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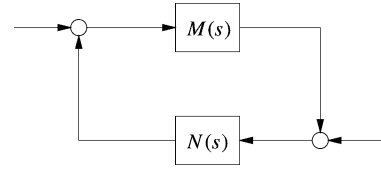


Fig. 1. Positive feedback interconnection.

Stability Robustness of a Feedback Interconnection of Systems With Negative Imaginary Frequency Response

Alexander Lanzon and Ian R. Petersen

Abstract—A necessary and sufficient condition, expressed simply as the dc loop gain (i.e., the loop gain at zero frequency) being less than unity, is given in this note to guarantee the internal stability of a feedback interconnection of linear time-invariant (LTI) multiple-input multiple-output systems with negative imaginary frequency response. Systems with negative imaginary frequency response arise, for example, when considering transfer functions from force actuators to colocated position sensors, and are commonly important in, for example, lightly damped structures. The key result presented here has similar application to the small-gain theorem, which refers to the stability of feedback interconnections of contractive gain systems, and the passivity theorem, which refers to the stability of feedback interconnections of positive real (or passive) systems. A complete state–space characterization of systems with negative imaginary frequency response is also given in this note and also an example that demonstrates the application of the key result is provided.

Index Terms—Bounded-real systems, passivity, positive position feedback, positive-real systems, small-gain theorem.

NOMENCLATURE

$\mathcal{RH}_\infty^{n \times n}$	Set of real-rational stable transfer function matrices of dimension $(n \times n)$.
\mathbb{R} and \mathbb{C}	Fields of real and complex numbers, respectively.
$\mathbb{R}^{n \times n}$ and $\mathbb{C}^{n \times n}$	Real and complex matrices, respectively, of dimension $(n \times n)$.

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$\lambda_i(A)$	The i th eigenvalue of a square complex matrix A
$\bar{\lambda}(A)$	Maximum eigenvalue for a square complex matrix A that has only real eigenvalues.
$\Re(a)$ and $\Im(a)$	Real and imaginary parts, respectively, of $a \in \mathbb{C}$.
A^T and A^*	The transpose and the complex conjugate transpose of a complex matrix A .
$M^\sim(s)$	Adjoint of transfer function matrix $M(s)$ given by $M(-s)^T$.
$\text{diag}(a, b)$	Shorthand for $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$.
A^{-*}	Shorthand for $(A^{-1})^*$.

I. INTRODUCTION

Consider a positive feedback interconnection of two linear time-invariant (LTI) multiple-input multiple-output (MIMO) systems, $M(s)$ and $N(s)$, as shown in Fig. 1, denoted by $[M(s), N(s)]$. The Nyquist stability theorem [1] gives necessary and sufficient conditions under which this interconnection is internally stable, using much information of $M(s)$ and $N(s)$. However, when $M(s)$ and $N(s)$ satisfy certain known properties (e.g., they are both bounded real with product of gains less than unity, or they are both positive real, etc.), it is also possible to derive powerful theorems (such as the small-gain theorem [2], [3], or the passivity theorem [4], [5], etc.) that use only limited information on $M(s)$ and $N(s)$ to establish the internal stability of this feedback interconnection. This is powerful and interesting because it provides a mechanism to derive robust stability results when systems are perturbed by uncertain dynamics that are quantified only in terms of restricted information (e.g., stable and contractive gain for the small-gain theorem, or stable and positive real for the passivity theorem, etc.).

In this note, we derive a new result of a similar flavor. We assume that both $M(s)$ and $N(s)$ are LTI MIMO stable systems with “negative imaginary frequency response”¹ and use this information to derive a necessary and sufficient internal stability condition using only limited information on $M(s)$ and $N(s)$.

