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Towards Controller Synthesis for Systems with Negative Imaginary Frequency Response

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Abstract—This technical note provides a reformulation of closed-loop systems that have negative imaginary frequency response into closed-loop systems that have bounded gain, so that theory and results from $\mathcal{H}_\infty$ control can be borrowed to enable controller synthesis for the former class of systems. Systems with negative imaginary frequency response (negative-imaginary systems for short) arise for example in structures with collocated position sensors and force actuators, and finding a systematic controller synthesis treatment for such systems has important applications in, for example, lightly damped large space structure problems. The key result in this technical note assists by reformulating such systems into a bounded-real framework. This technical note also addresses a controller synthesis problem in an $\mathcal{H}_\infty$ optimal control framework for a generalized plant with an invariant zero at the origin in its (1,2) element which is due to the reformulation of the closed-loop system from a negative-imaginary system to a bounded-real framework. An example demonstrates the feasibility of the reformulation given herein.

Index Terms—Bounded real, $\mathcal{H}_\infty$ control, lightly damped systems, negative imaginary frequency response, passivity, positive real, small-gain theorem.

NOTATION AND SYMBOLS

Let $\mathcal{RH}_\infty$ denote the set of real-rational stable transfer function matrices and $\mathcal{RH}_\infty^{n \times m}$ denote the systems in $\mathcal{RH}_\infty$ that have m columns and n rows. Let $\mathbb{R}$ and $\mathbb{C}$ denote fields of real and complex numbers respectively, $\mathbb{C}_-$ and $\bar{\mathbb{C}}_-$ denote the open and closed left-half planes respectively. Let A^* denote the complex conjugate transpose of matrix A. Let $\bar{\sigma}(A)$ denote the largest singular value of matrix A. Let $\lambda_i(A)$ denote the i-th eigenvalue of a square complex matrix A. Let $\|P\|_\infty$ denote the $\mathcal{H}_\infty$-norm of $P \in \mathcal{RH}_\infty$. Let $F_\ell(G, K)$ denote the lower Linear Fractional Transformation (LFT) of transfer function matrices G and K [1]. Let $G \star K$ denote the Redheffer Star-Product of transfer function matrices G and K with respect to some appropriate partitioning [1]. Let $\mathrm{diag}(A, B)$ be shorthand for $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. Let $\langle G, K \rangle$ denote the feedback interconnection shown in Fig. 1 and correspondingly let $T(G, K)$ denote the transfer function from $\begin{pmatrix} w \\ \binom{u_1}{u_2} \end{pmatrix}$ to $\begin{pmatrix} z \\ \binom{v_1}{v_2} \end{pmatrix}$. We say $\langle G, K \rangle$ is internally stable when $T(G, K) \in \mathcal{RH}_\infty$.

I. INTRODUCTION

It is a well-known fact that mechanical systems modelled with force actuators and collocated velocity sensors give rise to positive-real (or passive) transfer functions [2], [3]. It is, however, a less known fact that the same systems modelled with force actuators and collocated position sensors, as opposed to velocity sensors, (i.e. when the variable to be controlled is position instead of velocity) give rise to systems with negative imaginary frequency response (negative-imaginary systems for short) [4] in the LTI case (or systems with counter-clockwise input-output dynamics [5], [6] in a nonlinear time-varying setting). This is a consequence of Newton's second law of motion that typically produces a transfer function of the form $K/(s^2 + 2\zeta\omega_n s + \omega_n^2)$ when collocated position measurements are used in conjunction with force actuation [7]. Broadly speaking, negative-imaginary systems are stable systems whose Nyquist plot has a phase lag between 0 and $-\pi$ for all frequencies, and hence have a Nyquist plot below the Real axis for all frequencies. Negative-imaginary systems are also closely connected to nonlinear/time-varying systems with counter-clockwise input-output dynamics [5], [6].

Studying negative-imaginary systems has important engineering applications in, for example, large space structures and multi-link robots. When these structures are lightly damped (i.e. flexible) with collocated position sensors and force actuators, they give rise to transfer functions which consist of the sum of a large number of second order terms with negative-imaginary properties. Normally, only a small number of modes are considered for control design so that the truncated modes will give rise to spill-over dynamics that can destabilize the systems, especially at high frequencies. The SISO LTI case of such systems have been studied extensively in positive position feedback literature, where much of the analysis and synthesis is based on graphical Nyquist approaches [8], [9]. Recently, non-convex optimization methods to damp structural vibrations of lightly damped structures with collocated actuators and sensors have also been proposed [10].

