Distance Measures for Uncertain Linear Systems: A General Theory

Alexander Lanzon, *Senior Member, IEEE*, and George Papageorgiou, *Member, IEEE*

Abstract—In this paper, we propose a generic notion of distance between systems that can be used to measure discrepancy between open-loop systems in a feedback sense under several uncertainty structures. When the uncertainty structure is chosen to be four-block (or equivalently, normalized coprime factor) uncertainty, then this generic distance measure reduces to the well-known ν-gap metric. Associated with this generic distance notion, we also define a generic stability margin notion that allows us to give the distance measure a feedback interpretation by deriving generic robust stability and robust performance results. The proposed distance notion and the corresponding results exploit a powerful generalization of the small-gain theorem which handles perturbations in $\mathcal{RL}_\infty$, rather than only in $\mathcal{RH}_\infty$. When the uncertainty structure is fixed to one of the standard structures (e.g., additive, multiplicative, inverse multiplicative, coprime factor, four-block or any mixtures of the above), we give a step-by-step procedure (based on model validation ideas) that shows how the generic notion of distance and the correspondingly generic winding number conditions can be reduced to simple formulae. This work provides a unified framework that captures and embeds previous results in this area and also completes the picture by showing how other results of a similar nature can be obtained from the same framework. The techniques used involve only basic linear algebra, so they also provide a simplification of previous advanced proofs. Furthermore, the various distance measures so created can be used for non-conservative model embedding into the smallest uncertain family. An illustrative example is also given that demonstrates the superior qualities, above the ν-gap metric, of a particular distance measure obtained from this work in situations where the plant is lightly-damped. All systems considered in this paper are linear time-invariant.

Index Terms—Distance measures, feedback systems, gap metric, $\mathcal{H}_\infty$ control, model validation, ν-gap metric, robust performance, robust stability, stability margin.

I. INTRODUCTION

T HE concept of measuring distance between systems was first proposed by Vidyasagar [2] and exploited therein to derive robustness results. Subsequently, El-Sakkary [3], Georgiou and Smith [4], and Qiu and Davison [5] looked at a similar distance notion and used advanced operator theory to create an elegant theory, based on the well-known gap metric, that measures discrepancy between systems around a *normalized* coprime factorization of the nominal plant. These authors (principally, but also others) derived powerful results that exploited the distance between two systems, as measured by the gap metric, to characterize when a perturbed feedback interconnection is stable and to also characterize the residual stability margin when one system was replaced by another in a feedback interconnection with a fixed controller.

Separately to the above efforts, Glover [6] had noted that the small-gain theorem, proposed by Zames [7], could be extended to also handle perturbations in $\mathcal{L}_\infty$, rather than only in $\mathcal{H}_\infty$. For this result to hold, an extra condition (related to a right-half plane (RHP) pole/zero count) had to be also satisfied. Glover's work [6] was restricted to the additive uncertainty structure, though his work set-out a new way of thinking of small-gain. In fact, Vinnicombe [8], [9] exploited this new philosophy of allowing perturbations to belong to $\mathcal{RL}_\infty$ to derive a new metric[1], known as the ν-gap metric, which measures discrepancy between systems around a *normalized* coprime factorization of the nominal plant (or equivalently a four-block uncertainty setting) but which allows for perturbations to belong to $\mathcal{RL}_\infty$, rather than just $\mathcal{RH}_\infty$ as in the case of the gap metric, provided a RHP pole/zero count was satisfied. This relaxation of allowing perturbations to belong to $\mathcal{RL}_\infty$, rather than just $\mathcal{RH}_\infty$, resulted in superior qualities of the ν-gap metric over the gap metric. For instance, the ν-gap metric is easier to calculate in the LTI case than the gap metric, gives less conservative results because the perturbation class ($\mathcal{RL}_\infty$) for the ν-gap metric is larger than the perturbation class ($\mathcal{RH}_\infty$) for the gap metric, and also allows for a certain strong necessity condition to hold [9, Theorem 3.10] (a result that does not hold in the gap metric theory). All other robustness guarantees, related to characterization of the residual stability margin when one system is replaced by another in a feedback interconnection with a fixed controller and related to discrepancies between the closed-loop transfer functions when one system is replaced by another, were basically mirrored in the ν-gap metric theory from the gap metric theory without significant changes. Vinnicombe has also nicely cast this idea of Glover on extending the small gain theorem to perturbations in $\mathcal{RL}_\infty$ in a formal theorem, see [9, Lemma 1.22]. This lemma will underpin our generic robust stability condition.

For decades, the control engineering community has known [12]–[15] that different uncertainty structures (e.g., additive, input or output multiplicative, input or output inverse multiplicative, not necessarily normalized coprime factor, four-block

[1]The works of Georgiou and Smith, and Vinnicombe have been used in the design of a number of experimental controllers, the most notable two being for a Harrier aircraft that were successfully flight tested (see Hyde [10] and Papageorgiou [11]).

and various mixtures of the above) possessed different properties and sound choice of which uncertainty structure to use was often shuffled from the theoreticians onto practitioners to use engineering insight and intuition. For example, it is well-known [12]–[15] that multiplicative uncertainty is good to capture high-frequency unmodelled dynamics (i.e., where the loop-gain is small), inverse multiplicative uncertainty is good to capture low-frequency modelling discrepancies (i.e., where the loop-gain is large), and that coprime factor uncertainty captures the general properties of both the above and actually even more. However, the difficulty of embedding an exclusively low-frequency discrepancy, for example, into a "catch all" uncertainty structure is that this would result in a larger family. The standard small-gain theorem (using perturbations in $\mathcal{RH}_\infty$) has been used to derive robust stability results for all the above-mentioned standard uncertainty structures (see [14, Table 9.1] for a brief exposition of all these standard results and also for a skim of where each uncertainty structure can be useful).

It remained open, however, how to use the idea of Glover [6] on the extension of the small-gain theorem to perturbations in $\mathcal{RL}_\infty$, rather than $\mathcal{RH}_\infty$, on these several uncertainty structures to derive more powerful robust stability and robust performance results. Also, due to their close link to the small-gain theorem and the corresponding robust stability results, one cannot but wonder whether there is a more generic notion of distance between systems (which of course specializes to the ν-gap metric in the four-block or *normalized* coprime factor uncertainty setting) that could handle all the specific different uncertainty structures (i.e., additive, input or output multiplicative, input or output inverse multiplicative, not necessarily normalized coprime factor, four-block and various mixtures of the above). This article presents a positive answer to the above two questions. The extension of the small-gain theorem to perturbations in $\mathcal{RL}_\infty$, rather than $\mathcal{RH}_\infty$, proposed by Glover and exploited by Vinnicombe in the ν-gap metric work will be explored on all standard uncertainty structures. The generic notion of distance proposed in this paper will simplify to easily computable norms and metrics in each of the standard uncertainty structures (and will reduced to the ν-gap metric in the four-block uncertainty setting) and robust stability/robust performance results will be derived for this generic setting that are of the same flavour as currently exists for only and specifically the four-block uncertainty structure. Consequently, this work completes missing areas in a mature theory, embeds all existing results under the proposed elegant framework, re-derives existing results using basic linear algebra rather than advanced operator theory, and sheds a new light on how these problems can be viewed. Our approach has been inspired by robust model validation theory (see Davis [16], Poolla *et al.* [17], Chen and Gu [18], and Newlin and Smith [19]).

Since this work completes the notion of distance measures for all standard uncertainty structures, this theory can now assist the practitioner (and hence alleviate the burden on engineering experience and intuition) on how to select which uncertainty modelling is most appropriate for the particular problem at hand. This is because the specific distance measures for the different uncertainty structures can be easily computed and the engineer can then choose to embed the different plant models into a family of smallest size (be it multiplicative, additive, coprime factor, four-block or other mixes). Given a number of plant models, the idea is to choose the smallest family that contains all plant models, the presumption being that it will be easier for a controller to achieve the desired closed-loop objectives with a small family. The resulting formulae and tests can hence be of significant practical importance in that they allow the control systems engineer to quantify the discrepancy between systems from a feedback perspective for any particular uncertainty structure. The corresponding specific distance measure can then be used to infer stability and performance properties of the perturbed feedback interconnection in relation to the nominal feedback loop.

This paper proposes a generic distance notion that captures all the above in one framework, a corresponding generic stability margin and derives generic robust stability and robust performance theorems that place under one framework all the main results of this kind. A systematic procedure is also provided that takes the reader through a step-by-step process on how to specialize the generic results for any particular standard uncertainty structure at hand. In this paper, the systematic procedure is applied specifically to a not necessarily normalized coprime factor uncertainty structure (which easily modifies to the four-block case when one insists on requiring the coprime factors to be normalized), and a brief indication is given of other results that could be obtained via similar application of the systematic procedure on other uncertainty structures. Finally, we show how weighting functions can be easily incorporated in this theory and we give an illustrative example that demonstrates the importance of having distance measures other than the standard ν-gap metric and gap metric if this theory is to be widely used in application. The superior qualities of techniques developed in this paper over ν-gap metric analysis will be readily apparent.

Notation

Notation is standard. Let $\mathcal{R}$ denote the set of proper real-rational transfer functions.[2] Also, let $P^*(s)$ denote the adjoint of $P(s) \in \mathcal{R}$ defined by $P^*(s) = P(-s)^T$. Let $\mathcal{RL}_\infty$ denote the space of proper real-rational functions bounded on $j\mathbb{R}$ including ∞, and $\mathcal{RH}_\infty$ denote the space of proper real-rational functions bounded and analytic in the open right half complex plane. Denote also the space of functions that are units in $\mathcal{RH}_\infty$ by $\mathcal{GH}_\infty$ (that is, $f \in \mathcal{GH}_\infty \Leftrightarrow f, f^{-1} \in \mathcal{RH}_\infty$). Let $\mathcal{F}_l(\cdot, \cdot)$ (resp. $\mathcal{F}_u(\cdot, \cdot)$) denote a lower (resp. upper) linear fractional transformation [13], [14]. Also, let $(\cdot \star \cdot)$ denote the Redheffer star product of two transfer functions with respect to some partition [13], [14]. For a scalar $p(s) \in \mathcal{R}$, its winding number wno $p(s)$ is defined as the number of encirclements of the origin made by $p(s)$ as s follows the standard Nyquist D-contour, indented into the right half plane around any imaginary axis poles of $p(s)$. Furthermore, let $\eta(P)$ denote the number of closed right half plane poles of $P \in \mathcal{R}$.

The ordered pair $\{N, M\}$, with $M, N \in \mathcal{RH}_\infty$, is a right-coprime factorization (*rcf*) of $P \in \mathcal{R}$ if M is invertible in $\mathcal{R}$,

---

[2] $P \in \mathcal{R}$ implies that $\overline{\sigma}(P(\infty)) < \infty$. Also, $P \in \mathcal{R}$ is invertible in $\mathcal{R}$ if, and only if, P is square and $\det P(\infty) \neq 0$.
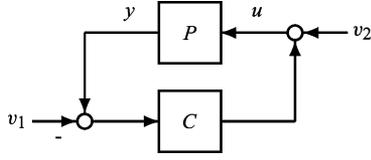
Fig. 1.   Standard feedback interconnection.

Fig. 2.   Standard linear fractional interconnection.

$P = NM^{-1}$, and N and M are right-coprime over $\mathcal{RH}_\infty$. Furthermore, the ordered pair $\{N, M\}$ is a normalized *rcf* of P if $\{N, M\}$ is a *rcf* of P and $M^*M + N^*N = I$. Likewise, the ordered pair $\{\tilde{N}, \tilde{M}\}$, with $\tilde{M}, \tilde{N} \in \mathcal{RH}_\infty$, is a left-coprime factorization (*lcf*) of $P \in \mathcal{R}$ if $\tilde{M}$ is invertible in $\mathcal{R}$, $P = \tilde{M}^{-1}\tilde{N}$, and $\tilde{N}$ and $\tilde{M}$ are left-coprime over $\mathcal{RH}_\infty$. Furthermore, the ordered pair $\{\tilde{N}, \tilde{M}\}$ is a normalized *lcf* of P if $\{\tilde{N}, \tilde{M}\}$ is a *lcf* and $\tilde{M}\tilde{M}^* + \tilde{N}\tilde{N}^* = I$. Let $\{N, M\}$ be a *rcf* and $\{\tilde{N}, \tilde{M}\}$ be a *lcf* of a plant P. Also, let $\{U, V\}$ be a *rcf* and $\{\tilde{U}, \tilde{V}\}$ be a *lcf* of a controller C. Define

$$G := \begin{bmatrix} N \\ M \end{bmatrix}, \quad \tilde{G} := \begin{bmatrix} -\tilde{M} & \tilde{N} \end{bmatrix}$$

$$K := \begin{bmatrix} V \\ U \end{bmatrix}, \quad \tilde{K} := \begin{bmatrix} -\tilde{U} & \tilde{V} \end{bmatrix}$$

where G and $\tilde{G}$ will be referred to as the graph symbols of P, and K and $\tilde{K}$ will be referred to as the inverse graph symbols of C.[3]

II. DEFINITION OF A GENERIC DISTANCE MEASURE

In this section, we define a generic distance measure for uncertain systems. Associated with this distance measure, we also define a stability margin which is used to interpret the distance measure from a feedback perspective.

Let $[P, C]$ denote the standard feedback interconnection illustrated in Fig. 1. From Fig. 1 and after some algebra

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} P \\ I \end{bmatrix} (I - CP)^{-1} \begin{bmatrix} -C & I \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}. \quad (1)$$

Definition 1: Given a plant $P \in \mathcal{R}$ and a controller $C \in \mathcal{R}$. $[P, C]$ is said to be well-posed if the transfer function in (1) belongs to $\mathcal{R}$, that is, if $(I - CP)^{-1} \in \mathcal{R}$. Also, $[P, C]$ is said to be internally stable if, in addition, the transfer function in (1) belongs to $\mathcal{RH}_\infty$.

[3]For normalized graph symbols

$$\begin{bmatrix} G^* \\ \bar{G} \end{bmatrix} \begin{bmatrix} G & \bar{G}^* \end{bmatrix} = I, \quad \begin{bmatrix} G & \bar{G}^* \end{bmatrix} \begin{bmatrix} G^* \\ \bar{G} \end{bmatrix} = I.$$

Similarly for K and $\bar{K}$.
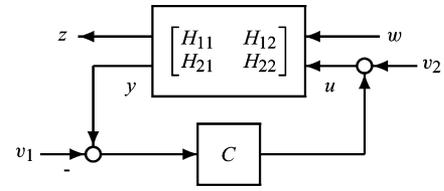
Let $\langle H, C \rangle$ denote the standard linear fractional interconnection depicted in Fig. 2. Equation (2) at the bottom of the page can be derived from Fig. 2 after some algebra.