We now show why systems with negative imaginary frequency response are important in engineering applications. We will do this via a simple example. Consider a lightly damped structure with colocated position sensors and force actuators. Lightly damped structures with colocated position sensors and force actuators can typically be modeled by a (possibly infinite) sum of second-order transfer functions as follows:

$$P_\Delta(s) := \sum_{i=1}^H \frac{k_i \omega_{n,i}^2}{s^2 + 2\zeta_i \omega_{n,i} s + \omega_{n,i}^2}.$$

¹Broadly, we say that (see Section II for precise set definitions) a system $R(s)$ has “negative imaginary frequency response” when $j[R(j\omega) - R(j\omega)^*] \geq 0$ (or > 0) for all $\omega \in (0, \infty)$. This is because, for single-input single-output (SISO) systems, $-2\Im(R(j\omega)) = j[R(j\omega) - R(j\omega)^*]$. Note that, at $\omega = 0$ or $\omega = \infty$, $\Im(R(j\omega)) = 0$ as $R(s)$ is real rational.

For the purpose of control systems design, however, one typically tends to include only a small finite number of modes ($h \ll H$) in the modeling of such systems, thereby giving rise to spillover unmodeled dynamics (i.e., unmodeled dynamics due to the lightly damped modes not included in the plant model). That is, let $P(s)$ be the truncated plant model used for control systems design and $\Delta(s)$ be the spillover dynamics, both given by

$$P(s) := \sum_{i=1}^h \frac{k_i \omega_{n,i}^2}{s^2 + 2\zeta_i \omega_{n,i} s + \omega_{n,i}^2}$$

$$\Delta(s) := \sum_{i=(h+1)}^H \frac{k_i \omega_{n,i}^2}{s^2 + 2\zeta_i \omega_{n,i} s + \omega_{n,i}^2}.$$

It is typically an important, though difficult, design specification to ensure that the closed-loop system retains stability in the presence of such spillover dynamics $\Delta(s)$. Since the relative degree of such spillover dynamics is more than unity, standard positive real analysis [4], [5] will not be very helpful in establishing robust stability and since these systems tend to be highly resonant, application of the small-gain theorem [2], [3] would typically be very conservative. However, it is readily noticed that such spillover dynamics $\Delta(s)$ are stable and satisfy a negative imaginary frequency response property. The dc gain of the spillover dynamics is simply $\sum_{i=(h+1)}^H k_i$. Also, satisfaction of stability and the negative imaginary frequency response property is invariant to values of $\zeta_i > 0$ and $\omega_{n,i} > 0$ for all $i \in [h+1, H]$. Consequently, provided that a controller $C(s)$ is designed so as to make the closed-loop transfer function $C/(1+PC)$ from plant output disturbances to plant input satisfy a negative imaginary frequency response property with dc gain strictly less than $1/(\sum_{i=(h+1)}^H k_i)$, then robust stability to all spillover dynamics $\Delta(s)$, and more, will hold for any value of $\zeta_i > 0$ and $\omega_{n,i} > 0$ for all $i \in [h+1, H]$, and hence, $C(s)$ will also robustly stabilize $P_\Delta(s)$.

Note that a similar condition *specifically for a subclass of SISO systems* has existed in the positive position feedback control literature [6] and [7] for some time. It is also not difficult to see how such a condition arises in SISO systems via a Nyquist diagram sketch. In fact, most controller synthesis and analysis in positive position feedback control is based on graphical techniques using Nyquist plots, or nonconvex parameter optimization [8]. In this note, we: 1) formalize the robustness qualities of feedback interconnections of systems with negative imaginary frequency response via a mathematical theorem and corollary (as opposed to graphical Nyquist sketches); 2) extend the ideas to MIMO LTI systems and show that, even in that case, a necessary and sufficient condition for the internal stability of such systems is that the dc loop gain (measured in a precise sense) is less than unity; and 3) give a complete state–space characterization of MIMO LTI systems with negative imaginary frequency response that may, in future work, underpin controller synthesis.