In [4], [11], an *analysis* result was proposed for MIMO LTI negative-imaginary systems via a theorem and a corollary which capture the graphical design methods in the positive position feedback literature. It shows that a positive feedback interconnection of two systems, where one system is negative-imaginary and the other system is strictly negative-imaginary, is internally stable if and only if the DC loop gain is less than unity. This theory is well suited to address the control of lightly damped structures with collocated force actuators and position sensors that cannot be tackled adequately by means of the bounded-real or positive-real theories. Since the unmodeled dynamics is highly resonant, small-gain based controller synthesis methods would typically be conservative. On the other hand, as the relative degree of the unmodeled dynamics is more than unity, passivity based synthesis methods are not applicable to tackle this physical problem. To this end, the analysis result of [4], [11] facilitates and is underpinning to controller synthesis for this class of systems. In this regard, this technical note will take a first step towards providing a systematic framework for controller synthesis for MIMO negative-imaginary systems based on the newly developed analysis result in [4], [11]. More specifically, for uncertain systems where the perturbation belongs to the class of strictly negative-imaginary systems (for example the above mentioned spill-over dynamics), it is natural to seek to design a stabilizing controller such that the closed-loop system satisfies the negative-imaginary condition. Via results in [4], it is then possible to quantify the largest family of perturbations that have strictly negative-imaginary properties in terms of the reciprocal of the DC gain of the nominal system. This technical note will reformulate the problem of finding such a controller into an equivalent problem of finding an internally stabilizing controller for a transformed system such that the closed-loop is bounded-real.
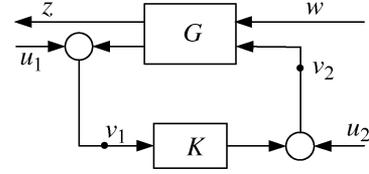

Fig. 1. LFT Interconnection.

Although the reformulation technique posed in this technical note is based on a well-established bilinear transformation technique, which is normally used to reformulate the closed-loop system from the positive-real system to bounded-real framework, the proposed work is not just a straightforward extension of previous results. We have to be mindful of significant difficulties that are introduced at zero frequency when a negative-imaginary system is transformed into a positive-real system by differentiating the output with respect to time. In this regard, one may argue that the reformulation from a negative-imaginary system to a bounded-real system via an intermediate positive-real system may provide an easy solution to the proposed problem by multiplying the negative-imaginary transfer function matrix with sI, and indeed, this idea is explored in the present work. However, the transformation from a negative-imaginary to a positive-real system results in significant technical difficulties relating to the preservation of stability due to the pole/zero cancellation at the origin resulting from the transformation. Furthermore, this transformation results in a strict negative-imaginary system being reformulated as a non-strict bounded-real system and hence, $\mathcal{H}_\infty$ optimal theory is required for controller synthesis instead of suboptimal $\mathcal{H}_\infty$ theory. In suboptimal $\mathcal{H}_\infty$ theory, the standard assumptions on the invariant zeros on the $j\omega$-axis can be lifted (see for example [12]–[14], and references therein). However, the transformed system violates the assumption of 'no invariant zeros on the imaginary axis', which is a necessary assumption for optimal Riccati equation based $\mathcal{H}_\infty$ controller synthesis methods. Indeed, the reformulated (1,2) subsystem block of the generalized plant contains an invariant zero at the origin. To the best of our knowledge, there is no optimal $\mathcal{H}_\infty$ controller synthesis theory available in literature that can give a straightforward solution to this synthesis problem when such an invariant zero at the origin is present. In this context, [15], [16] may give some directions and this technical note flags an open research problem in $\mathcal{H}_\infty$ optimal control synthesis for a generalized plant with an invariant zero on the imaginary axis as this leads to a synthesis technique for negative-imaginary systems.

II. PRELIMINARIES

First we define some sets for compactness of notation.

Definition 1: Let the set of square stable negative-imaginary transfer functions [4] be defined by

$$\mathcal{I} := \left\{ X \in \mathcal{RH}_\infty^{n \times n} : j\left[X(j\omega) - X(j\omega)^*\right] \geq 0 \; \forall \omega \in (0, \infty) \right\}$$

and the set of square stable strictly negative-imaginary transfer functions be defined as

$$\mathcal{I}_s := \left\{ X \in \mathcal{RH}_\infty^{n \times n} : j\left[X(j\omega) - X(j\omega)^*\right] > 0 \; \forall \omega \in (0, \infty) \right\} \subset \mathcal{I}.$$

Definition 2: Let the set of square stable positive real transfer functions be defined by

$$\mathcal{P} := \left\{ X \in \mathcal{RH}_\infty^{n \times n} : \left[X(j\omega) + X(j\omega)^*\right] \geq 0 \; \forall \omega \in \mathbb{R} \right\}.$$

It is easy to see that $X \in \mathcal{I}$ implies $s[X(s) - X(\infty)] \in \mathcal{P}$ [4].

Definition 3: Let the set of square stable contractive transfer functions whose Nyquist plot does not pass through $-1 + j0$ point be defined by

$$\mathcal{B} := \{ X \in \mathcal{RH}_{\infty}^{n \times n} : \|X\|_{\infty} \leq 1,$$
$$\det (I + X(j\omega)) \neq 0 \; \forall \omega \in \mathbb{R} \cup \{\infty\} \}.$$

The next three simple technical lemmas are given here to streamline presentation of the proof of the main result in the next section. The first two lemmas state that if $X \in \mathcal{P}$ or $X \in \mathcal{B}$, then $(I + X)^{-1}$ is stable. These standard properties will be used to make a connection amongst different parts of the proof of the main result.