Definition 2: Given a generalized plant $H \in \mathcal{R}$ and a controller $C \in \mathcal{R}$. $\langle H, C \rangle$ is said to be well-posed if the transfer function in (2) belongs to $\mathcal{R}$, that is, if $(I - H_{22}C)^{-1} \in \mathcal{R}$. Also, $\langle H, C \rangle$ is said to be internally stable if, in addition, the transfer function in (2) belongs to $\mathcal{RH}_\infty$. Finally, H is said to be stabilizable if there exists a C such that $\langle H, C \rangle$ is internally stable.[4]

It will be shown in the sequel, that if $[H_{22}, C]$ is internally stable, then the smaller the size of $\|\mathcal{F}_l(H, C)\|_\infty$, the larger the size of the set of plants that C is guaranteed to stabilize. Therefore, it can be argued that $\|\mathcal{F}_l(H, C)\|_\infty^{-1}$ is a measure of the robust stability of a feedback system (i.e., a stability margin). Also, the nominal performance of a feedback system, that is the performance with no plant uncertainty, can be related to H and $\|\mathcal{F}_l(H, C)\|_\infty^{-1}$ (by performance it is typically meant reference tracking, disturbance rejection and noise rejection). This is because the size of any closed-loop transfer function can be bounded above in terms of the size of any weighting functions absorbed in H and $\|\mathcal{F}_l(H, C)\|_\infty^{-1}$.[5] Therefore, $\|\mathcal{F}_l(H, C)\|_\infty^{-1}$ is an important quantity that summarizes the robustness and performance properties of a feedback system.

Definition 3: Given a plant $P \in \mathcal{R}^{p \times q}$, a generalized plant $H \in \mathcal{R}$ with $H_{22} = P$, and a controller $C \in \mathcal{R}^{q \times p}$. Define the stability margin $b^H(P, C)$ of the feedback interconnection $\langle H, C \rangle$ as given in the equation at the bottom of the next page.

Depending on H, $[P, C]$ is internally stable may or may not imply that $\mathcal{F}_l(H, C) \in \mathcal{RL}_\infty$.[6] Therefore, in general, we need to assume $\mathcal{F}_l(H, C) \in \mathcal{RL}_\infty$ to ensure that

[4]If $\left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array} \right]$ is the state-space realization of H_{22} inherited from a stabilizable and detectable state-space realization of H, then this definition of "H is stabilizable" is equivalent to (A, B_2) being stabilizable and (C_2, A) being detectable [13].

[5]For an example of how nominal performance can be related to H and $\|\mathcal{F}_l(H, C)\|_\infty^{-1}$ see [14, p. 493, Theorem 18.11].

[6]For example, when H is stabilizable, $[P, C]$ is internally stable is equivalent to $\langle H, C \rangle$ is internally stable [13] which implies that $\mathcal{F}_l(H, C) \in \mathcal{RL}_\infty$.

$$\begin{bmatrix} z \\ y \\ u \end{bmatrix} = \left(\begin{bmatrix} H_{11} & 0 & H_{12} \\ H_{21} & 0 & H_{22} \\ 0 & 0 & I \end{bmatrix} + \begin{bmatrix} H_{12} \\ H_{22} \\ I \end{bmatrix} C(I - H_{22}C)^{-1} \begin{bmatrix} H_{21} & -I & H_{22} \end{bmatrix} \right) \begin{bmatrix} w \\ v_1 \\ v_2 \end{bmatrix} \quad (2)$$

$\|\mathcal{F}_l(H,C)\|_\infty^{-1} > 0$ in the definition of the stability margin. Note also that if $\mathcal{F}_l(H,C) \neq 0$, then $b^H(P,C) < \infty$ (we exclude $\mathcal{F}_l(H,C) = 0$ from the meaningful part of the definition of $b^H(P,C)$ since this is a case of no practical interest). The choice of generalized plant H and the required size of the stability margin are application specific, as we will see in subsequent sections.

We will now define a distance measure between a nominal plant and a perturbed plant.

Definition 4: Given a plant $P \in \mathcal{R}^{p \times q}$, a generalized plant $H \in \mathcal{R}$ with $H_{22} = P$, and a perturbed plant $P_\Delta \in \mathcal{R}^{p \times q}$. Let the set of all admissible perturbations be given by

$$\Delta = \left\{ \Delta \in \mathcal{RL}_\infty : (I - H_{11}\Delta)^{-1} \in \mathcal{R}, P_\Delta = \mathcal{F}_u(H,\Delta) \right\}.$$

Define the distance measure $d^H(P,P_\Delta)$ between plants P and P_Δ for the uncertainty structure implied by H as

$$d^H(P,P_\Delta) := \begin{cases} \inf_{\Delta \in \Delta} \|\Delta\|_\infty & \text{if } \Delta \neq \emptyset, \\ \infty & \text{otherwise.} \end{cases}$$

Note that all $\Delta \in \Delta$ satisfy a well-posedness condition and a consistency of equations condition, and that if $\Delta \neq \emptyset$, then the infimum always exists in $\mathbb{R}$ since $\Delta \in \mathcal{RL}_\infty$. Also note that $d^H(P,P_\Delta)$ is nonnegative and that $d^H(P,P) = 0$ since $P = \mathcal{F}_u(H,0)$. Whether or not $d^H(\cdot,\cdot)$ is a metric on $\mathcal{R}^{p \times q}$ or some subset of $\mathcal{R}$ depends on the generalized plant H. The distance measure will in fact not be a metric for *only* specific uncertainty structures such as multiplicative uncertainty because it only violates the symmetry property of metrics. All other metric properties will typically be always fulfilled. Such distances are commonly referred to as *directed distances* in that the distance is measured with respect to one of the two systems. In our opinion, this is not a limitation as distance in robust control is typically measured with respect to a nominal plant. If one still wants to insist on forcing the distance measure to be a metric also in these situations (note that metrics are important when mathematical results on metric spaces need to be used, for example when measuring distance to non-rational objects), then the symmetry property can be fulfilled by considering $\max \left\{ d^H(P,P_\Delta), d^{H_\Delta}(P_\Delta,P) \right\}$, just as is done in gap-metric theory using directed gaps.

It will be shown in the sequel, that the smaller the size of $d^H(P,P_\Delta)$, the smaller the worst-case degradation of the stability margin due to a plant perturbation. Therefore, we will say that $d^H(P,P_\Delta)$ can be interpreted as the distance between two plants from a feedback perspective. The generic distance notion defined in Definition 4 will be specialized to easily computable formulae via a systematic procedure discussed in Section VI. This generic distance notion will reduce to a simple formula such as that given in (20) for the (not necessarily normalized) left coprime factor uncertainty case and will reduce to the simple formulae given in the third column of Table II in Section VIII in

the additive, input multiplicative, output inverse multiplicative and left four-block uncertainty cases.

Definition 5: Given a plant $P \in \mathcal{R}^{p \times q}$, a generalized plant $H \in \mathcal{R}$ with $H_{22} = P$, and a perturbed plant $P_\Delta \in \mathcal{R}^{p \times q}$. Define

$$\Delta^{\min} := \left\{ \Delta \in \Delta : \|\Delta\|_\infty = d^H(P,P_\Delta) \right\}.$$

Note that $\Delta^{\min} \subset \Delta$ and that $\Delta^{\min} = \emptyset$ when the infimum in the definition of $d^H(P,P_\Delta)$ is achieved on the boundary of Δ. In the sequel and whenever necessary, we shall be imposing conditions on P and P_Δ to ensure that $\Delta^{\min}$ is not empty (i.e., the infimum is attained inside the set Δ). The set $\Delta^{\min}$ is the set of the smallest-sized $\Delta \in \mathcal{RL}_\infty$ that satisfy $(I - H_{11}\Delta)^{-1} \in \mathcal{R}$ (well-posedness) and $P_\Delta = \mathcal{F}_u(H,\Delta)$ (consistency of equations). We will provide a systematic procedure in Section VI that explicitly characterizes the set $\Delta^{\min}$ in terms of nominal plant P, generalized plant H, and perturbed plant P_Δ only.

III. ROBUST STABILITY

When $\Delta^{\min} \neq \emptyset$, the following theorem gives necessary and sufficient conditions for the stability of a perturbed feedback system given a bound on the distance between the nominal and perturbed plants.

Theorem 1 (Robust Stability): Given a plant $P \in \mathcal{R}^{p \times q}$, a stabilizable generalized plant $H \in \mathcal{R}$ with $H_{22} = P$, a perturbed plant $P_\Delta \in \mathcal{R}^{p \times q}$ and a controller $C \in \mathcal{R}^{q \times p}$ such that $d^H(P,P_\Delta) < b^H(P,C)$ and $\Delta^{\min} \neq \emptyset$, then the following statements are equivalent:

(a) $[P_\Delta, C]$ is internally stable;
(b) $\forall \Delta \in \Delta^{\min}, \eta(P_\Delta) = \eta(P) + \text{wno} \det(I - H_{11}\Delta)$;
(c) $\exists \Delta \in \Delta^{\min} : \eta(P_\Delta) = \eta(P) + \text{wno} \det(I - H_{11}\Delta)$ (3)

where the winding number is evaluated on a contour indented to the right around any imaginary axis poles of P and P_Δ.

Proof: If $d^H(P,P_\Delta) < b^H(P,C)$, then $b^H(P,C) > 0$ and therefore by definition $[P,C]$ is internally stable. By supposition $\Delta^{\min} \neq \emptyset$. Then

$$d^H(P,P_\Delta) < b^H(P,C)$$
$$\Rightarrow \|\Delta\|_\infty < \|\mathcal{F}_l(H,C)\|_\infty^{-1} \quad \forall \Delta \in \Delta^{\min}$$
$$\Rightarrow \|\Delta \mathcal{F}_l(H,C)\|_\infty < 1 \quad \forall \Delta \in \Delta^{\min}.$$

Therefore, for all $\Delta \in \Delta^{\min} \subset \Delta$:
• $\Delta \in \mathcal{RL}_\infty$;
• $\|\Delta \mathcal{F}_l(H,C)\|_\infty < 1$;
• $(I - H_{11}\Delta)^{-1} \in \mathcal{R}$; and
• $P_\Delta = \mathcal{F}_u(H,\Delta)$.
The proof follows from [9, p. 45, Lemma 1.22]. Since H is stabilizable and $[P,C]$ is internally stable, from [9, Lemma 1.22],

$$b^H(P,C) := \begin{cases} \|\mathcal{F}_l(H,C)\|_\infty^{-1} & \text{if } 0 \neq \mathcal{F}_l(H,C) \in \mathcal{RL}_\infty \text{ and } [P,C] \text{ is internally stable,} \\ 0 & \text{otherwise.} \end{cases}$$
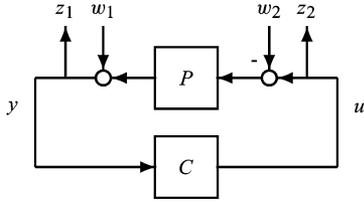
Fig. 3. Left 4-block interconnection.

TABLE I
THE UNCERTAINTY STRUCTURES CONSIDERED. $\{\bar{N}, \bar{M}\}$ IS A NORMALIZED
LEFT COPRIME FACTORIZATION OF P AND $R \in \mathcal{GH}_\infty$

Uncertainty structure	S_w	S_z
Additive	$\begin{bmatrix} I \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 & I \end{bmatrix}$
Inverse additive	$\begin{bmatrix} 0 \\ I \end{bmatrix}$	$\begin{bmatrix} I & 0 \end{bmatrix}$
Input multiplicative	$\begin{bmatrix} 0 \\ I \end{bmatrix}$	$\begin{bmatrix} 0 & I \end{bmatrix}$
Output inverse multiplicative	$\begin{bmatrix} I \\ 0 \end{bmatrix}$	$\begin{bmatrix} I & 0 \end{bmatrix}$
Output inverse multiplicative and additive	$\begin{bmatrix} I \\ 0 \end{bmatrix}$	I
Inverse additive and input multiplicative	$\begin{bmatrix} 0 \\ I \end{bmatrix}$	I
Output inverse multiplicative and inverse additive	I	$\begin{bmatrix} I & 0 \end{bmatrix}$
Additive and input multiplicative	I	$\begin{bmatrix} 0 & I \end{bmatrix}$
Left 4-block	I	I
Non-normalised left coprime factor	$\begin{bmatrix} (R\bar{M})^{-1} \\ 0 \end{bmatrix}$	I

we get (a) $\Rightarrow$ (b). It is obvious that (b) $\Rightarrow$ (c). Also from [9, Lemma 1.22], we get (c) $\Rightarrow$ (a) which concludes the proof. ∎

A very powerful implication of (c) $\Rightarrow$ (b), or equivalently its contrapositive

$$\exists \Delta \in \mathbf{\Delta}^{\min} : \eta(P_\Delta) \neq \eta(P) + \text{wno} \det(I - H_{11}\Delta)$$
$$\Rightarrow \eta(P_\Delta) \neq \eta(P) + \text{wno} \det(I - H_{11}\Delta) \quad \forall \Delta \in \mathbf{\Delta}^{\min}$$

is that winding number condition (3) is either true for all $\Delta \in \mathbf{\Delta}^{\min}$ or false for all $\Delta \in \mathbf{\Delta}^{\min}$. Therefore, to check whether or not $[P_\Delta, C]$ is internally stable it suffices to evaluate the winding number in (3) on a single arbitrary $\Delta \in \mathbf{\Delta}^{\min}$.[7] Condition (3) will be specialized to easily computable formulae through the systematic procedure discussed in Section VI. This procedure will allow us to evaluate condition (3) on a convenient $\Delta \in \mathbf{\Delta}^{\min}$ to produce a simple compact formula such as that given in (21) for the (not necessarily normalized) left coprime factor uncertainty case and such as the simple formulae given in the second column of Table II in Section VIII for the additive, input multiplicative, output inverse multiplicative and left four-block uncertainty cases.

The robust stability theorem also holds if, instead of supposing that $d^H(P, P_\Delta) < b^H(P, C)$, we suppose that $0 \neq \mathcal{F}_l(H, C) \in \mathcal{RL}_\infty$, that $[P, C]$ is internally stable and that

$$\bar{\sigma}(\Delta(j\omega)) < \frac{1}{\bar{\sigma}(\mathcal{F}_l(H, C))(j\omega)} \quad \forall \omega \text{ and } \forall \Delta \in \mathbf{\Delta}^{\min}.$$

[7]Actually, winding number condition (3) is also true (or false) for all $\Delta \in \mathbf{\Delta}$ that satisfy $\|\Delta\|_\infty < b^H(P, C)$. Consequently, choosing a perturbation Δ that satisfies $\|\Delta\|_\infty \in [d^H(P, P_\Delta), b^H(P, C))$ does not change the winding number.

If $b^H(P, C) > \epsilon (\geq 0)$ and H is stabilizable, then $[P_\Delta, C]$ is internally stable for all P_Δ that belong to the set

$$\mathcal{P} = \Big\{ P_\Delta = \mathcal{F}_u(H, \Delta) : d^H(P, P_\Delta) \leq \epsilon,$$
$$\Delta \in \mathcal{RL}_\infty, (I - H_{11}\Delta)^{-1} \in \mathcal{R},$$
$$\eta(P_\Delta) = \eta(P) + \text{wno} \det(I - H_{11}\Delta) \Big\}.$$

Therefore, a controller C that achieves a stability margin greater than ϵ with a nominal plant P, is guaranteed to stabilize all the plants in the set $\mathcal{P}$.

