both systems where $\epsilon_1, \delta_1 \geq 0$ and $\epsilon_2, \delta_2 > 0$. By assumption, we know $M_1(s)$ and $M_2(s)$ are strictly proper which implies $M_1(\infty) = M_2(\infty) = 0$. Then using Lemma 2 of [1], it can be shown that $\bar{\lambda}(M_1(0)) = \bar{\sigma}(M_1(0))$ as $M_1(0) \geq 0$ and $\bar{\lambda}(M_2(0)) = \bar{\sigma}(M_2(0))$ as $M_2(0) \geq 0$. Hence $\bar{\lambda}(M_1(0))\bar{\lambda}(M_2(0)) \leq k_1k_2$ using the suppositions, and thus $k_1k_2 < 1$ implies $\bar{\lambda}(M_1(0))\bar{\lambda}(M_2(0)) < 1$. This then satisfies the internal stability condition of the stability analysis result of Theorem 5 in [1].

Corollary 3: Let the suppositions of Theorem 1 hold but furthermore assume $\mathcal{B}^j = \mathbb{R}$, $j = 1, 2$. Then the interconnected system in Fig. 1 with positive feedback is finite-gain stable if $k_1k_2 < 1$.

Proof: Trivial reduction of Theorem 1 to the case when $M_1(s) \in \mathcal{RH}_\infty$ has $\|M_1\|_\infty \leq k_1$ and $M_2(s) \in \mathcal{RH}_\infty$ has $\|M_2\|_\infty \leq k_2$ (i.e small-gain theorem). ■

Remark 5: Corollary 3 captures the result of small-gain theorem.

One may wonder whether IQC theory [23] captures the result of this work or not? The short answer is "No, it does not" for the following reasons:

a) The proposed framework can capture the conditional stability result of [1], the condition being a restriction of the largest eigenvalue of the dc loop-gain, whereas IQC theory cannot. IQC theory deals with the full frequency range $\omega \in \mathbb{R}$ whereas strictly negative-imaginary system $M_2(s)$ satisfies the frequency-domain inequality $j[M_2(j\omega) - M_2(j\omega)^*] > 0$ only on an open frequency interval $\omega \in (0, \infty)$. This strict inequality cannot be satisfied at $\omega = 0$ because via Lemma 2 of [1], we know that $M_2(0) = M_2(0)^*$. This fact causes the main theorem in [23], which underpins IQC theory, to be inapplicable by violation of its supposition.

b) The $\Pi(j\omega)$ matrix used in IQC theory needs to be the same for both systems making up the feedback interconnection. This is different from the results given in this letter where we can allow two different triplets $(Q(\omega), S(\omega), R(\omega))$ by allowing different k, ϵ and δ for each individual system. It is indeed correct that the flag function $\alpha(\omega)$ needs to be the same so that both systems share complementary properties, but the constants k, ϵ and δ can be different. This then results in our work being able to capture a larger class of interconnected system (see Remark 3) compared to [1].

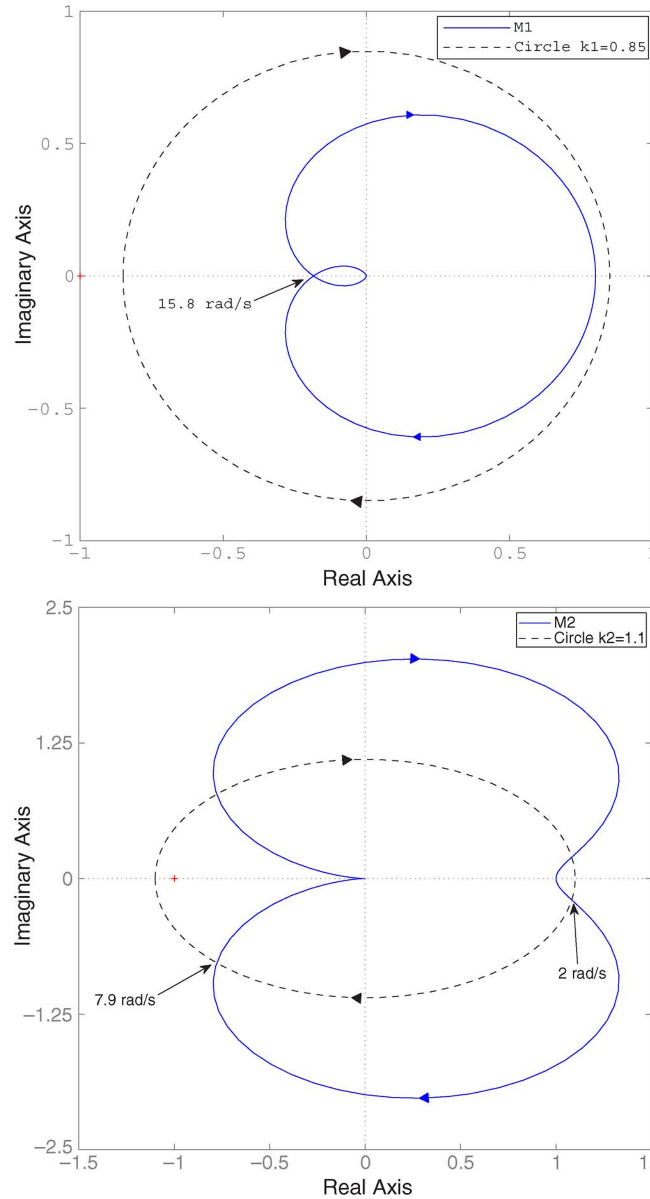

Fig. 4. Nyquist plots of M_1 and M_2.