Lemma 1: Given $X \in \mathcal{P}$, then $(I + X)^{-1} \in \mathcal{RH}_{\infty}$.

Proof: This lemma is trivially established via simple application of the passivity theorem [17]. ∎

Lemma 2: Given $Y \in \mathcal{B}$. Then $(I + Y)^{-1} \in \mathcal{RH}_{\infty}$.

Proof: First note that $(I + \alpha Y)^{-1} \in \mathcal{RH}_{\infty} \; \forall \alpha \in (0, 1)$ and also $Y \in \mathcal{B}$ gives $\det(I + Y(j\omega)) \neq 0 \; \forall \omega \in \mathbb{R} \cup \{\infty\}$. As α increases continuously to unity, the transmission zeros of $(I + \alpha Y(s))$ vary continuously and are in $\mathbb{C}_{-}$ for all $\alpha \in (0, 1)$, and they do not intersect the $j\omega$-axis at $\alpha = 1$. Therefore, at $\alpha = 1$, they must remain in $\mathbb{C}_{-}$. Thus $(I + Y)^{-1} \in \mathcal{RH}_{\infty}$. ∎

The third technical lemma gives a simple necessary and sufficient condition for input-output stability of a particular Redheffer Star-Product. It will be used in the next section to make a connection between systems in $\mathcal{P}$ and $\mathcal{B}$.

Lemma 3: Given $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \in \mathcal{RH}_{\infty}$. Then

$$\begin{pmatrix} I & -2I \\ I & -I \end{pmatrix} \star T \in \mathcal{RH}_{\infty} \quad \Leftrightarrow \quad (I + T_{11})^{-1} \in \mathcal{RH}_{\infty}. \quad (1)$$

Proof: This equivalence can be seen directly from an expansion [1] of the Redheffer Star-Product of $\begin{pmatrix} I & -2I \\ I & -I \end{pmatrix} \star T$. ∎

III. MAIN RESULT

The main result of this technical note is given in Theorem 4 below. The theorem broadly states that a controller internally stabilizes a generalized plant Σ and makes the input-output map satisfy a negative imaginary frequency response property if and only if the same controller internally stabilizes a different generalized plant G (constructed from Σ) and makes the input-output map contractive.

Theorem 4: Given a controller $K \in \mathcal{R}^{q \times p}$ and a generalized plant

$$\Sigma = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array} \right] \quad (2)$$

with $A \in \mathbb{R}^{n \times n}$, $D_{11} = D_{11}^* \in \mathbb{R}^{m \times m}$, $D_{22} = 0 \in \mathbb{R}^{p \times q}$, $D_{12} = 0$, (A, B_2) stabilizable and (C_2, A) detectable. Let $U = I + C_1 B_1$ and $V = I + B_1 C_1$. Also, suppose U and V are invertible. Then, $\langle \Sigma, K \rangle$ is internally stable and $j[F_\ell(\Sigma, K)(j\omega) - F_\ell(\Sigma, K)(j\omega)^*] \geq 0 \; \forall \omega \in (0, \infty)$ if and only if $\langle G, K \rangle$ is internally stable, $\|F_\ell(G, K)\|_{\infty} \leq 1$ and $\det(I + F_\ell(G, K)(j\omega)) \neq 0 \; \forall \omega \in \mathbb{R} \cup \{\infty\}$, where

$$G = \left[\begin{array}{c|cc} V^{-1}A & B_1 U^{-1} & V^{-1}B_2 \\ -2U^{-1}C_1A & (I - C_1 B_1)U^{-1} & -2U^{-1}C_1 B_2 \\ \hline C_2 - D_{21}U^{-1}C_1A & D_{21}U^{-1} & -D_{21}U^{-1}C_1 B_2 \end{array} \right]. \quad (3)$$

Proof: We will prove the result via a sequence of equivalent reformulations:

(a) $\langle \Sigma, K \rangle$ is internally stable and $j[F_\ell(\Sigma, K)(j\omega) - F_\ell(\Sigma, K)(j\omega)^*] \geq 0 \; \forall \omega \in (0, \infty)$.

(b) $\langle \bar{\Sigma}, K \rangle$ is internally stable and $j[F_\ell(\bar{\Sigma}, K)(j\omega) - F_\ell(\bar{\Sigma}, K)(j\omega)^*] \geq 0 \; \forall \omega \in (0, \infty)$, where

$$\bar{\Sigma} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & 0 \\ C_2 & D_{21} & 0 \end{array} \right]. \quad (4)$$

[The equivalence (a) $\Leftrightarrow$ (b) follows on noting that $F_\ell(\Sigma, K)(\infty) = D_{11} = D_{11}^*$.]