IV. GENERALIZED PLANTS CONSIDERED

Given a plant $P \in \mathcal{R}$ and a controller $C \in \mathcal{R}$. Consider the feedback interconnection illustrated in Fig. 3 that will be referred to as the left 4-block interconnection. From Fig. 3

$$\begin{bmatrix} z_1 \\ z_2 \\ - \\ y \end{bmatrix} = \begin{bmatrix} I & -P & | & P \\ 0 & 0 & | & I \\ - & - & - & | & - & - \\ I & -P & | & P \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ - \\ u \end{bmatrix}.$$

In the remainder of this paper and in order to pin down the technical machinery, we will only consider generalized plants that can be derived from Fig. 3. Let

$$z = S_z \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \text{ and } \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = S_w w$$

where $S_w, S_z \in \mathcal{R}$ select and possibly filter the inputs and outputs of interest. Also, let

$$H = \begin{bmatrix} S_z & | & \\ - & - & | & - & - \\ & | & I \end{bmatrix} \begin{bmatrix} I & -P & | & P \\ 0 & 0 & | & I \\ - & - & - & | & - & - \\ I & -P & | & P \end{bmatrix}$$
$$\times \begin{bmatrix} S_w & | & \\ - & - & | & - & - \\ & | & I \end{bmatrix} \quad (4)$$

and

$$w = \Delta z$$

where $\Delta \in \mathcal{R}$.

If $\left(I - S_z \begin{bmatrix} I & -P \\ 0 & 0 \end{bmatrix} S_w \Delta \right)^{-1} \in \mathcal{R}$, then $\mathcal{F}_u(H, \Delta) \in \mathcal{R}$ and

$$P_\Delta = \mathcal{F}_u(H, \Delta)$$
$$\Leftrightarrow P_\Delta = P + \begin{bmatrix} I & -P \end{bmatrix} S_w \Delta$$
$$\times \left(I - S_z \begin{bmatrix} I & -P \\ 0 & 0 \end{bmatrix} S_w \Delta \right)^{-1} S_z \begin{bmatrix} P \\ I \end{bmatrix}$$
$$\Leftrightarrow P_\Delta - P = \left(I - \begin{bmatrix} I & -P \end{bmatrix} S_w \Delta S_z \begin{bmatrix} I \\ 0 \end{bmatrix} \right)^{-1}$$
$$\times \begin{bmatrix} I & -P \end{bmatrix} S_w \Delta S_z \begin{bmatrix} P \\ I \end{bmatrix}$$
$$\Leftrightarrow P_\Delta - P = \begin{bmatrix} I & -P \end{bmatrix} S_w \Delta S_z \left(\begin{bmatrix} P \\ I \end{bmatrix} + \begin{bmatrix} P_\Delta - P \\ 0 \end{bmatrix} \right)$$
$$\Leftrightarrow P_\Delta - P = \begin{bmatrix} I & -P \end{bmatrix} S_w \Delta S_z \begin{bmatrix} P_\Delta \\ I \end{bmatrix}. \quad (5)$$

Also, if $(I - PC)^{-1} \in \mathcal{R}$, then $\mathcal{F}_l(H, C) \in \mathcal{R}$ and

$$
\begin{aligned}
\mathcal{F}_l&(H, C) \\
&= S_z \begin{bmatrix} I & -P \\ 0 & 0 \end{bmatrix} S_w \\
&\quad + S_z \begin{bmatrix} P \\ I \end{bmatrix} C(I - PC)^{-1} \begin{bmatrix} I & -P \end{bmatrix} S_w \\
&= S_z \left(\begin{bmatrix} I - PC \\ 0 \end{bmatrix} + \begin{bmatrix} PC \\ C \end{bmatrix} \right) (I - PC)^{-1} \begin{bmatrix} I & -P \end{bmatrix} S_w \\
&= S_z \begin{bmatrix} I \\ C \end{bmatrix} (I - PC)^{-1} \begin{bmatrix} I & -P \end{bmatrix} S_w. \qquad (6)
\end{aligned}
$$

Table I lists the uncertainty structures that will henceforth be considered. The theory and tools developed in this paper readily apply to *any* uncertainty structure that can be derived from the left and right 4-block interconnections.[8]

V. ROBUST PERFORMANCE

The robust stability theorem gives necessary and sufficient conditions for the stability of a perturbed feedback system. In addition to checking whether $[P_\Delta, C]$ is internally stable, an engineer typically wishes to know the residual stability margin $b^{H_\Delta}(P_\Delta, C)$, where H_Δ is the generalized plant that corresponds to the perturbed plant P_Δ. We will now derive an upper and a lower bound for the stability margin of a perturbed feedback system, and an upper bound for $\|\mathcal{F}_l(H_\Delta, C) - \mathcal{F}_l(H, C)\|_\infty$ which is a measure of the difference in performance between the nominal and perturbed feedback systems.

To derive the bounds, we will need the following technical lemma.

Lemma 2: Given a nominal plant $P \in \mathcal{R}^{p \times q}$, a generalized plant

$$
H = \begin{bmatrix} S_z & \vline & \\ \hline & \vline & \\ & \vline & I \end{bmatrix} \begin{bmatrix} I & -P & \vline & P \\ 0 & 0 & \vline & I \\ \hline I & -P & \vline & P \end{bmatrix} \times \begin{bmatrix} S_w & \vline & \\ \hline & \vline & \\ & \vline & I \end{bmatrix}
$$

where $S_w, S_z \in \mathcal{R}$, a controller $C \in \mathcal{R}^{q \times p}$ and a perturbation $\Delta \in \mathcal{R}$ such that:

- $(I - PC)^{-1} \in \mathcal{R}$;
- $(I - \mathcal{F}_l(H, C)\Delta)^{-1} \in \mathcal{R}$;
- $(I - H_{11}\Delta)^{-1} \in \mathcal{R}$.

[8]The generalized plant of the right 4-block interconnection is equal to
$$
\begin{bmatrix} 0 & P & \vline & P \\ 0 & I & \vline & I \\ \hline -I & P & \vline & P \end{bmatrix}
$$
and can be derived from Fig. 1 with $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} y \\ u \end{bmatrix}$. Hence, it is a *dual* problem and can be treated easily using similar techniques.

Suppose furthermore that $S_w(I - k\Delta S_z S_w)^{-1} \in \mathcal{R}$ for a given $k \in \{0, 1\}$ and let $S_{w_\Delta} = S_w(I - k\Delta S_z S_w)^{-1}$, $P_\Delta = \mathcal{F}_u(H, \Delta)$, the generalized plant

$$
H_\Delta = \begin{bmatrix} S_z & \vline & \\ \hline & \vline & \\ & \vline & I \end{bmatrix} \begin{bmatrix} I & -P_\Delta & \vline & P_\Delta \\ 0 & 0 & \vline & I \\ \hline I & -P_\Delta & \vline & P_\Delta \end{bmatrix} \times \begin{bmatrix} S_{w_\Delta} & \vline & \\ \hline & \vline & \\ & \vline & I \end{bmatrix}
$$

and $S = (1 - k)S_z S_w$. Then

$$
\mathcal{F}_l(H_\Delta, C) = \mathcal{F}_l(H, C) + \mathcal{F}_l(H, C)\Delta(\mathcal{F}_l(H_\Delta, C) - S).
$$

Proof: First consider the case when $k = 0$. For simplicity, we will only prove the case when $S_w = I$ and $S_z = I$ (that is, left 4-block uncertainty), as the proof for general S_w and S_z is almost identical. Now note that

$$
\begin{aligned}
H_\Delta &= \begin{bmatrix} I & -P_\Delta & P_\Delta \\ 0 & 0 & I \\ I & -P_\Delta & P_\Delta \end{bmatrix} \\
&= \mathcal{F}_u \left(\begin{bmatrix} 0 & \vline & 0 & -I & I \\ \hline I & \vline & I & 0 & 0 \\ 0 & \vline & 0 & 0 & I \\ I & \vline & I & 0 & 0 \end{bmatrix}, \mathcal{F}_u(H, \Delta) \right) \\
&= \mathcal{F}_u \left(\underbrace{\begin{bmatrix} I & -P & \vline & 0 & -P & P \\ 0 & 0 & \vline & 0 & -I & I \\ \hline I & -P & \vline & I & -P & P \\ 0 & 0 & \vline & 0 & 0 & I \\ I & -P & \vline & I & -P & P \end{bmatrix}}_{J}, \Delta \right).
\end{aligned}
$$

By expanding $\left(I - H \begin{bmatrix} \Delta & 0 \\ 0 & C \end{bmatrix} \right)$ in terms of its Schur complements, we get

$$
\begin{aligned}
\det &\left(I - H \begin{bmatrix} \Delta & 0 \\ 0 & C \end{bmatrix} \right) \\
&= \det(I - H_{11}\Delta)\det(I - P_\Delta C) \\
&= \det(I - PC)\det(I - \mathcal{F}_l(H, C)\Delta).
\end{aligned}
$$

Since $(I - PC)$ and $(I - \mathcal{F}_l(H, C)\Delta)$ are invertible in $\mathcal{R}$ and $(I - H_{11}\Delta) \in \mathcal{R}$, it follows that $\det(I - P_\Delta C)(\infty) \neq 0$ and thus $(I - P_\Delta C)$ is also invertible in $\mathcal{R}$. Now observe that

$$
\begin{aligned}
\mathcal{F}_l&(H_\Delta, C) \\
&= \mathcal{F}_l(\mathcal{F}_u(J, \Delta), C) \\
&= \mathcal{F}_l \left(\mathcal{F}_u \left(\begin{bmatrix} 0 & \vline & -I & I \\ \hline 0 & \vline & 0 & I \\ I & \vline & I & 0 \end{bmatrix} \star H, \Delta \right), C \right)
\end{aligned}
$$

$$
= \mathcal{F}_l \left(\left[\begin{array}{c|c} 0 & I \\ \hline I - \Delta & \Delta \end{array} \right], \mathcal{F}_l(H,C) \right)
$$
$$
= \mathcal{F}_l(H,C)(I - \Delta \mathcal{F}_l(H,C))^{-1}(I - \Delta)
$$
$$
= (I - \mathcal{F}_l(H,C)\Delta)^{-1}\mathcal{F}_l(H,C)(I - \Delta).
$$

Therefore, $\mathcal{F}_l(H_\Delta, C) \in \mathcal{R}$ and

$$
\mathcal{F}_l(H_\Delta, C) = (I - \mathcal{F}_l(H,C)\Delta)^{-1}\mathcal{F}_l(H,C)(I - \Delta)
$$
$$
\Leftrightarrow (I - \mathcal{F}_l(H,C)\Delta)\mathcal{F}_l(H_\Delta, C) = \mathcal{F}_l(H,C)(I - \Delta)
$$
$$
\Leftrightarrow \mathcal{F}_l(H_\Delta, C) = \mathcal{F}_l(H,C)
$$
$$
+ \mathcal{F}_l(H,C)\Delta(\mathcal{F}_l(H_\Delta, C) - I).
$$

This concludes the proof for the first case.

Now consider the case when $k = 1$. From the first case (for $k = 0$) with general S_w and S_z, we have the equation shown at the bottom of the page, which can be rearranged as

$$
S_z \mathcal{F}_l \left(\left[\begin{array}{cc|c} I & -P_\Delta & P_\Delta \\ 0 & 0 & I \\ \hline I & -P_\Delta & P_\Delta \end{array} \right], C \right) S_w
$$
$$
= \mathcal{F}_l(H,C)(I - \Delta S_z S_w) + \mathcal{F}_l(H,C)\Delta
$$
$$
\times S_z \mathcal{F}_l \left(\left[\begin{array}{cc|c} I & -P_\Delta & P_\Delta \\ 0 & 0 & I \\ \hline I & -P_\Delta & P_\Delta \end{array} \right], C \right) S_w.
$$

The result follows by multiplying both sides of the above equation by $(I - \Delta S_z S_w)^{-1}$ and noting that in this case

$$
\mathcal{F}_l(H_\Delta, C) = S_z \mathcal{F}_l \left(\left[\begin{array}{cc|c} I & -P_\Delta & P_\Delta \\ 0 & 0 & I \\ \hline I & -P_\Delta & P_\Delta \end{array} \right], C \right)
$$
$$
\times S_w(I - \Delta S_z S_w)^{-1}.
$$

■

Note that from Lemma 2 we see that as long as $(I - \Delta S)^{-1} \in \mathcal{R}$, then $\mathcal{F}_l(H,C) = 0$ if and only if $\mathcal{F}_l(H_\Delta, C) = 0$. This is important for the definition of $b^{H_\Delta}(P_\Delta, C)$ (see Definition 3).

The following theorem bounds the stability margin of a perturbed feedback system, and the performance difference between the nominal and perturbed feedback systems.

Theorem 3 (Robust Performance): Given a nominal plant $P \in \mathcal{R}^{p \times q}$, a stabilizable generalized plant

$$
H = \left[\begin{array}{c|c} S_z & \\ \hline & I \end{array} \right] \left[\begin{array}{cc|c} I & -P & P \\ 0 & 0 & I \\ \hline I & -P & P \end{array} \right]
$$
$$
\times \left[\begin{array}{c|c} S_w & \\ \hline & I \end{array} \right]
$$

where $S_w, S_z \in \mathcal{R}$, a perturbed plant $P_\Delta \in \mathcal{R}^{p \times q}$ and a controller $C \in \mathcal{R}^{q \times p}$ such that $d^H(P, P_\Delta) < b^H(P, C)$ and $\mathbf{\Delta}^{\min} \neq \emptyset$. Assume furthermore that

$$
\exists \Delta \in \mathbf{\Delta}^{\min} : \eta(P_\Delta) = \eta(P) + \mathrm{wno} \det(I - H_{11}\Delta) \quad (7)
$$

where the winding number is evaluated on a contour indented to the right around any imaginary axis poles of P and P_Δ. Suppose furthermore that $S_w(I - kS_z S_w)^{-1} \in \mathcal{R}$ for a given $k \in \{0, 1\}$ and let $S_{w_\Delta} = S_w(I - kS_z S_w)^{-1}$, the generalized plant

$$
H_\Delta = \left[\begin{array}{c|c} S_z & \\ \hline & I \end{array} \right] \left[\begin{array}{cc|c} I & -P_\Delta & P_\Delta \\ 0 & 0 & I \\ \hline I & -P_\Delta & P_\Delta \end{array} \right]
$$
$$
\times \left[\begin{array}{c|c} S_{w_\Delta} & \\ \hline & I \end{array} \right]
$$

and $S = (1 - k)S_z S_w$.

Then the following results hold when $S \in \mathcal{RL}_\infty$ and $(I - \Delta S)^{-1} \in \mathcal{R}$:

(a) $0 \neq \mathcal{F}_l(H_\Delta, C) \in \mathcal{RL}_\infty$ and $[P_\Delta, C]$ is internally stable;

(b) $\left| b^{H_\Delta}(P_\Delta, C) - b^H(P, C) \right|$
      $\leq \|\mathcal{F}_l(H_\Delta, C) - S\|_\infty b^{H_\Delta}(P_\Delta, C)d^H(P, P_\Delta)$; and

(c) $\|\mathcal{F}_l(H_\Delta, C) - \mathcal{F}_l(H, C)\|_\infty$
      $\leq \dfrac{\|\mathcal{F}_l(H_\Delta, C) - S\|_\infty d^H(P, P_\Delta)}{b^H(P, C)}$.