V. Numerical Example

In Fig. 3, two systems M_1 and M_2 have been shown which are interconnected via positive feedback. M_1 is a third-order Sallen-key low pass filter cascaded with a gain multiplier circuit with gain R_6R_8/R_7R_9, whereas M_2 is a second-order low pass Sallen-key filter [25], [26]. Considering $R_1 = 10\ \mathrm{K\Omega}$, $R_2 = 110\ \mathrm{K\Omega}$, $R_3 = 27\ \mathrm{K\Omega}$, $C_1 = 15\ \mu\mathrm{F}$, $C_2 = 3.3\ \mu\mathrm{F}$, $C_3 = 0.68\ \mu\mathrm{F}$ and $R_6R_8/R_7R_9 = 0.8$, the Nyquist plot of the system M_1 is shown in Fig. 4. For system M_2, the values are chosen as follows: $R_5 = 13\ \mathrm{K\Omega}$, $R_4 = 4.3\ \mathrm{K\Omega}$, $C_4 = 100\ \mu\mathrm{F}$ and $C_5 = 4.7\ \mu\mathrm{F}$. The Nyquist plot of the system M_2 is shown in Fig. 4. For these chosen values, $\|M_1\|_\infty = 0.8$ and $\|M_2\|_\infty = 2.0593$. None of the small-gain, passivity, or negative-imaginary theories alone can provide any guarantee

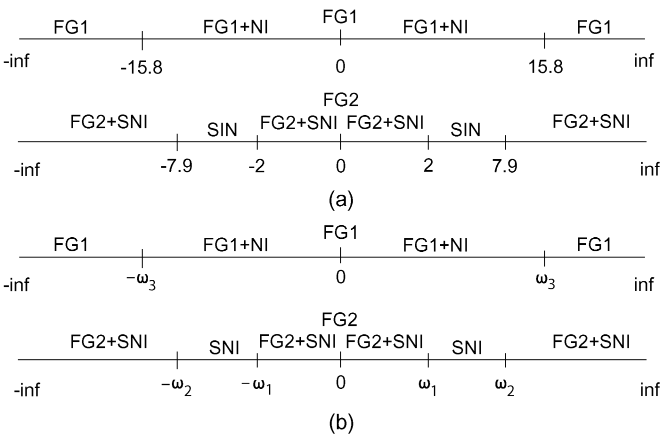

Fig. 5. Frequency intervals (NI: negative-imaginary, SNI: strictly negative-imaginary, FG1: finite-gain with gain ≤ 0.85, FG2: finite-gain with gain ≤ 1.1).

of stability. We now analyze stability of this interconnected systems using the proposed result of this letter.

Selecting $k_1 = 0.85(> \bar{\sigma}(M_1(0)) = 0.8)$ and $k_2 = 1.1(> \bar{\sigma}(M_2(0)) = 1)$, respectively, for M_1 and M_2, we get mixed properties as shown in Fig. 5(a). One could also select different bounds so that $k_1 > \bar{\sigma}(M_1(0))$ and $k_2 > \bar{\sigma}(M_2(0))$ and the condition $k_1 k_2 < 1$ is satisfied. If there is no bound that satisfies these conditions, then the interconnection of systems with positive feedback is not finite-gain stable. One of such instability conditions appears when $(R_6 R_8 / R_7 R_9) \geq 1$.

The values of $\alpha(\omega)$ for interconnected systems are as follows: $\alpha = 0$ when $\omega = 0$ or $\omega \in (-\infty, -15.8]$ or $[15.8, \infty)$; $0 \leq \alpha \leq 1$ when $\omega \in (-15.8, -7.9]$ or $[7.9, -15.8)$ or $[-2, 0)$ or $(0, 2]$; and $\alpha = 1$ when $\omega \in (-7.9, -2)$ or $(2, 7.9)$.

The interconnected systems is dissipative with respect to $(\bar{Q}(\omega), \bar{S}(\omega), \bar{R}(\omega))$ where $\alpha(\omega)$ is defined as above. Since $k_1 k_2 < 1$ and one of the systems, i.e. $M_2(s)$ is strictly negative-imaginary i.e. ϵ_2 and δ_2 are both >0, internal stability follows via Theorem 1 which can be confirmed via direct computation of the poles of $\left[\begin{smallmatrix} M_1 \\ I \end{smallmatrix}\right](I - M_2 M_1)^{-1}[-M_2 \ I]$ for the positive feedback interconnection.

While Fig. 5(a) gives the exact frequencies for this example, a designer can use Fig. 5(b) instead with any $\omega_1 \leq 2$ and $7.9 \leq \omega_2 < \omega_3 \leq 15.8$. So exact estimation of frequencies is not needed.

VI. Conclusion

In this letter, using the notion of dissipativity the finite-gain stability of interconnected systems is guaranteed when two causal, stable and linear time invariant strictly proper systems are connected via a positive feedback and each of them has "mixed" properties of the negative-imaginary and finite-gain system. To show dissipativity, frequency dependent triplets have been exploited which are function of a continuous flag variable that captures the correct balance of "mixed" properties of the systems. This letter combines the conditional stability result of the negative-imaginary system with the unconditional stability result of the small-gain theorem and it results in a general finite-gain stability framework for analysis. From this framework, individually the small-gain theorem and the stability analysis result of negative-imaginary systems can be obtained when systems are restricted to the specific corresponding classes, i.e., respectively, to small-gain and negative-imaginary systems. A numerical example is given to show effectiveness of the proposed framework.

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