(c) $\langle \hat{\Sigma}, K \rangle$ is internally stable and $[F_\ell(\hat{\Sigma}, K)(j\omega) + F_\ell(\hat{\Sigma}, K)(j\omega)^*] \geq 0 \; \forall \omega \in \mathbb{R}$, where

$$\hat{\Sigma} = \begin{pmatrix} sI & 0 \\ 0 & I \end{pmatrix} \bar{\Sigma} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 A & C_1 B_1 & C_1 B_2 \\ C_2 & D_{21} & 0 \end{array} \right]. \quad (5)$$

[The internal stability parts can be seen to be equivalent on noting that K is the same, $\hat{\Sigma}$ and $\bar{\Sigma}$ are both stabilizable and detectable, and that

$$\bar{\Sigma}_{22} = \hat{\Sigma}_{22} = \left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & 0 \end{array} \right] \quad (6)$$

thus allowing use of [18, Lemma A.4.1]. The frequency domain inequalities are also equivalent since $F_\ell(\hat{\Sigma}, K)(j\omega) = j\omega \cdot F_\ell(\bar{\Sigma}, K)(j\omega)$.]

(d) $\langle G, K \rangle$ is internally stable, $\bar{\sigma}[F_\ell(G, K)(j\omega)] \leq 1 \; \forall \omega \in \mathbb{R}$, and $\det(I + F_\ell(G, K)(j\omega)) \neq 0 \; \forall \omega \in \mathbb{R} \cup \{\infty\}$, where

$$G = \left[\begin{pmatrix} I & -2I \\ I & -I \end{pmatrix} \star \hat{\Sigma} \right]$$
$$= \left[\begin{array}{c|cc} V^{-1}A & B_1 U^{-1} & V^{-1}B_2 \\ -2U^{-1}C_1A & (I - C_1 B_1)U^{-1} & -2U^{-1}C_1 B_2 \\ \hline C_2 - D_{21}U^{-1}C_1A & D_{21}U^{-1} & -D_{21}U^{-1}C_1 B_2 \end{array} \right].$$

[(c) $\Rightarrow$ (d): Since $F_\ell(\hat{\Sigma}, K) \in \mathcal{P}$, it follows that $(I + F_\ell(\hat{\Sigma}, K))^{-1} \in \mathcal{RH}_{\infty}$ via Lemma 1. Then define $Y = (I - F_\ell(\hat{\Sigma}, K))(I + F_\ell(\hat{\Sigma}, K))^{-1}$ and note that $Y = F_\ell(G, K) \in \mathcal{RH}_{\infty}$. Also, since $(I + Y)^{-1} = (1/2)(I + F_\ell(\hat{\Sigma}, K)) \in \mathcal{RH}_{\infty}$, it follows that $\det(I + Y(j\omega)) \neq 0 \; \forall \omega \in \mathbb{R} \cup \{\infty\}$. Also, $[F_\ell(\hat{\Sigma}, K)(j\omega) + F_\ell(\hat{\Sigma}, K)(j\omega)^*] \geq 0 \; \forall \omega \in \mathbb{R}$ implies $\bar{\sigma}[Y(j\omega)] \leq 1 \; \forall \omega \in \mathbb{R}$. Finally, since $\langle \hat{\Sigma}, K \rangle$ is internally stable, we have $T(\hat{\Sigma}, K) \in \mathcal{RH}_{\infty}$. Noting that $T_{11}(\hat{\Sigma}, K) = F_\ell(\hat{\Sigma}, K) \in \mathcal{P}$, we get $(I + T_{11}(\hat{\Sigma}, K))^{-1} \in \mathcal{RH}_{\infty}$ via Lemma 1 and this in turn gives

$$\begin{pmatrix} I & -2I \\ I & -I \end{pmatrix} \star T(\hat{\Sigma}, K) \in \mathcal{RH}_{\infty} \quad (7)$$

via Lemma 3, which implies $\langle G, K \rangle$ is internally stable as

$$T(G, K) = T \left(\begin{bmatrix} I & -2I \\ I & -I \end{bmatrix} \star \hat{\Sigma}, K \right)$$
$$= \begin{pmatrix} I & -2I \\ I & -I \end{pmatrix} \star T(\hat{\Sigma}, K). \quad (8)$$