Proof: We will only prove the case when $k = 0$, as the proof for the other case is almost identical. If $d^H(P, P_\Delta) < b^H(P, C)$, then $b^H(P, C) > 0$ and therefore by definition $0 \neq \mathcal{F}_l(H, C) \in \mathcal{RL}_\infty$ and $[P, C]$ is internally stable. By supposition $\mathbf{\Delta}^{\min} \neq \emptyset$. Thus

$$
d^H(P, P_\Delta) < b^H(P, C)
$$
$$
\Rightarrow \|\Delta \mathcal{F}_l(H, C)\|_\infty < 1 \quad \forall \Delta \in \mathbf{\Delta}^{\min}
$$

$$
S_z \mathcal{F}_l \left(\left[\begin{array}{cc|c} I & -P_\Delta & P_\Delta \\ 0 & 0 & I \\ \hline I & -P_\Delta & P_\Delta \end{array} \right], C \right) S_w = \mathcal{F}_l(H,C) + \mathcal{F}_l(H,C)\Delta S_z \left(\mathcal{F}_l \left(\left[\begin{array}{cc|c} I & -P_\Delta & P_\Delta \\ 0 & 0 & I \\ \hline I & -P_\Delta & P_\Delta \end{array} \right], C \right) - I \right) S_w
$$

$$\Leftrightarrow 0 < 1 - \overline{\sigma}(\Delta \mathcal{F}_l(H,C))(j\omega) \; \forall \, \omega, \; \forall \, \Delta \in \boldsymbol{\Delta}^{\min}$$
$$\Rightarrow 0 < \underline{\sigma}(I - \Delta \mathcal{F}_l(H,C))(j\omega) \; \forall \, \omega, \; \forall \Delta \in \boldsymbol{\Delta}^{\min}$$
$$\Leftrightarrow \det(I - \Delta \mathcal{F}_l(H,C))(j\omega) \neq 0 \; \forall \, \omega, \; \forall \, \Delta \in \boldsymbol{\Delta}^{\min}.$$

Thus, $(I - \Delta \mathcal{F}_l(H,C))^{-1} \in \mathcal{RL}_\infty$ for all $\Delta \in \boldsymbol{\Delta}^{\min}$. Since internal stability of $[P,C]$ implies that $(I - PC)^{-1} \in \mathcal{RH}_\infty$ and $\Delta \in \boldsymbol{\Delta}^{\min}$ implies that $(I - H_{11}\Delta)^{-1} \in \mathcal{R}$, from Lemma 2 and for all $\Delta \in \boldsymbol{\Delta}^{\min}$, we have

$$\mathcal{F}_l(H_\Delta, C) = \mathcal{F}_l(H,C)$$
$$+ \mathcal{F}_l(H,C)\Delta(\mathcal{F}_l(H_\Delta, C) - S_z S_w) \quad (8)$$
$$\Leftrightarrow (I - \mathcal{F}_l(H,C)\Delta)\mathcal{F}_l(H_\Delta, C) = \mathcal{F}_l(H,C)(I - \Delta S_z S_w)$$
$$\Leftrightarrow \mathcal{F}_l(H_\Delta, C) = \mathcal{F}_l(H,C)(I - \Delta \mathcal{F}_l(H,C))^{-1}$$
$$\times (I - \Delta S_z S_w). \quad (9)$$

Since $\Delta, (I - \Delta \mathcal{F}_l(H,C))^{-1} \in \mathcal{RL}_\infty$ for all $\Delta \in \boldsymbol{\Delta}^{\min}$, $S = S_z S_w \in \mathcal{RL}_\infty$ via suppositions and $b^H(P,C) > 0$ implies that $\mathcal{F}_l(H,C) \in \mathcal{RL}_\infty$, then from (9), $\mathcal{F}_l(H_\Delta, C) \in \mathcal{RL}_\infty$ for all $\Delta \in \boldsymbol{\Delta}^{\min}$. Also, $\mathcal{F}_l(H_\Delta, C) \neq 0$, for if it were equal to zero, then also $\mathcal{F}_l(H,C) = 0$ via (8) and the supposition $(I - \Delta S)^{-1} \in \mathcal{R}$, which is not possible as $b^H(P,C) > 0$.

From Theorem 1 and because by supposition there exists a $\Delta \in \boldsymbol{\Delta}^{\min}$ that satisfies the winding number condition, $[P_\Delta, C]$ is internally stable. This concludes the proof of part (a). Inequality (b) follows from the properties of the $\mathcal{L}_\infty$-norm and (8) as follows:

$$\|\mathcal{F}_l(H_\Delta, C)\|_\infty \leq \|\mathcal{F}_l(H,C)\|_\infty + \|\mathcal{F}_l(H,C)\|_\infty$$
$$\times \|\Delta\|_\infty \|\mathcal{F}_l(H_\Delta, C) - S_z S_w\|_\infty$$
$$\Leftrightarrow \|\mathcal{F}_l(H,C)\|_\infty^{-1} \leq \|\mathcal{F}_l(H_\Delta, C)\|_\infty^{-1}$$
$$+ \|\mathcal{F}_l(H_\Delta, C) - S_z S_w\|_\infty \|\Delta\|_\infty \|\mathcal{F}_l(H_\Delta, C)\|_\infty^{-1}$$
$$(\text{dividing both sides by } \|\mathcal{F}_l(H,C)\|_\infty \|\mathcal{F}_l(H_\Delta, C)\|_\infty)$$
$$\Leftrightarrow \|\mathcal{F}_l(H,C)\|_\infty^{-1} - \|\mathcal{F}_l(H_\Delta, C)\|_\infty^{-1}$$
$$\leq \|\mathcal{F}_l(H_\Delta, C) - S_z S_w\|_\infty \|\Delta\|_\infty \|\mathcal{F}_l(H_\Delta, C)\|_\infty^{-1}.$$

Similarly [via (8)]

$$\|\mathcal{F}_l(H_\Delta, C)\|_\infty^{-1} - \|\mathcal{F}_l(H,C)\|_\infty^{-1}$$
$$\leq \|\mathcal{F}_l(H_\Delta, C) - S_z S_w\|_\infty \|\Delta\|_\infty \|\mathcal{F}_l(H_\Delta, C)\|_\infty^{-1}$$

and therefore

$$\left| \|\mathcal{F}_l(H_\Delta, C)\|_\infty^{-1} - \|\mathcal{F}_l(H,C)\|_\infty^{-1} \right|$$
$$\leq \|\mathcal{F}_l(H_\Delta, C) - S_z S_w\|_\infty \|\Delta\|_\infty \|\mathcal{F}_l(H_\Delta, C)\|_\infty^{-1}.$$

Since $b^H(P,C) > 0$, $\|\mathcal{F}_l(H,C)\|_\infty = 1/b^H(P,C)$ and since $\Delta \in \boldsymbol{\Delta}^{\min}$, $\|\Delta\|_\infty = d^H(P, P_\Delta)$. The required inequality (b) follows by noting that since $0 \neq \mathcal{F}_l(H_\Delta, C) \in \mathcal{RL}_\infty$ and $[P_\Delta, C]$ is internally stable, $\|\mathcal{F}_l(H_\Delta, C)\|_\infty = 1/b^{H_\Delta}(P_\Delta, C)$.

We will now prove inequality (c). Again from the properties of the $\mathcal{L}_\infty$-norm and (8), we get

$$\|\mathcal{F}_l(H_\Delta, C) - \mathcal{F}_l(H,C)\|_\infty$$
$$\leq \|\mathcal{F}_l(H,C)\|_\infty \|\Delta\|_\infty \|\mathcal{F}_l(H_\Delta, C) - S_z S_w\|_\infty.$$

The result then follows. ∎

Note that it is straightforward to derive analogues of both inequalities that are valid pointwise in frequency. Condition (7) and the distance measure $d^H(P, P_\Delta)$ in Theorem 3 will be specialized to easily computable formulae through the systematic procedure discussed in Section VI. This procedure will allow us to evaluate condition (7) and the distance measure $d^H(P, P_\Delta)$ on a convenient $\Delta \in \boldsymbol{\Delta}^{\min}$ to produce simple compact formulae such as those given in (20) and (21) for the (not necessarily normalized) left coprime factor uncertainty case and such as the simple formulae given in the second and third columns of Table II in Section VIII in the additive, input multiplicative, output inverse multiplicative and left four-block uncertainty cases.

If $\|\mathcal{F}_l(H_\Delta, C) - S\|_\infty \, b^{H_\Delta}(P_\Delta, C) \leq 1$, as is the case for additive, inverse additive, left 4-block and coprime factor uncertainties, we can state the following corollary which is a more concise version of the robust performance theorem.

Corollary 4: Given a nominal plant $P \in \mathcal{R}^{p \times q}$, a perturbed plant $P_\Delta \in \mathcal{R}^{p \times q}$ and a controller $C \in \mathcal{R}^{q \times p}$. Let

- $S_w = \begin{bmatrix} I \\ 0 \end{bmatrix}$, $S_z = \begin{bmatrix} 0 & I \end{bmatrix}$ and $k = 0$ (that is, additive uncertainty); or
- $S_w = \begin{bmatrix} 0 \\ I \end{bmatrix}$, $S_z = \begin{bmatrix} I & 0 \end{bmatrix}$ and $k = 0$ (that is, inverse additive uncertainty); or
- $S_w = I$, $S_z = I$ and $k = 0$ (that is, left 4-block uncertainty); or
- $S_w = \begin{bmatrix} (R\tilde{M})^{-1} \\ 0 \end{bmatrix}$, $S_z = I$ and $k = 1$ (that is, non-normalized coprime factor uncertainty).[9]

Define

$$H = \left[\begin{array}{cc|c} S_z & & \\ \hline & & I \end{array} \right] \left[\begin{array}{cc|c} I & -P & P \\ 0 & 0 & I \\ \hline I & -P & P \end{array} \right] \\ \times \left[\begin{array}{c|c} S_w & \\ \hline & I \end{array} \right]$$

and suppose that $\boldsymbol{\Delta}^{\min} \neq \emptyset$ and

$$\exists \, \Delta \in \boldsymbol{\Delta}^{\min} : \eta(P_\Delta) = \eta(P) + \mathrm{wno}\det(I - H_{11}\Delta) \quad (10)$$

where the winding number is evaluated on a contour indented to the right around any imaginary axis poles of P and P_Δ. Also, let

$$H_\Delta = \left[\begin{array}{cc|c} S_z & & \\ \hline & & I \end{array} \right] \left[\begin{array}{cc|c} I & -P_\Delta & P_\Delta \\ 0 & 0 & I \\ \hline I & -P_\Delta & P_\Delta \end{array} \right] \\ \times \left[\begin{array}{c|c} S_{w_\Delta} & \\ \hline & I \end{array} \right]$$

where $S_{w_\Delta} = S_w(I - k\Delta S_z S_w)^{-1}$. Then

$$b^{H_\Delta}(P_\Delta, C) \geq b^H(P,C) - d^H(P, P_\Delta). \quad (11)$$

[9]Here, $\{\bar{N}, \bar{M}\}$ is a normalized left coprime factorization of P and $R \in \mathcal{GH}_\infty$.

Furthermore, when $b^H(P,C) > d^H(P,P_\Delta)$, it follows that:

$$\|\mathcal{F}_l(H_\Delta,C) - \mathcal{F}_l(H,C)\|_\infty \leq \frac{d^H(P,P_\Delta)}{b^H(P,C)b^{H_\Delta}(P_\Delta,C)}. \quad (12)$$

Proof: First suppose that $b^H(P,C) \leq d^H(P,P_\Delta)$. Then the right-hand-side of inequality (11) is ≤ 0. But from Definition 3, $b^{H_\Delta}(P_\Delta,C) \geq 0$ and hence (11) is automatically true.

Now suppose that $b^H(P,C) > d^H(P,P_\Delta)$. The proof follows from Theorem 3 on noting that the special structure of H implies that it is stabilizable, and that in the case of coprime factor uncertainty

$$S_{w_\Delta} = S_w(I - \Delta S_z S_w)^{-1}$$
$$= \begin{bmatrix} (R\tilde{M})^{-1} \\ 0 \end{bmatrix} \left(I - \Delta \begin{bmatrix} (R\tilde{M})^{-1} \\ 0 \end{bmatrix}\right)^{-1}$$

belongs to $\mathcal{R}$ for all $\Delta \in \boldsymbol{\Delta}^{\min}$ because by definition

$$(I - H_{11}\Delta)^{-1} = \left(I - \begin{bmatrix} (R\tilde{M})^{-1} \\ 0 \end{bmatrix} \Delta\right)^{-1}$$

belongs to $\mathcal{R}$ for all $\Delta \in \boldsymbol{\Delta}^{\min}$. Inequalities (11) and (12) immediately follow from conditions (b) and (c) of Theorem 3 on noting that for each of the cases considered $\|\mathcal{F}_l(H_\Delta,C) - S\|_\infty b^{H_\Delta}(P_\Delta,C) \leq 1$. Note that in the four-block uncertainty case, $\|\mathcal{F}_l(H_\Delta,C) - I\|_\infty = \|\mathcal{F}_l(H_\Delta,C)\|_\infty$ (see for example [9]) and $\forall \Delta \in \boldsymbol{\Delta}^{\min}, \|\Delta\|_\infty = d^H(P,P_\Delta) < b^H(P,C) \leq 1$ (see [20] for last-most inequality) implying $(I-\Delta)^{-1} \in \mathcal{R}$. ∎

VI. Systematic Procedure for Rewriting the Formula for $d^H(P,P_\Delta)$ and the Winding Number Condition Independent of Δ

It was foreshadowed earlier that we would give a systematic procedure that explicitly characterizes the set $\boldsymbol{\Delta}^{\min}$ in terms of P, H and P_Δ only. Such an explicit characterization then allows us to pick a convenient $\Delta \in \boldsymbol{\Delta}^{\min}$ to express the distance measure $d^H(P,P_\Delta)$ and winding number conditions (3), (7) and (10) independent of Δ, thereby yielding simple, compact and easily computable formulae. The procedure in this section produces precisely this explicit characterization.

Given $P \in \mathcal{R}^{p\times q}$, $H \in \mathcal{R}$ with $H_{22} = P$, and $P_\Delta \in \mathcal{R}^{p\times q}$:
1) Assume that $(I - H_{11}\Delta)^{-1} \in \mathcal{R}$ and rewrite

$$P_\Delta = \mathcal{F}_u(H,\Delta) = H_{22} + H_{21}\Delta(I-H_{11}\Delta)^{-1}H_{12}$$

as $A = B\Delta C$ for some $A, B, C \in \mathcal{R}$ independent of Δ. This is possible if:
   i) H_{21} is invertible in $\mathcal{R}$; or
   ii) H_{12} is invertible in $\mathcal{R}$; or
   iii) $H_{11} = DH_{21}$; or
   iv) $H_{11} = H_{12}D$;
   for some $D \in \mathcal{R}$ independent of Δ.
2) Solve $A = B\Delta C$ for all $\Delta \in \mathcal{RL}_\infty$. To do this we may have to impose restrictions on P and P_Δ. The set of all $\Delta \in \mathcal{RL}_\infty$ that satisfy consistency of equations ($A = B\Delta C$) is typically parameterized in terms of free parameters $Q_i \in \mathcal{RL}_\infty$, if any.