It is also worth pointing out that while a transfer function from force actuators to collocated position sensors has typically negative imaginary frequency response, the corresponding transfer function from force actuators to collocated velocity sensors has typically a positive real response. Consequently, it is legitimate to wonder whether simply replacing $M(s)$ with $sM(s)$ and $N(s)$ with $-(1/s)N(s)$ in Fig. 1 to obtain a negative feedback interconnection and using standard positive real analysis [4], [5] would do the trick? The answer to this question is “no, it does not” as $(1/s)N(s)$ is not stable, $sM(s)$ is not always guaranteed to be proper, and most importantly positive real analysis yields an unconditional stability result whereas the interconnection of two systems with negative imaginary frequency response

will always be conditionally stable (see Theorem 5). Of course, there are some connections between positive real systems and systems with negative imaginary frequency response that, in fact, will be exploited in Lemma 1 where we give a complete state–space characterization of systems with negative imaginary frequency response, but the differences are also important and should not be discounted (e.g., arising from frequencies $\omega = 0$ and $\omega = \infty$).

II. SOME TECHNICAL RESULTS

In this section, we generate the technical machinery that will enable us to concisely prove the main result in the next section. First, for the sake of brevity, let us define the following two sets of “stable systems with negative imaginary frequency response” as

$$\mathcal{C} := \left\{ R(s) \in \mathcal{RH}_\infty^{n \times n} : \right.$$

$$\left. j[R(j\omega) - R(j\omega)^*] \geq 0 \forall \omega \in (0, \infty) \right\} \quad (1)$$

$$\mathcal{C}_s := \left\{ R(s) \in \mathcal{RH}_\infty^{n \times n} : \right.$$

$$\left. j[R(j\omega) - R(j\omega)^*] > 0 \forall \omega \in (0, \infty) \right\} \subset \mathcal{C}. \quad (2)$$

The first lemma gives a complete state–space characterization of elements in \mathcal{C} . Hence, it also provides a test to easily check whether a transfer function matrix belongs to set \mathcal{C} or not. Testing whether a transfer function belongs to set \mathcal{C}_s or not requires an additional check on transmission zeros of $R(s) - \hat{R}(s)$ in the open frequency region $(0, \infty)$.

Lemma 1: Let

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

be a minimal state–space realization of a transfer matrix $R(s)$. Then, $R(s) \in \mathcal{C}$ if and only if A is Hurwitz, $D = D^*$ and there exists a real matrix $Y > 0$ such that

$$AY + YA^* \leq 0 \quad \text{and} \quad B = -AYC^*.$$

Proof: The two statements are connected via a sequence of equivalent reformulations:

- 1) $R(s) = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{C}$.
- 2) $\hat{R}(s) := (R(s) - D) = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \in \mathcal{C}$ and $D = D^*$. This equivalence follows on noting that $R(s) \in \mathcal{C}$ implies $j[R(\infty) - R(\infty)^T] \geq 0$ via continuity and a limiting argument, which in turn implies $-j[R(\infty) - R(\infty)^T] = (j[R(\infty) - R(\infty)^T])^T \geq 0$. Then, these two inequalities together imply $R(\infty) = R(\infty)^T$.
- 3) $F(s) := s\hat{R}(s) = \begin{bmatrix} A & B \\ CA & CB \end{bmatrix} \in \mathcal{RH}_\infty^{n \times n}$, $F(j\omega) + F(j\omega)^* \geq 0 \forall \omega \in \mathbb{R}$, A is Hurwitz, and $D = D^*$.
- 4) A is Hurwitz, $D = D^*$, and $\exists X > 0, L, W$ such that

$$XA + A^*X = -L^*L$$

$$B^*X + W^*L = CA$$

$$CB + (CB)^* = W^*W.$$

This equivalence is via the positive real lemma [see, for example, [4] together with the fact that (CA, A) is observable,

or [2, Ths. 13.25,13.26] together with the fact that (L^*L, A) is observable].