(d) $\Rightarrow$ (c): Since $F_\ell(G, K) \in \mathcal{B}$, $(I + F_\ell(G, K))^{-1} \in \mathcal{RH}_{\infty}$ via Lemma 2. Then define $X = (I - F_\ell(G, K))(I + F_\ell(G, K))^{-1} \in \mathcal{RH}_{\infty}$ and note that $X = F_\ell(\hat{\Sigma}, K)$. Then $\bar{\sigma}[F_\ell(G, K)(j\omega)] \leq 1 \; \forall \omega \in \mathbb{R}$ implies $[X(j\omega) + X(j\omega)^*] \geq 0 \; \forall \omega \in \mathbb{R}$. Finally, since $\langle G, K \rangle$ is internally stable, we have $T(G, K) \in \mathcal{RH}_{\infty}$. Noting that $T_{11}(G, K) = F_\ell(G, K) \in \mathcal{B}$, we get $(I + T_{11}(G, K))^{-1} \in \mathcal{RH}_{\infty}$ via Lemma 2 and this in turn gives

$$\begin{pmatrix} I & -2I \\ I & -I \end{pmatrix} \star T(G, K) \in \mathcal{RH}_{\infty} \quad (9)$$

via Lemma 3, which implies $\langle \hat{\Sigma}, K \rangle$ is internally stable as

$$\begin{pmatrix} 0 & 2I \\ \frac{1}{2}I & 0 \end{pmatrix} \star T(\hat{\Sigma}, K) = \begin{pmatrix} I & -2I \\ I & -I \end{pmatrix} \star T(G, K) \text{ via (8).}]$$

∎

This theorem states that the original problem of synthesizing an internally stabilizing controller such that a closed-loop LFT has negative imaginary frequency response can be transformed to an equivalent bounded-real problem. This is a first step towards a controller synthesis method for closed-loop systems with negative imaginary frequency response, allowing results to be borrowed from $\mathcal{H}_\infty$ control theory.

The following lemma shows that the restriction of $D_{12} = 0$ and $D_{22} = 0$ in the realization of the generalized plant for Σ in (2) can be easily circumvented.

Lemma 5: Given a strictly proper controller K and a generalized plant

$$\Sigma = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array} \right] \qquad (10)$$

with (A, B_2) stabilizable, (C_2, A) detectable, and $D_{11} = D_{11}^*$. Then, $\langle \Sigma, K \rangle$ is internally stable and $j[F_\ell(\Sigma, K)(j\omega) - F_\ell(\Sigma, K)(j\omega)^*] \geq 0 \; \forall \omega \in (0, \infty)$ if and only if $\langle \check{\Sigma}, \check{K} \rangle$ is internally stable and $j[F_\ell(\check{\Sigma}, \check{K})(j\omega) - F_\ell(\check{\Sigma}, \check{K})(j\omega)^*] \geq 0 \; \forall \omega \in (0, \infty)$, where

$$\check{\Sigma} = \left[\begin{array}{cc|cc} A & B_2 & B_1 & 0 \\ 0 & -\tau I & 0 & \tau I \\ \hline C_1 & D_{12} & D_{11} & 0 \\ C_2 & D_{22} & D_{21} & 0 \end{array} \right], \qquad (11)$$

$$\check{K}(s) = \left(\frac{s}{\tau} + 1 \right) K(s), \text{ and any arbitrary } \tau > 0. \qquad (12)$$

Proof: Easily follows on noting that:

$$T(\Sigma, K) \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \frac{1}{s/\tau+1}I \end{pmatrix} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \frac{1}{s/\tau+1}I \end{pmatrix} T(\check{\Sigma}, \check{K})$$

because

$$\check{\Sigma} = \Sigma \begin{pmatrix} I & 0 \\ 0 & \frac{1}{s/\tau+1}I \end{pmatrix}.$$

∎

The following remarks are appropriate on Theorem 4 at this stage.

Remark 1: The assumption on the invertibility of U and V in the theorem statement is imposed to ensure well-posedness of the transformed system G while the reformulation takes place from step (c) to (d) shown in the proof of the theorem.

Remark 2: Under the suppositions (A, B_2) stabilizable and (C_2, A) detectable, the state-space realization for G given in Theorem 4 is also stabilizable and detectable.

Proof: This can be easily shown via PBH test [1]. ∎

The following corollary is an immediate consequence of Theorem 4. It gives an equivalent bounded-real condition where an internally stabilizing controller renders the closed-loop LFT structure to be strictly negative-imaginary.

Corollary 1: Given the suppositions of Theorem 4. Then, $\langle \Sigma, K \rangle$ is internally stable and $j[F_\ell(\Sigma, K)(j\omega) - F_\ell(\Sigma, K)(j\omega)^*] > 0 \; \forall \omega \in (0, \infty)$ if and only if $\langle G, K \rangle$ is internally stable, $F_\ell(G, K)(0) = I$, $\bar{\sigma}(F_\ell(G, K)(j\omega)) < 1 \; \forall \omega \in (0, \infty)$ and $\det(I + F_\ell(G, K)(j\infty)) \neq 0$, where G is given in (3).