3) Derive conditions on P, P_Δ and Q_i so that $(I - H_{11}\Delta)^{-1} \in \mathcal{R}$; that is, well-posedness holds.[10]
4) Explicitly characterize the set $\boldsymbol{\Delta}$ in terms of P, P_Δ and Q_i and provide a simple compact formula in terms of P and P_Δ for the distance measure $d^H(P,P_\Delta)$ independent of Δ.[11]
5) Derive conditions on P, P_Δ and Q_i so that $\boldsymbol{\Delta}^{\min} \neq \emptyset$, and rewrite the winding number condition independent of $\Delta \in \boldsymbol{\Delta}^{\min}$. The set $\boldsymbol{\Delta}^{\min}$ is empty when the well-posedness condition is not satisfied. Since it suffices to evaluate the winding number condition for any single $\Delta \in \boldsymbol{\Delta}^{\min}$, choose an easy $\Delta \in \boldsymbol{\Delta}^{\min}$ that leads to a simple winding number condition, e.g., by setting $Q_i = 0$.
6) State the robust stability and robust performance theorems. Ensure that the suppositions of these theorems include any restrictions on P and P_Δ from steps 2, 3 and 5.

It is worth pointing out practical design cases where the above procedure can be used. Indeed, this procedure can be used on all ten cases given in Table I in Section IV and also their duals. Consequently, all of the standard additive, multiplicative, inverse multiplicative, coprime factor and four-block uncertainty structures can be handled by this systematic procedure to explicitly characterize set $\boldsymbol{\Delta}$ in terms of only P, P_Δ and arbitrary parameters Q_i, if any, and also various mixes of these uncertainty structures. Examples of simple, compact and easily computable formulae for the distance measure $d^H(P,P_\Delta)$ and winding number conditions (3), (7) and (10) when expressed independently of Δ through application of the above procedure are those given in (20) and (21) for the (not necessarily normalized) left coprime factor uncertainty case and those given in the second and third columns of Table II in Section VIII in the additive, input multiplicative, output inverse multiplicative and left four-block uncertainty cases.

VII. Coprime Factor Uncertainty Characterization with a Coprime Factor Performance Measure

In this section, we illustrate the novel generic procedure described in Section VI on a (not necessarily normalized) coprime factor uncertainty characterization with a coprime factor performance measure and derive results specific for this case. If the coprime factors were restricted to be normalized, then the results of this section would be closely linked to a four-block uncertainty characterization with a four-block performance measure (see [1] for the results in that case). However, we make no such assumption here and consider coprime factors that may or may not be normalized. In so doing, we also point out that non-normalized coprime factors can be of interest because they give rise to extra degrees of freedom (beyond the 4-block results) which could be exploited in design (see illustrative example Section X).

We will derive the results for the left coprime factor uncertainty characterization with left coprime factor performance

[10]Steps 2 and 3 parameterize all $\Delta \in \mathcal{RL}_\infty$ that satisfy $(I - H_{11}\Delta)^{-1} \in \mathcal{R}$ and $A = B\Delta C$. But since $(I-H_{11}\Delta)^{-1} \in \mathcal{R}$, then from step 1 $P_\Delta = \mathcal{F}_u(H,\Delta) \Leftrightarrow A = B\Delta C$. Therefore, steps 2 and 3 parameterize all $\Delta \in \mathcal{RL}_\infty$ that satisfy $(I - H_{11}\Delta)^{-1} \in \mathcal{R}$ and $P_\Delta = \mathcal{F}_u(H,\Delta)$.

[11]There is evidence that $d^H(P,P_\Delta)$ can be rewritten independently of Δ also when $\boldsymbol{\Delta}$ is a structured uncertainty set. This will be reported in a separate paper.

measure only and then show how these results can be easily modified to get dual results for the right coprime factor uncertainty characterization with right coprime factor performance measure.

A. Both Left Coprime

Consider Fig. 3 again and note that for this particular case $S_w = \begin{bmatrix} \tilde{M}_0^{-1} \\ 0 \end{bmatrix}$ and $S_z = I$ (where $\{\tilde{N}_0, \tilde{M}_0\}$ are left coprime factors (not necessarily normalized) over $\mathcal{RH}_\infty$ of P giving $P = \tilde{M}_0^{-1}\tilde{N}_0$)[12], as also seen in Table I in Section IV; that is we are considering the representation

$$P_\Delta = \tilde{M}_{\Delta,0}^{-1}\tilde{N}_{\Delta,0} = (\tilde{M}_0 - \Delta_{\tilde{M}})^{-1}(\tilde{N}_0 + \Delta_{\tilde{N}}). \quad (13)$$

It will be shown later in this section that when certain mild assumptions are fulfilled, $\{\tilde{N}_{\Delta,0}, \tilde{M}_{\Delta,0}\}$ are left coprime factors over $\mathcal{RL}_\infty$ of P_Δ for all $\Delta \in \mathbf{\Delta}^{\min}$ (see Lemma 8).

1) Define the Stability Margin $b_{lcf}(P,C)$: Plugging in $S_w = \begin{bmatrix} \tilde{M}_0^{-1} \\ 0 \end{bmatrix}$ and $S_z = I$ in (4) gives

$$H = \left[\begin{array}{ccc} \tilde{M}_0^{-1} & | & P \\ 0 & | & I \\ -\!-\!- & | & -\!-\!- \\ \tilde{M}_0^{-1} & | & P \end{array}\right]$$

and hence via (6), we get

$$\mathcal{F}_l(H,C) = \begin{bmatrix} I \\ C \end{bmatrix}(I - PC)^{-1}\tilde{M}_0^{-1}.$$

Consequently, straight from Definition 3, we see that given P, C and any (not necessarily normalized) left coprime factors $\{\tilde{N}_0, \tilde{M}_0\}$ over $\mathcal{RL}_\infty$ of P, the stability margin $b_{lcf}(P,C)$ for left coprime factor uncertainty characterizations is given by

$$b_{lcf}(P,C) := \begin{cases} \left\|\begin{bmatrix} I \\ C \end{bmatrix}(I - PC)^{-1}\tilde{M}_0^{-1}\right\|_\infty^{-1} \\ \qquad \text{if } [P,C] \text{ is internally stable,} \\ 0 \qquad \text{otherwise.} \end{cases} \quad (14)$$

In this specific case, $0 \neq \mathcal{F}_l(H,C) \in \mathcal{RL}_\infty$ is dropped in (14) since $[P,C]$ is internally stable automatically guarantees this via the following argument:

$$[P,C] \text{ is internally stable}$$
$$\Leftrightarrow \begin{bmatrix} P \\ I \end{bmatrix}(I - CP)^{-1}[-C \quad I] \in \mathcal{RH}_\infty$$
$$\Leftrightarrow \begin{bmatrix} I \\ C \end{bmatrix}(I - PC)^{-1}[I \quad -P] \in \mathcal{RH}_\infty$$
$$\Leftrightarrow \mathcal{F}_l(H,C)[-\tilde{M}_0 \quad \tilde{N}_0] \in \mathcal{RH}_\infty$$
$$\Rightarrow \mathcal{F}_l(H,C) \in \mathcal{RL}_\infty.$$

The last implication follows by noting that $[-\tilde{M}_0 \quad \tilde{N}_0]$ has a right inverse in $\mathcal{RL}_\infty$ whenever $\{\tilde{N}_0, \tilde{M}_0\}$ are left coprime

[12]One can easily rewrite this section using (not necessarily normalized) left coprime factors over $\mathcal{RL}_\infty$, rather than $\mathcal{RH}_\infty$, for the nominal plant P. It is unclear, though, whether this would be useful or not!

factors over $\mathcal{RL}_\infty$. Also, $\mathcal{F}_l(H,C) = \begin{bmatrix} I \\ C \end{bmatrix}(I - PC)^{-1}\tilde{M}_0^{-1}$ is clearly never equal to zero for all $P, C, \tilde{M}_0 \in \mathcal{R}$.

2) Solve Consistency Equation "$A = B\Delta C$" for all $\Delta \in \mathcal{RL}_\infty$: In this specific case, (5) reduces to

$$P_\Delta - P = \tilde{M}_0^{-1}\Delta\begin{bmatrix} P_\Delta \\ I \end{bmatrix} \quad (15)$$

on substituting $S_w = \begin{bmatrix} \tilde{M}_0^{-1} \\ 0 \end{bmatrix}$ and $S_z = I$.

In order to manipulate the subsequent equations and in the sequel of this section, let $\{\tilde{N}, \tilde{M}\}$ be normalized left coprime factors over $\mathcal{RH}_\infty$ of P and $\{\tilde{N}_0 = R\tilde{N}, \hat{M}_0 = R\tilde{M}\}$ be (not necessarily normalized) left coprime factors over $\mathcal{RH}_\infty$ of P related to the normalized factors via $R \in \mathcal{GH}_\infty$. Also, let $\{N_\Delta, M_\Delta\}$ be normalized right coprime factors over $\mathcal{RH}_\infty$ of P_Δ. Equation (15) can then be equivalently rearranged as

$$P_\Delta - P = \tilde{M}_0^{-1}\Delta\begin{bmatrix} P_\Delta \\ I \end{bmatrix}$$
$$\Leftrightarrow R\tilde{M}(P - P_\Delta)M_\Delta = -\Delta\begin{bmatrix} N_\Delta \\ M_\Delta \end{bmatrix}$$
$$\Leftrightarrow R\tilde{G}G_\Delta = -\Delta G_\Delta \quad (16)$$
$$\Leftrightarrow R\tilde{G}G_\Delta = -\Delta[\tilde{G}_\Delta^* \quad G_\Delta]\begin{bmatrix} \tilde{G}_\Delta \\ G_\Delta^* \end{bmatrix}G_\Delta$$
$$\Leftrightarrow R\tilde{G}G_\Delta = -\Delta[\tilde{G}_\Delta^* \quad G_\Delta]\begin{bmatrix} 0 \\ I \end{bmatrix}$$
$$\Leftrightarrow \Delta = [Q \quad -R\tilde{G}G_\Delta]\begin{bmatrix} \tilde{G}_\Delta \\ G_\Delta^* \end{bmatrix} \quad (17)$$

where $Q \in \mathcal{R}$ is an arbitrary object. Note that since $\Delta \in \mathcal{RL}_\infty$ if and only if $Q \in \mathcal{RL}_\infty$, the arbitrary object $Q \in \mathcal{R}$ will be restricted to $Q \in \mathcal{RL}_\infty$.

Thus, there always exist a $\Delta \in \mathcal{RL}_\infty$ that satisfies consistency of equations for any given P, P_Δ data. Also, given any P, P_Δ pair, there in fact always exist multiple solutions for $\Delta \in \mathcal{RL}_\infty$, and the non-uniqueness in the solution Δ is parameterized above by the arbitrary object $Q \in \mathcal{RL}_\infty$.

3) Derive Conditions to Guarantee Well-Posedness of $\mathcal{F}_u(H,\Delta)$: We now wish to make a connection between the consistency equation given in (15) and the uncertainty characterization $P_\Delta = \mathcal{F}_u(H,\Delta)$. To this end, observe that

$$\det(I - H_{11}\Delta)$$
$$= \det\left(I - \begin{bmatrix} \tilde{M}_0^{-1} \\ 0 \end{bmatrix}\Delta\right)$$
$$= \det\left(I + \begin{bmatrix} \tilde{M}^{-1} \\ 0 \end{bmatrix}R^{-1}[-Q \quad R\tilde{G}G_\Delta]\begin{bmatrix} \tilde{G}_\Delta \\ G_\Delta^* \end{bmatrix}\right)$$
$$= \det\left(I + \begin{bmatrix} \tilde{G}_\Delta \\ G_\Delta^* \end{bmatrix}\right.$$
$$\left. \times \left[\begin{pmatrix} -\tilde{M}^{-1}(R^{-1}Q) \\ 0 \end{pmatrix} \quad (GM^{-1}M_\Delta - G_\Delta)\right]\right)$$
$$= \det\left(\begin{bmatrix} \tilde{G}_\Delta \\ G_\Delta^* \end{bmatrix}\left[\tilde{G}_\Delta^* - \begin{pmatrix} \tilde{M}^{-1}(R^{-1}Q) \\ 0 \end{pmatrix} \quad GM^{-1}M_\Delta\right]\right).$$
$$\quad (18)$$

Consequently

$$\det(I - H_{11}\Delta)(\infty) \neq 0$$
$$\Leftrightarrow \det\left(\left[\tilde{G}_\Delta^* - \begin{pmatrix} \tilde{M}^{-1}(R^{-1}Q) \\ 0 \end{pmatrix} \quad G\right]\right)(\infty) \neq 0$$
$$\Leftrightarrow \det\left(\begin{bmatrix} -\tilde{M}_\Delta^* - \tilde{M}^{-1}(R^{-1}Q) & N \\ \tilde{N}_\Delta^* & M \end{bmatrix}\right)(\infty) \neq 0$$
$$\Leftrightarrow \det\left(-\tilde{M}_\Delta^* - \tilde{M}^{-1}(R^{-1}Q) - P\tilde{N}_\Delta^*\right)(\infty) \neq 0$$
$$\Leftrightarrow \det\left((R^{-1}Q) + \tilde{G}\tilde{G}_\Delta^*\right)(\infty) \neq 0. \tag{19}$$

This restricts the allowable $Q(\infty), R(\infty), P(\infty), P_\Delta(\infty)$ for well-posedness of the linear fractional transformation $\mathcal{F}_u(H, \Delta)$. Note also that there always exists a $Q \in \mathcal{RL}_\infty$ so that condition (19) is fulfilled.

Lemma 5: The choice $Q(\infty) = 0$ in equation (17) gives a $\Delta(\infty)$ that satisfies $\det(I - H_{11}\Delta)(\infty) \neq 0$ if and only if $\overline{\sigma}(\tilde{G}G_\Delta)(\infty) < 1$.

Proof: Choose $Q(\infty) = 0$ in (19). Then

$$\det(I - H_{11}\Delta)(\infty) \neq 0$$
$$\Leftrightarrow \underline{\sigma}\left(\tilde{G}\tilde{G}_\Delta^*\right)(\infty) > 0$$
$$\Leftrightarrow \overline{\sigma}(\tilde{G}G_\Delta)(\infty) < 1.$$

The last equivalence is due to $\underline{\sigma}(\tilde{G}\tilde{G}_\Delta^*)^2 = 1 - \overline{\sigma}(\tilde{G}G_\Delta)^2$ via [9, p. 121]. ∎

Under the restrictions imposed by (19), we can now make the connection

$$P_\Delta - P = \tilde{M}_0^{-1}\Delta\begin{bmatrix} P_\Delta \\ I \end{bmatrix} \Leftrightarrow P_\Delta = \mathcal{F}_u(H, \Delta)$$

as shown above (5).

Consequently, given a nominal plant P and a perturbed plant P_Δ, we have shown above that condition (19) guarantees that there exists a $\Delta \in \mathcal{RL}_\infty$ satisfying $P_\Delta = \mathcal{F}_u(H, \Delta)$. Also, when this condition is satisfied, there exist multiple solutions $\Delta \in \mathcal{RL}_\infty$ for $P_\Delta = \mathcal{F}_u(H, \Delta)$ given by (17) with $Q \in \mathcal{RL}_\infty$.

4) Define Solution Set $\mathbf{\Delta}$ and the Distance Measure $d_{lcf}(P, P_\Delta)$: We are now in a position to define the solution set $\mathbf{\Delta}$, which contains all feasible solutions $\Delta \in \mathcal{RL}_\infty$ satisfying $P_\Delta = \mathcal{F}_u(H, \Delta)$, for left coprime factor uncertainty characterizations. It is clear from Definition 4 and the preceding derivations that set $\mathbf{\Delta}$ reduces to

$$\mathbf{\Delta} = \left\{ \Delta = [Q \quad -R\tilde{G}G_\Delta]\begin{bmatrix} \tilde{G}_\Delta \\ G_\Delta^* \end{bmatrix} : \right.$$
$$\left. Q \in \mathcal{RL}_\infty, \det\left((R^{-1}Q) + \tilde{G}\tilde{G}_\Delta^*\right)(\infty) \neq 0 \right\}$$

in this specific case. The set $\mathbf{\Delta}$ is clearly never empty. Consequently, straight from Definition 4, the distance measure $d_{lcf}(P, P_\Delta)$ for left coprime factor uncertainty characterizations is given by

$$d_{lcf}(P, P_\Delta) = \inf_{\Delta \in \mathbf{\Delta}} \|\Delta\|_\infty$$
$$= \inf_{Q \in \mathcal{RL}_\infty} \left\| [Q \quad -R\tilde{G}G_\Delta]\begin{bmatrix} \tilde{G}_\Delta \\ G_\Delta^* \end{bmatrix} \right\|_\infty$$
$$= \inf_{Q \in \mathcal{RL}_\infty} \|[Q \quad -R\tilde{G}G_\Delta]\|_\infty$$
$$= \left\| R\tilde{G}G_\Delta \right\|_\infty \tag{20}$$

since $\begin{bmatrix} \tilde{G}_\Delta \\ G_\Delta^* \end{bmatrix}$ is all-pass. It is important to appreciate that the generic distance notion in Definition 4 reduced to a simple easy-to-compute formula given by (20) for the left coprime factor uncertainty setting. Similar specialisms will indeed be possible for all cases described in Table I (and their duals), some of which will be reported in Section VIII.

5) Write the Winding Number Condition Independent of Δ: An immediate corollary to Lemma 5 follows:

Corollary 6: The choice $Q = 0$ in equation (17) gives a $\Delta \in \mathbf{\Delta}^{\min}$ if and only if $\overline{\sigma}(\tilde{G}G_\Delta)(\infty) < 1$.

Consequently, whenever the $\overline{\sigma}(\tilde{G}G_\Delta)(\infty) < 1$ is assumed, winding number condition (3) reduces to

$$\eta(P_\Delta) - \eta(P) = \text{wno} \det\left(\begin{bmatrix} \tilde{G}_\Delta \\ G_\Delta^* \end{bmatrix} [\tilde{G}_\Delta^* \quad GM^{-1}M_\Delta]\right)$$

via (17) and (18) on choosing $Q = 0$. It can be easily seen that this condition can be rewritten more compactly as

$$\text{wno} \det(G_\Delta^* G) = 0 \tag{21}$$

which is easily computable as it is independent of Δ. It is important to appreciate that winding number condition (3) in Theorem 1 reduced to a simple easy-to-compute formula given by (21) for the left coprime factor uncertainty setting. Similar specialisms will indeed be possible for all cases described in Table I in Section IV (and their duals), some of which will be reported in Section VIII.

Notice that the assumption $\overline{\sigma}(\tilde{G}G_\Delta)(\infty) < 1$ excludes a thin set of plants, because it is a restriction at only infinite frequency and also only excludes the situation when $\overline{\sigma}(\tilde{G}G_\Delta)(\infty) = 1$ (e.g., $P(\infty) = 1$ and $P_\Delta(\infty) = -1$). Note also that if any one of P or P_Δ is strictly proper or the product of their gains at infinite frequency is strictly less than unity, as will typically happen in practice, this assumption is trivially satisfied.

6) State Robust Stability and Robust Performance Theorems: In this subsection, we rewrite the generic Robust Stability Theorem 1 and the generic Robust Performance Theorem 3 for this specific design case using the results derived in Section VII-A, 1)–5).

Theorem 7 (Robust Stability — Left Coprime Factor): Given a plant $P \in \mathcal{R}^{p \times q}$, a perturbed plant $P_\Delta \in \mathcal{R}^{p \times q}$, a controller $C \in \mathcal{R}^{q \times p}$ and (not necessarily normalized) left coprime factors $\{\tilde{N}_0, \tilde{M}_0\}$ over $\mathcal{RH}_\infty$ of P. Define normalized graph symbols $G, \tilde{G}, G_\Delta, \tilde{G}_\Delta$ as in the notation subsection and let $R \in \mathcal{GH}_\infty$ satisfy $[-\tilde{M}_0 \quad \tilde{N}_0] = R\tilde{G}$. Define a stability margin $b_{lcf}(P, C)$ as in (14) and a distance measure $d_{lcf}(P, P_\Delta)$ as in (20). Furthermore, suppose $d_{lcf}(P, P_\Delta) < b_{lcf}(P, C)$ and $\overline{\sigma}(\tilde{G}G_\Delta)(\infty) < 1$.

Then

$$[P_\Delta, C] \text{ is internally stable} \Leftrightarrow \text{wno} \det(G_\Delta^* G) = 0$$

where the winding number is evaluated on a contour indented to the right around any imaginary axis poles of P and P_Δ.

Proof: Trivial reduction of Theorem 1. The assumption that

$$H = \begin{bmatrix} \tilde{M}_0^{-1} & | & P \\ 0 & | & I \\ -- & | & -- \\ \tilde{M}_0^{-1} & | & P \end{bmatrix} \tag{22}$$

needs to be stabilizable is dropped since it is automatically guaranteed in this case via the following argument: Since $b_{lcf}(P, C) > d_{lcf}(P, P_\Delta) \geq 0$, it follows that $[P, C]$ is internally stable. But

$$[P, C] \text{ is internally stable}$$

$$\Leftrightarrow \begin{bmatrix} P \\ I \end{bmatrix} (I - CP)^{-1} [-C \quad I] \in \mathcal{RH}_\infty$$

$$\Leftrightarrow \begin{bmatrix} I \\ C \end{bmatrix} (I - PC)^{-1} [I \quad -P] \in \mathcal{RH}_\infty$$

$$\Leftrightarrow \mathcal{F}_l(H, C) [-\tilde{M}_0 \quad \tilde{N}_0] \in \mathcal{RH}_\infty$$

$$\Rightarrow \mathcal{F}_l(H, C) \in \mathcal{RH}_\infty$$

since $\{\tilde{N}_0, \tilde{M}_0\}$ are left coprime factors of P over $\mathcal{RH}_\infty$. Then, using the above observations in (2) with direct substitution from the specific entries of (22), one easily notices that $\langle H, C \rangle$ is internally stable. This shows that H is stabilizable.

Also, the supposition $\overline{\sigma}(\tilde{G} G_\Delta)(\infty) < 1$ is required to enable use of Lemma 5 and Corollary 6 for ease of characterization of the winding number condition. ∎

We will now show that when $d_{lcf}(P, P_\Delta) < b_{lcf}(P, C)$, $\{\tilde{N}_{\Delta,0}, \tilde{M}_{\Delta,0}\}$ as defined by (13) are left coprime factors over $\mathcal{RL}_\infty$ of P_Δ for all $\Delta \in \mathbf{\Delta}^{\min}$.

Lemma 8: Given the suppositions of Theorem 7. Then $\{\tilde{N}_{\Delta,0} = \tilde{N}_0 + \Delta_{\tilde{N}}, \tilde{M}_{\Delta,0} = \tilde{M}_0 - \Delta_{\tilde{M}}\}$ are left coprime factors over $\mathcal{RL}_\infty$ of $P_\Delta = \tilde{M}_{\Delta,0}^{-1} \tilde{N}_{\Delta,0}$ for all $\Delta = [\Delta_{\tilde{M}} \quad \Delta_{\tilde{N}}] \in \mathbf{\Delta}^{\min}$.

Proof: For all $\Delta \in \mathbf{\Delta}^{\min}$

$$(0 \leq) \sup_{\omega \in \mathbb{R}} \overline{\sigma}(\Delta) = \|\Delta\|_\infty = d_{lcf}(P, P_\Delta)$$

$$< b_{lcf}(P, C)$$

$$= \left\| (R\tilde{G}K)^{-1} \right\|_\infty^{-1} = \inf_{\omega \in \mathbb{R}} \underline{\sigma}(R\tilde{G}K)$$

$$\leq \inf_{\omega \in \mathbb{R}} \underline{\sigma}(R\tilde{G}).$$

This implies that $\overline{\sigma}(\Delta)(j\omega) < \underline{\sigma}(R\tilde{G})(j\omega) \ \forall \ \omega \in \mathbb{R} \cup \{\infty\}, \Delta \in \mathbf{\Delta}^{\min}$, which in turn implies that

$$[-\tilde{M}_{\Delta,0} \quad \tilde{N}_{\Delta,0}] = (R\tilde{G} + \Delta)$$

is right invertible for all $\omega \in \mathbb{R} \cup \{\infty\}$ and all $\Delta \in \mathbf{\Delta}^{\min}$. ∎

We now reduce Theorem 3 specifically for left coprime factor uncertainty characterizations with left coprime factor performance measures using the results derived in Section VII–A, 1)–5).

Theorem 9 (Robust Performance — Left Coprime Factor): Given the suppositions of Theorem 7 and furthermore assuming wno $\det(G_\Delta^* G) = 0$, where the winding number is evaluated on a contour indented to the right around any imaginary axis poles of P and P_Δ. Then

$$|b_{lcf}(P_\Delta, C) - b_{lcf}(P, C)| \leq d_{lcf}(P, P_\Delta) \qquad (23)$$

and

$$\|\mathcal{F}_l(H_\Delta, C) - \mathcal{F}_l(H, C)\|_\infty \leq \frac{d_{lcf}(P, P_\Delta)}{b_{lcf}(P, C) b_{lcf}(P_\Delta, C)} \qquad (24)$$

where

$$H = \begin{bmatrix} \tilde{M}_0^{-1} & | & P \\ 0 & | & I \\ -- & | & -- \\ \tilde{M}_0^{-1} & | & P \end{bmatrix} \quad \text{and}$$

$$H_\Delta = \begin{bmatrix} \tilde{M}_{\Delta,0}^{-1} & | & P_\Delta \\ 0 & | & I \\ -- & | & -- \\ \tilde{M}_{\Delta,0}^{-1} & | & P_\Delta \end{bmatrix}.$$

Proof: Trivial reduction of Theorem 3 on choosing $k = 1$ and noting that $S = 0 \in \mathcal{RL}_\infty$ thereby giving

$$\|\mathcal{F}_l(H_\Delta, C) - S\|_\infty = \|\mathcal{F}_l(H_\Delta, C)\|_\infty = \frac{1}{b_{lcf}(P_\Delta, C)}.$$

The second equality follows from definition (14) of $b_{lcf}(P_\Delta, C)$ on noting that $[P_\Delta, C]$ is internally stable via Theorem 7 and that $\{\tilde{N}_{\Delta,0}, \tilde{M}_{\Delta,0}\}$ are left coprime factors over $\mathcal{RL}_\infty$ of P_Δ for all $\Delta \in \mathbf{\Delta}^{\min}$ via Lemma 8. ∎

B. Both Right Coprime

The problem of right coprime factor uncertainty characterization, i.e.

$$P_\Delta = \mathcal{F}_u \left(\begin{bmatrix} 0 & -M_0^{-1} & | & M_0^{-1} \\ - & -- & | & -- \\ I & -P & | & P \end{bmatrix}, \Delta \right)$$

with a right coprime factor performance measure is simply a *dual problem* to that derived above for left coprime factor uncertainty characterization, i.e.,

$$P_\Delta = \mathcal{F}_u \left(\begin{bmatrix} \tilde{M}_0^{-1} & | & P \\ 0 & | & I \\ -- & | & -- \\ \tilde{M}_0^{-1} & | & P \end{bmatrix}, \Delta \right)$$

with a left coprime factor performance measure.

Consequently, the results in this case follow trivially by performing the following substitutions:

Left coprime factor	$\longrightarrow$	Right coprime factor
P	$\longrightarrow$	P^T
$\{\tilde{N}_0, \tilde{M}_0\}$	$\longrightarrow$	$\{N_0^T, M_0^T\}$
P_Δ	$\longrightarrow$	P_Δ^T
$\{\tilde{N}_{\Delta,0}, \tilde{M}_{\Delta,0}\}$	$\longrightarrow$	$\{N_{\Delta,0}^T, M_{\Delta,0}^T\}$
Δ	$\longrightarrow$	$\Delta^T \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$
C	$\longrightarrow$	C^T

in the definitions, theorems and results of Section VII-A.