Proof: The result follows via a straightforward modification of the proof of Theorem 4 from steps (a) to (d) by replacing the frequency domain non-strict inequalities with strict inequalities for all $\omega \in (0, \infty)$. Note that $F_\ell(\hat{\Sigma}, K) = sF_\ell(\bar{\Sigma}, K)$ and $F_\ell(G, K) = (I - F_\ell(\hat{\Sigma}, K))(I + F_\ell(\hat{\Sigma}, K))^{-1}$, this implies $F_\ell(\hat{\Sigma}, K)(0) = 0$ and $F_\ell(G, K)(0) = I$. Hence the reformulated $\mathcal{H}_\infty$ constraint in step (d) can be restated as $F_\ell(G, K)(0) = I$, $\bar{\sigma}(F_\ell(G, K)(j\omega)) < 1 \; \forall \omega \in (0, \infty)$ and the determinant condition reduces to $\det(I +$

$F_\ell(G, K)(j\infty)) \neq 0$. The proof of internal stability part remains the same. ∎

The reformulated closed-loop system $F_\ell(G, K)(s)$ in Theorem 4 and Corollary 1 is always identity at zero frequency i.e. $F_\ell(G, K)(0) = I$. This fact points out a significant technical difficulty in controller synthesis for negative-imaginary systems. It indicates, the reformulated problem is always converted into a non-strict bounded-real problem because $\bar{\sigma}(F_\ell(G, K)(0)) = 1$ independent whether we started from a strict negative-imaginary system or a non-strict negative-imaginary system. This then necessarily invokes optimal $\mathcal{H}_\infty$ theory for controller synthesis. This problem occurs because the transformation from negative-imaginary system to positive-real system, shown in step (b) to (c) in proof of the main theorem, involves multiplying the negative-imaginary transfer function matrix with sI that always makes the transformed positive-real system as non-strict. Hence, controller synthesis methods for strict positive-real systems [19], [20] are not applicable in this context.

Remark 3: The subsystem $G_{12}(s)$ of the transformed system G shown in (3) has an invariant zero at $s = j0$ and this violates a required assumption for $\mathcal{H}_\infty$ optimal controller synthesis, and hence Riccati equation based methods [1], [21] cannot be used.

Proof: Note that since (A, B_2) is stabilizable, it follows $\text{rank}(A \; B_2) = n$. Hence

$$\begin{aligned} &\text{rank} \begin{pmatrix} V^{-1}A & V^{-1}B_2 \\ -2U^{-1}C_1A & -2U^{-1}C_1B_2 \end{pmatrix} \\ &= \text{rank} \left(\begin{pmatrix} V^{-1} & 0 \\ 0 & -2U^{-1} \end{pmatrix} \begin{pmatrix} I \\ C_1 \end{pmatrix} (A \; B_2) \right) \\ &= n < n + m. \end{aligned}$$

∎

The following remark states that under an additional assumption, the invariant zero in Remark 3 becomes a blocking zero, which may be a useful fact for controller synthesis for negative-imaginary systems.

Remark 4: When $\det(A) \neq 0$, $G_{12}(0) = 0$, (i.e. $s = 0$ becomes a blocking zero at zero frequency for $G_{12}(s)$), and hence $F_\ell(G, K)(0) = G_{11}(0) = I$.

Proof: This result is trivial on noting that

$$G_{12}(s) = -2U^{-1}C_1B_2 - 2U^{-1}C_1A(sI - V^{-1}A)^{-1}V^{-1}B_2,$$

$$G_{11}(s) = (I - C_1B_1)U^{-1} - 2U^{-1}C_1A(sI - V^{-1}A)^{-1}B_1U^{-1}$$

and hence, it follows that whenever $\det(A) \neq 0$, $G_{12}(0) = 0$, and $F_\ell(G, K)(0) = G_{11}(0) = I$. ∎

Remark 4 states when A is nonsingular, the reformulated closed-loop system is the identity at zero frequency; i.e. $F_\ell(G, K)(0) = G_{11}(0) = I$ which is independent of the controller K. Also, it is shown via Corollary 1, that if we want to transform the controller synthesis problem for strictly negative imaginary systems, the reformulated system hits the boundary of $gain = 1$ only at zero frequency. In this regard, it might be possible to obtain a solution of the posed problem by extending the generalized KYP lemma to tackle the frequency domain inequality at all frequencies excluding specific points (in this case, zero and infinity) as for example [22]–[24] have done on a given frequency interval. However, this is a nontrivial problem.

IV. ILLUSTRATIVE EXAMPLE

As mentioned in the introduction, lightly damped flexible structures with collocated position sensors and force actuators typically give rise to the negative-imaginary systems considered in this technical note. Hence, the uncertain lightly damped mechanical system depicted in [4] with collocated position sensors and force actuators is a typical example of a negative-imaginary system. This physically motivated example was explored in [4] to illustrate the analysis results of [4]. As
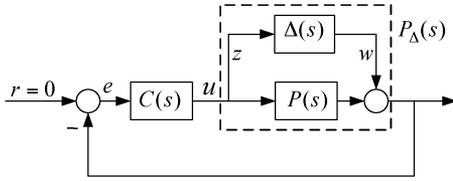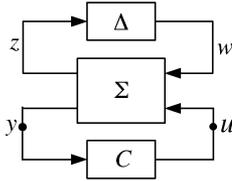
Fig. 2. Controlled closed-loop system.