VIII. SUMMARY OF OTHER RESULTS

While Theorems 1 and 3 are stated in a very powerful and generic form that lends itself to various different uncertainty structures, such generality restricts applicability because the generic distance notion $d^H(P, P_\Delta)$ involves an infimum over $\Delta \in \mathbf{\Delta}$ but the set $\mathbf{\Delta}$ has not been explicitly characterized in those general theorems. A systematic procedure was consequently given in Section VI that states how to explicitly characterize the set $\mathbf{\Delta}$ using basic linear algebra manipulations,

TABLE II
SPECIALIZATIONS FOR VARIOUS OTHER UNCERTAINTY
STRUCTURES AND PERFORMANCE MEASURES

Uncertainty structure with identical performance measure	Winding number conditions (3), (7) and (10)	Distance measure $d^H(P, P_\Delta)$	Stability margin $b^H(P, C)$
Additive $(k = 0)$	$\eta(P_\Delta) = \eta(P)$	$\|P_\Delta - P\|_\infty$ if $(P_\Delta - P) \in \mathscr{RL}_\infty$, ∞ otherwise.	$\|C(I - PC)^{-1}\|_\infty^{-1}$ if $[P, C]$ is internally stable, 0 otherwise.
Input Multiplicative $(k = 0)$	$\eta(P_\Delta) = \eta(P)$	$\|X^{-1}(P - P_\Delta)\|_\infty$ if *Condition I or II* holds, $\|(P^*P)^{-1}P^*(P - P_\Delta)\|_\infty$ if *Condition III* holds, ∞ otherwise.	$\|C(I - PC)^{-1}P\|_\infty^{-1}$ if $[P, C]$ is internally stable, 0 otherwise.
Output Inverse Multiplicative $(k = 0)$	wno $\det(PP_\Delta^\star) = \eta(P_\Delta) - \eta(P)$ if $p \leq q$, wno $\det(P_\Delta^\star P) = \eta(P_\Delta) - \eta(P)$ otherwise.	$\|(P_\Delta - P)U^{-1}\|_\infty$ if *Condition I or II* holds, $\|(P_\Delta - P)P_\Delta^\star(P_\Delta P_\Delta^\star)^{-1}\|_\infty$ if *Condition III* holds, ∞ otherwise.	$\|(I - PC)^{-1}\|_\infty^{-1}$ if $[P, C]$ is internally stable, 0 otherwise.
Left four-block $(k = 0)$	wno $\det(G_\Delta^\star G) = 0$	$\|\breve{G}G_\Delta\|_\infty$	$\left\|\begin{bmatrix} I \\ C \end{bmatrix}(I - PC)^{-1}[I \ \ -P]\right\|_\infty^{-1}$ if $[P, C]$ is internally stable, 0 otherwise.

and then the same procedure also states how to use this characterization to specialize the definition of the generic distance notion $d^H(P, P_\Delta)$ to obtain simple easily computable formulae for each of the standard uncertainty structures (e.g., additive, inverse additive, input or output multiplicative, input or output inverse multiplicative, coprime factor, four-block or other mixes of the above). Once the set $\boldsymbol{\Delta}$ is characterized, winding number conditions (3), (7) and (10) also reduce to a simpler condition by picking any arbitrarily convenient $\Delta \in \boldsymbol{\Delta}^{\min}$.[13]

The systematic procedure of Section VI was applied only (due to space constraints) to not necessarily normalized coprime factor uncertainty structure with identical performance measure to exemplify use of the step-by-step procedure. This procedure can be equally easily (actually, it will be easier in most other cases) applied to other standard uncertainty structures to give structure-by-structure specific distance measures, stability margins, winding number conditions, robust stability results and robust performance results that are *extremely* easy to compute. In this section, we state some of these "other results" without proof to assist the reader in the application of this material to other uncertainty structures not explored in detail herein. Note also that Corollary 4 nicely collects the robust performance results for additive, inverse additive, coprime factor and four-block uncertainty structures with identical performance measure in a simpler form than Theorem 3. Table II collects the foreshadowed results. Winding number conditions (3), (7) and (10) specialize to the simple easy-to-compute formulae in the second column of Table II in the additive, input multiplicative, output inverse multiplicative and four-block uncertainty settings whereas the generic distance notion $d^H(P, P_\Delta)$ of Definition 4 specializes to the simple easy-to-compute formulae in the third column of Table II in the same uncertainty cases. Each formula in Table II can be easily evaluated using standard Matlab commands.

In the "Input Multiplicative" case:
- *Condition I* means $P, P_\Delta \in \mathcal{R}^{p \times q}$ with $p = q$ satisfying $P(\infty)$ having full rank and $P^{-1}P_\Delta \in \mathcal{RL}_\infty$;
- *Condition II* means $P, P_\Delta \in \mathcal{R}^{p \times q}$ with $p < q$ satisfying $P(\infty)$ having full rank and $X^{-1}P_\Delta \in \mathcal{RL}_\infty$ (where $X \in \mathcal{R}^{p \times p}$ satisfies $XX^* = PP^*$);

[13]The argument immediately following the proof of Theorem 1 explains why *any* arbitrary $\Delta \in \boldsymbol{\Delta}^{\min}$ can be chosen.

- *Condition III* means $P, P_\Delta \in \mathcal{R}^{p \times q}$ with $p > q$ satisfying $P(\infty)$ having full rank, $(P^*P)^{-1}P^*P_\Delta \in \mathcal{RL}_\infty$ and $\breve{P}P_\Delta = 0$ (where $\breve{P} = \left[\begin{array}{c|c} A - BD^\dagger C & -BD^\dagger \\ \hline D_\perp^* C & D_\perp^* \end{array}\right] \in \mathcal{R}^{(p-q) \times p}$ when $P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$).

In the "Output Inverse Multiplicative" case:
- *Condition I* means $P, P_\Delta \in \mathcal{R}^{p \times q}$ with $p = q$ satisfying $P(\infty), P_\Delta(\infty)$ having full rank and $PP_\Delta^{-1} \in \mathcal{RL}_\infty$;
- *Condition II* means $P, P_\Delta \in \mathcal{R}^{p \times q}$ with $p > q$ satisfying $P(\infty), P_\Delta(\infty)$ having full rank and $PU^{-1} \in \mathcal{RL}_\infty$ (where $U \in \mathcal{R}^{q \times q}$ satisfies $U^*U = P_\Delta^* P_\Delta$);
- *Condition III* means $P, P_\Delta \in \mathcal{R}^{p \times q}$ with $p < q$ satisfying $P(\infty), P_\Delta(\infty)$ having full rank, $PP_\Delta^*(P_\Delta P_\Delta^*)^{-1} \in \mathcal{RL}_\infty$ and $P\bar{P}_\Delta = 0$ (where $\bar{P}_\Delta = \left[\begin{array}{c|c} A - BD^\dagger C & -BD_\perp^* \\ \hline D^\dagger C & D_\perp^* \end{array}\right] \in \mathcal{R}^{q \times (q-p)}$ when $P_\Delta = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$).

IX. INCLUDING WEIGHTS

Up to now, we have not discussed whether or not the robust stability and the robust performance theorems, and the procedure for computing the distance between two plants hold for generalized plants that include weighting functions. The short answer to this question is "Yes, they hold" as outlined below. In this section, we will discuss two weighting schemes. The first scheme, that will be referred to as the open loop weighting scheme, is to pre- and post-compensate the nominal and perturbed plants with weights W_1 and W_2 respectively. Therefore, the nominal plant becomes $W_2 P W_1$ and the perturbed plant becomes $W_2 P_\Delta W_1$. This is the weighting scheme that is used in $\mathcal{H}_\infty$ loop shaping [14, p. 486, Section 18.2], [21]. It should be evident that this open loop weighting scheme does not affect the applicability of the above-presented results. The second scheme, that will be referred to as the closed-loop weighting scheme, is to use the weights to directly shape closed-loop transfer functions, that is, in our framework to include the weights in S_w and S_z (see (6)). So for example in the case of input multiplicative uncertainty, we would replace S_w and S_z in Table I in Section IV with $S_w = \begin{bmatrix} 0 \\ W_1 \end{bmatrix}$ and $S_z = [0 \ \ W_2]$.

It should be clear that Theorem 1 (robust stability) and Theorem 3 (robust performance) hold for both the open and the closed-loop weighting schemes. Note though that the suppositions of the theorems may impose restrictions on the weights, for example, in the case of input multiplicative uncertainty and closed-loop weights, $W_1, W_2 \in \mathcal{RL}_\infty$ guarantees that $S \in \mathcal{RL}_\infty$ (see Theorem 3). Additional restrictions may be required to ensure that H is stabilizable for instance. Also, Corollary 4 holds as is for both open loop weights and with appropriate S_w and S_z for closed-loop weights in the cases of additive, inverse additive and coprime factor uncertainty. For the generalized plants listed in Table I in Section IV, that could include open or closed-loop weights, we can always rewrite $P_\Delta = \mathcal{F}_u(H, \Delta)$ as $A = B\Delta C$, see (5). Therefore, since the systematic procedure for computing the distance between

two plants (outlined in Section VI) can be applied with no weights, it will also hold for both weighting schemes (since the weights do not change the structure of the consistency equation $A = B\Delta C$). Observe also that in order to ensure well-posedness of $\mathcal{F}_u(H, \Delta)$ and that $\mathbf{\Delta}^{\min} \neq \emptyset$, one may have to impose additional restrictions on the weights.

X. ILLUSTRATIVE EXAMPLE

In this section, we use an example to motivate why a designer may want to consider one of the novel distance measures proposed in this paper, namely that derived for a non-normalized left coprime factor uncertainty setting in Section VII, as opposed to the existing ν-gap metric (and gap-metric).

Consider, for instance, the problem of robustly stabilising a flexible booster rocket. The aerodynamic forces that a rocket encounters on its way to orbit tend to make the rocket tumble. This instability can be controlled with feedback of pitch rate to thrust vectoring control. However, the elasticity of the rocket complicates the feedback control. A booster rocket model from thrust vectoring control input to pitch rate measured output is described in [22]. The model is of tenth order, and has two poles referred to as the aerodynamic short period and eight poles referred to as elastic modes. The short period of the rocket is unstable, one pole being at +1.00rad/sec and the other at -1.09rad/sec. All the elastic modes have a damping ratio of 0.01 and the mode frequencies are 6.13, 16.8, 36.7 and 52.1rad/sec. The model also has nine zeros. What makes this booster rocket model interesting (and more difficult to control) is that the natural frequency of the first elastic mode is close to the unstable short period frequency. Thus, to robustly stabilize the booster rocket, the first elastic mode will have to be phase stabilized as opposed to gain stabilized. The gain of the loop transfer function will be smaller than unity before and after the first elastic mode frequency, and close to or larger than unity at the frequency of the first elastic mode. A simple abstraction of this problem is that there is a pair of lightly damped poles which make the gain suddenly large in a frequency region where the gain due to the remaining dynamics should have been comparatively small. The dual type of problem is that where models have a lightly damped pair of zeros in a frequency region where the gain is large, i.e., before and after the natural frequency of the zeros the gain is larger than unity, but the gain is close to or smaller than unity around the natural frequency of the zeros.

The practical value of Theorem 7 will now be demonstrated using the following nominal model that contains an abstraction of the key interesting features of the booster rocket model:

$$P = \begin{bmatrix} \dfrac{5}{s^2} \dfrac{s^2+2(0.005)0.4s+0.4^2}{s^2+2(0.005)0.33s+0.33^2} & 0.1 \\ 0.1 & \dfrac{1}{2s^2} \dfrac{s^2+2(0.005)3s+3^2}{s^2+2(0.005)4s+4^2} \end{bmatrix}.$$

We will also use a perturbed model P_Δ that is constructed by increasing the frequency of the zeros of the (1,1) element of P
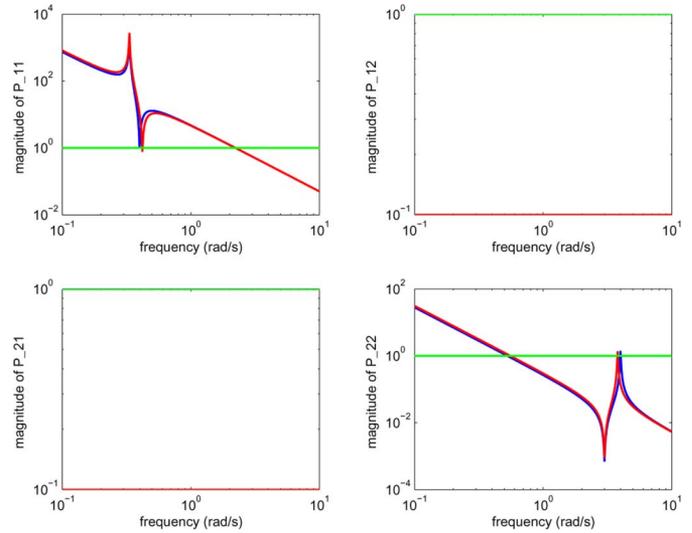

Fig. 4. Element-by-element magnitude Bode plots of the MIMO (2 input, 2 output) nominal (blue) and perturbed (red) plant models.

by 5% and decreasing the frequency of the poles of the (2,2) element by 5%. The element-by-element magnitude Bode plots of the two models are illustrated in Fig. 4. A controller C that robustly stabilizes P can be obtained in a number of ways, e.g., from optimal controller synthesis techniques like μ-synthesis or by using classical control concepts and engineering experience. For this example, we designed a controller using an appropriately weighted $\mathcal{H}_\infty$ cost function, the details of which are beyond the scope of this analysis paper. Matlab scripts for synthesising the controller and for performing all other computations in this example can be provided to the interested readers by emailing the authors.[14] The element-by-element magnitude Bode plot of the controller is given in Fig. 5. Note that the

[14]In this example, the 2-input and 2-output MIMO controller $C(s)$ is given by

$$C(s) = \begin{bmatrix} 0 & -0.1 \\ -0.1 & 0 \end{bmatrix}$$
$$+ \frac{1}{s + 537600} \begin{bmatrix} -23245 & 190930 \\ 217049 & -1782771 \end{bmatrix}$$
$$+ \frac{1}{s + 2.446} \begin{bmatrix} 0.237 & -1.876 \\ -1.823 & 14.407 \end{bmatrix}$$
$$+ \sum_{j=1}^{9} \begin{bmatrix} \psi_{4,(2j-1)} & \psi_{4,(2j)} \\ \psi_{5,(2j-1)} & \psi_{5,(2j)} \end{bmatrix}$$
$$\times \left(sI - \begin{bmatrix} \psi_{1,(2j-1)} & \psi_{1,(2j)} \\ -\psi_{1,(2j)} & \psi_{1,(2j-1)} \end{bmatrix} \right)^{-1}$$
$$\times \begin{bmatrix} \psi_{2,(2j-1)} & \psi_{3,(2j-1)} \\ \psi_{2,(2j)} & \psi_{3,(2j)} \end{bmatrix}$$

where $\psi_{i,j}$ is the ith row and jth column element of the matrix shown at the bottom of the page.

0.0120	0.389	−4.588	−4.927	−0.300	4.75	−0.790	3.881	−0.279	3.23	−0.447	1.721	−0.017	0.442	−0.023	0.355	−0.031	0.386
−0.961	−3.90	−4.23	−2.383	0.0488	1.34	0.809	1.155	0.940	0.506	0.006	−0.316	−1.591	−0.829	−0.987	1.759	0.042	−1.567
−0.242	0.073	−2.187	−3.65	−1.05	7.922	1.368	8.309	4.737	6.877	2.673	−4.058	0.0995	−0.007	0.195	−0.187	0.120	0.298
−0.005	0.025	3.881	−0.371	−0.029	0.022	−0.058	−0.015	−0.019	−0.021	0.0521	0.057	0.001	−0.024	−0.010	0.019	−0.008	0.024
0.001	0	2.636	−1.86	−0.153	0.148	−0.320	0.039	−0.168	−0.078	1.37	0.382	−0.002	0	0.001	0	0.002	0
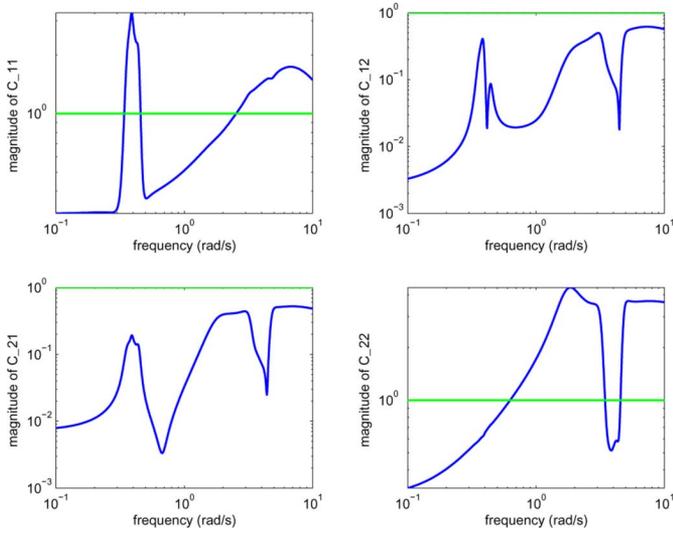
Fig. 5. Element-by-element magnitude Bode plots of the MIMO (2 input, 2 output) controller.