Fig. 3. Rearranged LFT interconnection.

Fig. 4. Largest singular value plot of $F_\ell(G, \check{C})(j\omega)$.

a consequence, the present work is a first step towards controller synthesis for this class of systems. For completeness, simplicity and also for ease of exposition, the same physically motivated example has been adopted in this section to illustrate the key reformulation of this technical note. The uncertain plant being considered is

$$P_\triangle(s) := p(s)\delta(s)$$
$$\times \begin{bmatrix} s^2 + (\alpha + 1)s + (k + 1) & (\alpha s + k) \\ (\alpha s + k) & s^2 + (\alpha + 1)s + (k + 1) \end{bmatrix}$$

where α and k are unknown real parameters, $p(s) = 1/(s^2 + s + 1)$ and $\delta(s) = 1/(s^2 + (2\alpha + 1)s + (2k + 1))$. For the purpose of robust controller synthesis, the controlled closed-loop system in Fig. 2 is rearranged in a standard LFT interconnection shown in Fig. 3. In these two figures, the generalized plant Σ, the nominal plant P and the uncertainty Δ are given respectively by

$$\Sigma = \begin{bmatrix} 0 & I \\ -I & -P \end{bmatrix}, \quad P(s) = \Psi \mathrm{diag}\left(\frac{1}{2}p(s), 0\right)\Psi^*,$$
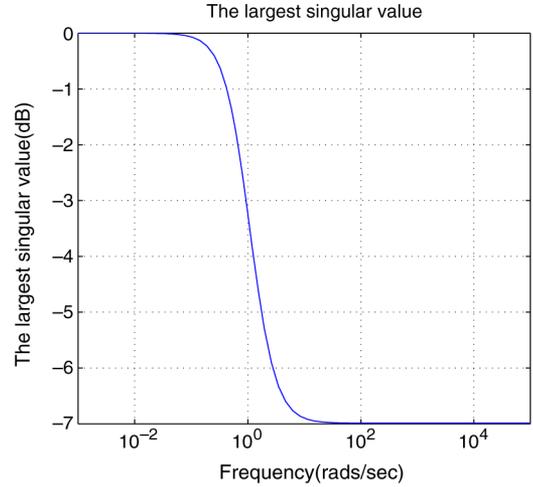$$\text{and } \Delta(s) = \Psi^{-1}\mathrm{diag}\left(\frac{1}{2}\delta(s), 0\right)(\Psi^{-1})^* \quad (13)$$

where $\Psi = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Since the uncertainty $\Delta(s)$ belongs to $\mathcal{I}$, a particular choice of controller $C(s)$ that internally stabilizes Σ and makes $F_\ell(\Sigma, C)$ belong to $\mathcal{I}_s$ was chosen in [4] as $C(s) = \Psi^{-*}\mathrm{diag}((-2(s^2+s+1)/(2s^3+4s^2+4s+3)), (-1/(s+1)))\Psi^{-1}$. This guarantees robust stability for all perturbations in $\mathcal{I}$ as long as the DC loop gain condition is also satisfied [4, Theorem 5].

Since $C(s)$ is strictly proper and the D_{12} matrix of Σ is nonzero, we first use Lemma 5 to give $\check{C}(s) = ((s/\tau) + 1)C(s)$ and $\check{\Sigma} = \begin{pmatrix} 0 & (1/(s/\tau + 1))I \\ -I & (1/(s/\tau + 1))P \end{pmatrix}$ where we arbitrarily set $\tau = 1$. Then, using the construction in Theorem 4, we obtain the transformed generalized plant $G(s)$ as

$$G = \left[\begin{array}{cccc|cccc} -0.9778 & -0.8526 & 0.6992 & 0.6992 & 0 & 0 & 0 & 0 \\ 1.1474 & -0.0222 & -0.1054 & -0.1054 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 2 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 & 0 & -2 \\ -0.1054 & -0.6992 & -1 & 0 & -1 & 0 & 1 & 0 \\ -0.1054 & -0.6992 & 0 & -1 & 0 & -1 & 0 & 1 \end{array}\right].$$

Since we know from [4] that the chosen $C(s)$ internally stabilizes $\Sigma(s)$ and makes $F_\ell(\Sigma, C) \in \mathcal{I}_s$, then via Lemma 5 and Corollary

1, we should get that $\check{C}(s)$ internally stabilizes $G(s)$ and satisfies $F_\ell(G, \check{C})(0) = I$, $\bar{\sigma}(F_\ell(G, \check{C})(j\omega)) < 1 \ \forall \omega \in (0, \infty)$ and $\det(I + F_\ell(G, \check{C})(j\infty)) \neq 0$.

A simple computation gives $F_\ell(G, \check{C})(0) = I$ and $\bar{\sigma}(F_\ell(G, \check{C})(j\omega)) < 1 \ \forall \omega \in (0, \infty)$ is satisfied as the plot of $\bar{\sigma}[F_\ell(G, \check{C})(j\omega)]$ lies below the 0 dB line shown in Fig. 4. Also, $\det(I + F_\ell(G, \check{C})(j\infty)) = 0.8 \neq 0$. Finally, it is observed that $\check{C}$ internally stabilizes G as the poles of $T(G, \check{C})$ are at: $-0.5 \pm j0.8660$, $-0.5 \pm j0.8660, -1, -0.7236, -0.2764$.

This illustrative example demonstrates that the problem of finding an internally stabilizing controller such that the input-output map has strictly negative imaginary frequency response can be reformulated into a bounded-real problem.

Although in this technical note, a controller synthesis method is not given for an optimal $\mathcal{H}_\infty$ design problem with an invariant zero which is due to the reformulation of a negative-imaginary synthesis problem into the bounded-real framework, the following sub-optimal $\mathcal{H}_\infty$ problem

$$\min_{\check{C} \text{ internally stabilizing } G} \gamma : \left\|F_\ell(G, \check{C})\right\|_\infty < \gamma \quad (14)$$

will always give γ tending to unity. Hence, from a controller synthesis point of view, Corollary 1 is useful and more specifically, one can first solve the above sub-optimal $\mathcal{H}_\infty$ problem and then, the conditions $F_\ell(G, \check{C})(0) = I$, $\bar{\sigma}(F_\ell(G, \check{C})(j\omega)) < 1 \ \forall \omega \in (0, \infty)$ and $\det(I + F_\ell(G, \check{C})(j\infty)) \neq 0$ can be checked to see if the conditions of Corollary 1 are fulfilled. Note that the A-matrix of $\check{\Sigma}(s)$ is nonsingular, and hence via Remark 4, $F_\ell(G, \check{C})(0) = I$ is always satisfied which is independent of the controller. In this case, we only need to check the remaining two conditions.

Note that the first element Σ_{11} of the generalized plant Σ rearranged in LFT structure as shown in Fig. 3 is zero, and hence the well-posedness assumptions are satisfied, i.e $U = I + C_1 B_1$ and $V = I + B_1 C_1$ are invertible, and $D_{11} = D_{11}^*$ in Theorem 4 and Corollary 1. It is worth pointing out that the above conditions are also true if the uncertain plant of the controlled closed-loop system considered in Fig. 2 is described as any of the following sets:

$$P_\Delta = P + W_1 \Delta W_2 \text{ (Additive perturbed system)}$$
$$P_\Delta = (I + W_1 \Delta W_2)P$$
$$\text{(Output Multiplicative perturbed system)}$$
$$P_\Delta = P(I + W_1 \Delta W_2)$$
$$\text{(Input Multiplicative perturbed system).}$$

V. CONCLUSION

The main result of this technical note is a first step towards a controller synthesis technique for negative-imaginary systems. In [4], an analysis result was proposed, similar to the small-gain or passivity theorem, for negative-imaginary systems. This technical note shows how an LFT interconnection that has negative-imaginary closed-loop properties can be reformulated into a bounded-real LFT interconnection. Although this technical note does not tackle the important step of explicit controller synthesis for such a class of systems, the main results in this technical note could constitute a first step in allowing results from $\mathcal{H}_\infty$ control synthesis to be borrowed for controller synthesis for closed-loop systems with negative-imaginary properties. However, the controller synthesis problem under consideration is not trivial and significant research effort is needed to solve it. To this end, important observations have been pointed out in the Corollary and Remarks in this note. It is hoped that this note highlights open technical problems in this area which need to be solved for effective controller synthesis for negative-imaginary systems.

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On the Sigma Exponent of Discrete Linear Systems

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Abstract—In this technical note, we introduce the concepts of sigma exponent of discrete linear time varying system. It is shown that these exponents describe the possible changes in the Lyapunov exponents under perturbation decreasing at infinity at exponential rate. Finally we present formulas for the exponential exponents in terms of transition matrix of the system.

Index Terms—Characteristic exponents, Lyapunov exponents, perturbation theory, time varying discrete linear systems.

I. INTRODUCTION

In the past two decades, considerable research efforts have been devoted to study of robust stability of uncertain dynamic systems with parametric uncertainties. A great number of significant results covering these issues have been reported in the literature (see, e.g., [7], [13], [16], and the references therein). A crucial problem in this area is to determine to what extent stability is preserved when the parameters of the nominal system are subject to perturbations. Since the different concepts of stability may be characterized by different numerical characteristics such as Lyapunov exponents, Bohl exponents or generalized spectral radius. Therefore, one may attack this problem by investigating the influence of perturbation on these characteristics. The goal of the technical note is to describe the changes in the Lyapunov exponents of linear time varying discrete system under perturbation decreasing at infinity at exponential rate.

For the linear time invariant system it follows from the continuity of matrix spectrum that the property of being stable is preserved under sufficiently small perturbation. The situation is much more complicated in time varying case, since the Lyapunov exponents are not continuous as

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