Fig. 7. Magnitude plot of $(\bar{G}K)^{-1}\bar{G}G_\Delta(j\omega)$.
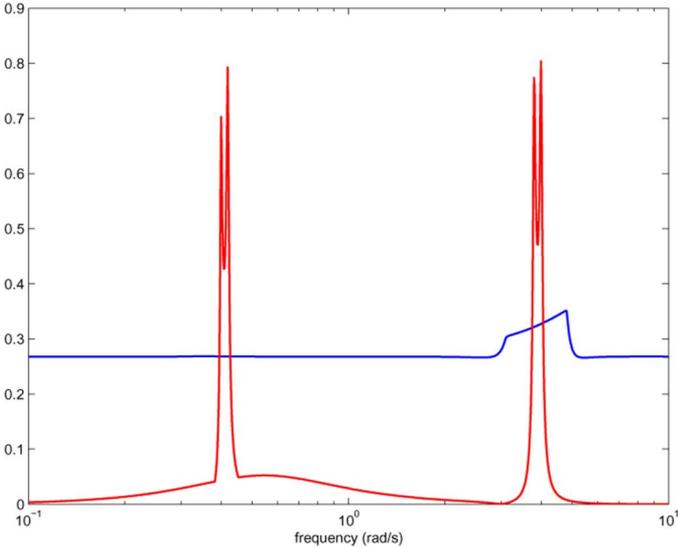

Fig. 6. Point-wise frequency analogues of $b(P,C)$ (blue) and $\delta_\nu(P,P_\Delta)$ (red).

controller does not invert the nominal model's lightly damped pair of zeros at 0.4 rad/sec and lightly damped pair of poles at 4 rad/sec. A controller that inverts these lightly damped modes would not be robust to small model perturbations.

Since $0.804 = \delta_\nu(P,P_\Delta) > b(P,C) = 0.266$, ν-gap analysis does not guarantee that $[P_\Delta,C]$ is stable. Fig. 6 illustrates the point-wise frequency analogues of $\delta_\nu(P,P_\Delta)$ and $b(P,C)$.

We will now apply Theorem 7. It is simple to check that $\overline{\sigma}(\tilde{G}G_\Delta)(\infty) < 1$ and $\operatorname{wno}\det(G_\Delta^*G) = 0$ are satisfied and this of course is also in agreement with $\delta_\nu(P,P_\Delta) < 1$. Thus, via Theorem 7, $[P_\Delta,C]$ is stable when

$$d_{lcf}(P,P_\Delta) < b_{lcf}(P,C)$$
$$\Leftrightarrow \left\|R\tilde{G}G_\Delta\right\|_\infty \left\|(R\tilde{G}K)^{-1}\right\|_\infty < 1$$

for any choice of $R \in \mathcal{G}\mathcal{H}_\infty$, recalling that R is the object that detunes normalization of coprime factors when it is different from a unitary matrix. Choose $R = (\tilde{G}K)^{-1}$ to minimize the
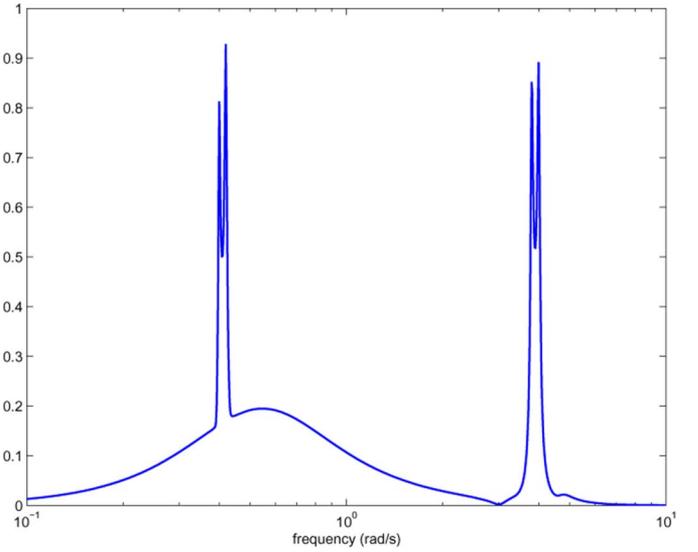
left-hand side of the last inequality. Note that $(\tilde{G}K)^{-1} \in \mathcal{G}\mathcal{H}_\infty$ is equivalent to $[P,C]$ is internally stable. Thus, the last inequality reduces to $\left\|(\tilde{G}K)^{-1}\tilde{G}G_\Delta\right\|_\infty < 1$. For the above example, $\left\|(\tilde{G}K)^{-1}\tilde{G}G_\Delta\right\|_\infty = 0.927$ and thus $[P_\Delta,C]$ is guaranteed to be stable through our analysis result of non-normalized coprime factor distance measure. Fig. 7 depicts the magnitude of $(\tilde{G}K)^{-1}\tilde{G}G_\Delta(j\omega)$ point-wise in frequency. Note that as expected and as in Fig. 6, the peaks are around the zero frequency of the (1,1) element of P and the pole frequency of the (2,2) element of P.

In summary, by applying Theorem 7 we were able to guarantee stability of $[P_\Delta,C]$ whereas with ν-gap analysis we were not. It is important to emphasis that Theorem 7 guarantees stability for a family of perturbed plants, not just for one perturbed plant. Also, this family of perturbed plants includes unstructured unmodelled dynamics in addition to parametric uncertainty. This family is given by

$$\mathcal{P}_{lcf} := \left\{ P_\Delta \in \mathcal{R} : \left\|(\tilde{G}K)^{-1}\tilde{G}G_\Delta\right\|_\infty < 1, \right.$$
$$\left. \overline{\sigma}(\tilde{G}G_\Delta)(\infty) < 1 \text{ and } \operatorname{wno}\det(G_\Delta^*G) = 0 \right\}.$$

Similarly ν-gap analysis also guarantees stability for a family of perturbed plants given by

$$\mathcal{P}_\nu := \left\{ P_\Delta \in \mathcal{R} : \left\|\tilde{G}G_\Delta\right\|_\infty \left\|(\tilde{G}K)^{-1}\right\|_\infty < 1, \right.$$
$$\left. \overline{\sigma}(\tilde{G}G_\Delta)(\infty) < 1 \text{ and } \operatorname{wno}\det(G_\Delta^*G) = 0 \right\}.$$

It is easy to see that $\mathcal{P}_\nu \subseteq \mathcal{P}_{lcf}$.[15] Thus, ν-gap analysis can never guarantee stability for a larger set of plants than Theorem 7. For our practically motivated example, $\mathcal{P}_\nu \subset \mathcal{P}_{lcf}$ (i.e., there are plants in $\mathcal{P}_{lcf}$ which are not in $\mathcal{P}_\nu$, e.g., the chosen perturbed plant P_Δ) and indeed the conservatism of $\mathcal{P}_\nu$ is significant compared to $\mathcal{P}_{lcf}$ because $\left\|\tilde{G}G_\Delta\right\|_\infty \left\|(\tilde{G}K)^{-1}\right\|_\infty =$

[15]For $\mathcal{P}_\nu$, the condition $\overline{\sigma}(\bar{G}G_\Delta)(\infty) < 1$ is always automatically fulfilled because of $\left\|\bar{G}G_\Delta\right\|_\infty \left\|(\tilde{G}K)^{-1}\right\|_\infty < 1$.

3.023 whereas $\left\|(\tilde{G}K)^{-1}\tilde{G}G_\Delta\right\|_\infty = 0.927$ implying that an engineer will get tighter guarantees using the distance measures developed in this paper.

XI. CONCLUSION

The material in this paper unifies under one elegant framework several disjoint pieces of work that appeared in the literature over the past 2–3 decades. We also show that the proposed framework can be specialized to situations not explored in the literature to hence also complement existing knowledge. The techniques used in this paper rely on basic linear algebra and hence also provide a simplification over previous advanced operator theory methods adopted in specific areas of the literature which our framework now embodies. The proposed theory also exploits a powerful generalization of the small-gain theorem to allow for perturbations in $\mathcal{RL}_\infty$, rather than only $\mathcal{RH}_\infty$. This produces significantly less conservative[16] results. We specialize the proposed generic notions of distance and stability margin with their correspondingly generic robust stability and robust performance results to not necessarily normalized coprime factor uncertainty (a situation not explored previously in the literature) and show via an illustrative example that the resulting distance measure, stability margin, and robustness theorems produce a design with far superior qualities than can be obtained by standard $\mathcal{H}_\infty$ design methods, such as $\mathcal{H}_\infty$ loop-shaping with the ν-gap metric measure.

REFERENCES

[1] G. Papageorgiou and A. Lanzon, "Distance measures, robust stability conditions and robust performance guarantees for uncertain feedback systems," in *Proc. Control Uncertain Systems: Modelling, Approximation Design—Workshop*, Cambridge, U.K., Apr. 2006, pp. 317–344.

[2] M. Vidyasagar, "The graph metric for unstable plants and robustness estimates for feedback stability," *IEEE Trans. Automat. Control*, vol. AC-29, no. 5, pp. 403–418, May 1984.

[3] A. K. El-Sakkary, "The gap metric: Robustness of stabilization of feedback systems," *IEEE Trans. Automat. Control*, vol. AC-30, no. 3, pp. 240–247, Mar. 1985.

[4] T. T. Georgiou and M. C. Smith, "Optimal robustness in the gap metric," *IEEE Trans. Automat. Control*, vol. 35, no. 6, pp. 673–686, Jun. 1990.

[5] L. Qiu and E. J. Davison, "Feedback stability under simultaneous gap metric uncertainties in plant and controller," *Syst. Control Lett.*, vol. 18, pp. 9–22, 1992.

[6] K. Glover, "All optimal Hankel-norm approximations of linear multivariable systems and their $\mathcal{L}_\infty$-error bounds," *Int. J. Control*, vol. 39, no. 6, pp. 1115–1193, 1984.

[7] G. Zames, "On the input-output stability for time-varying nonlinear feedback systems—Part I: Conditions using concepts of loop gain, conicity and positivity," *IEEE Trans. Automat. Control*, vol. AC-11, pp. 228–238, 1966.

[8] G. Vinnicombe, "Frequency domain uncertainty and the graph topology," *IEEE Trans. Automat. Control*, vol. 38, no. 9, pp. 1371–1383, Sep. 1993.

[9] G. Vinnicombe, *Uncertainty and Feedback $\mathcal{H}_\infty$ Loop-Shaping and the ν-Gap Metric*. London, U.K.: Imperial College Press, 2001.

[10] R. A. Hyde, $\mathcal{H}_\infty$ *Aerospace Control Design—A VSTOL Flight Application*, ser. Advances in Industrial Control Series. New York: Springer-Verlag, 1995.

[11] G. Papageorgiou, K. Glover, G. D'Mello, and Y. Patel, "Taking robust LPV control into flight on the VAAC Harrier," in *Proc. 39th IEEE Conf. Decision Control*, 2000, pp. 4558–4564.

[12] J. C. Doyle, B. A. Francis, and A. R. Tannenbaum, *Feedback Control Theory*. New York: Macmillan Publishing Company, 1992.

[13] M. Green and D. J. N. Limebeer, *Linear Robust Control*. Englewood Cliffs, NJ: Prentice-Hall, 1995.

[14] K. Zhou, J. C. Doyle, and K. Glover, *Robust and Optimal Control*. Englewood Cliffs, NJ: Prentice-Hall, 1996.

[15] S. Skogestad and I. Postlethwaite, *Multivariable Feedback Control: Analysis and Design*. New York: Wiley, 1998.

[16] R. A. Davis, "Model validation for robust control," Ph.D. dissertation, University of Cambridge, Cambridge, U.K., 1995.

[17] K. Poolla, P. Khargonekar, A. Tikku, J. Krause, and K. Nagpal, "A time-domain approach to model validation," *IEEE Trans. Automat. Control*, vol. 39, no. 5, pp. 951–959, May 1994.

[18] J. Chen and G. Gu, *Control-Oriented System Identification: An $\mathcal{H}_\infty$ Approach*. New York: Wiley, 2000.

[19] M. P. Newlin and R. S. Smith, "A generalization of the structured singular value and its application to model validation," *IEEE Trans. Automat. Control*, vol. 43, no. 7, pp. 901–907, Jul. 1998.

[20] K. Glover and D. McFarlane, "Robust stabilization of normalized coprime factor plant descriptions with $\mathcal{H}_\infty$-bounded uncertainty," *IEEE Trans. Automat. Control*, vol. AC-34, no. 8, pp. 821–830, Aug. 1989.

[21] A. Lanzon, "Weight optimization in $\mathcal{H}_\infty$ loop-shaping," *Automatica*, vol. 41, no. 7, pp. 1201–1208, Jul. 2005.

[22] D. F. Enns, "Rocket stabilization as a structured singular value synthesis design example," *IEEE Control Syst. Mag.*, vol. 11, no. 4, pp. 67–73, Jun. 1991.

[16]This can be see via a simple additive uncertainty perturbation example. If the nominal plant P is not stable, then $\Delta \in \mathcal{RH}_\infty$ (as required by standard small-gain theorem) implies that the perturbed plant $P_\Delta = P + \Delta$ has the exact same right-half plane pole locations as the nominal plant P. This is an unacceptable assumption. If however $\Delta \in \mathcal{RL}_\infty$ rather than $\mathcal{RH}_\infty$, then this restriction is lifted.

Alexander Lanzon (S'94–M'01–SM'07) was born in Malta. He received the B.Eng. degree (with honors) in electrical engineering from the University of Malta in 1995, and the M.S. and Ph.D. degrees in control engineering from the University of Cambridge, Cambridge, U.K., in 1997 and 2000, respectively.

Before joining the University of Manchester, Manchester, U.K., in 2006, he held academic positions at the Georgia Institute of Technology, Atlanta, and the Australian National University. He also received earlier research training at Bauman Moscow State Technical University, Moscow, Russia, and industrial training at ST-Microelectronics, Ltd., National ICT Australia Ltd. and Yaskawa Denki Tokyo Ltd, Japan. His research interests include fundamental theory of feedback systems, robust control and applications to aerospace control (including UAVs and control of new vehicle concepts).

Dr Lanzon is also a Fellow of the IET and a member of AIAA.

George Papageorgiou (S'95–M'98) received the B.S. degree and the Ph.D. degree in control engineering from the University of Cambridge, Cambridge, U.K., in 1994 and 1998, respectively.

His research focused on robust control theory and aerospace applications. Since joining Honeywell Aerospace Advanced Technology, he has carried out research in advanced controls, system analysis and modeling, and applied it to aerospace vehicles, engines and MEMS devices. He is currently Technical Manager of the Communications, Navigation and Surveillance Systems group of Honeywell's European Aerospace Advanced Technology Center. His research interests include technologies for Air Traffic Management modernization and for small aircraft low-cost certifiable avionics.