5) A is Hurwitz, $D = D^*$, and $\exists X > 0, L, W$ such that

$$\begin{aligned} XA + A^*X &= -L^*L \\ B &= X^{-1}(A^*C^* - L^*W) \\ CX^{-1}A^*C^* + CAX^{-1}C^* \\ &= W^*W + CX^{-1}L^*W + W^*LX^{-1}C^*. \end{aligned}$$

6) A is Hurwitz, $D = D^*$, and $\exists X > 0, L, W$ such that

$$\begin{aligned} XA + A^*X &= -L^*L \\ B &= X^{-1}(A^*C^* - L^*W) \\ W &= -LX^{-1}C^* \quad \text{via a completion of squares.} \end{aligned}$$

7) A is Hurwitz, $D = D^*$, and $\exists X > 0, L$ such that $XA + A^*X = -L^*L$ and $B = X^{-1}(A^*C^* + L^*LX^{-1}C^*)$.

8) A is Hurwitz, $D = D^*$, and $\exists X > 0$ such that $XA + A^*X \leq 0$ and $B = -AX^{-1}C^*$. ■

The second lemma relates the gain at zero frequency and the gain at infinite frequency for systems with negative imaginary frequency response.

Lemma 2: Given $R(s) \in \mathcal{C}$ (respectively \mathcal{C}_s), then $R(0) - R(\infty) \geq 0$ (respectively > 0).

Proof: Given a minimal realization

$$R(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] \in \mathcal{C}$$

and applying Lemma 1, we get

$$R(0) - R(\infty) = -CA^{-1}B = CA^{-1}AYC^* = CYC^* \geq 0 \quad (3)$$

which concludes the proof for the nonstrict inequality.

Now, we focus on $R(s) \in \mathcal{C}_s \Leftrightarrow \hat{R}(s) := (R(s) - D) \in \mathcal{C}_s$ (since $D = D^*$ by Lemma 1) and suppose there exists an $x \in \mathbb{R}^{n \times n}$ such that $\hat{R}(0)x = 0$. Then, it follows that $CYC^*x = 0$ which implies that $C^*x = 0$ as $Y > 0$. This then also gives $Bx = 0$ via $B = -AYC^*$ which yields

$$\hat{R}(j\omega)x = C(j\omega I - A)^{-1}Bx = 0 \quad \forall \omega \in \mathbb{R}.$$

But $j[\hat{R}(j\omega) - \hat{R}(j\omega)^*] > 0 \forall \omega \in (0, \infty)$ implies that $\hat{R}(j\omega)$ is nonsingular for all $\omega \in (0, \infty)$, and hence, the only possible $x \in \mathbb{R}^{n \times n}$ such that $\hat{R}(0)x = 0$ is $x = 0$. This shows that $\hat{R}(0)$ is also nonsingular, and thus, $\hat{R}(0) > 0$. This concludes the proof. ■

The following lemma gathers some straightforward computations that help us understand properties of systems with negative imaginary frequency response.

Lemma 3: Given $R(s) \in \mathcal{C}$, $\Delta(s) \in \mathcal{C}$, and $R_s(s) \in \mathcal{C}_s$. Then,

$$R(s) + \Delta(s) \in \mathcal{C} \quad \text{and} \quad R_s(s) + \Delta(s) \in \mathcal{C}_s.$$

Proof: Trivial.

The final technical lemma provides a matrix result that states that unity is not in the spectrum of matrix AB when matrices A and B satisfy certain negative imaginary properties.

Lemma 4: Given $A \in \mathbb{C}^{n \times n}$ with $j[A - A^*] \geq 0$ and $B \in \mathbb{C}^{n \times n}$ with $j[B - B^*] > 0$, then,

$$\det(I - AB) \neq 0.$$

Proof: The suppositions can be rewritten as $(jA) + (jA)^* \geq 0$ and $(jB)^{-1} + (jB)^{-*} > 0$. Then, $\det(I - AB) = \det(I + (jA)(jB)) = \det((jA) + (jB)^{-1}) \det(jB) \neq 0$. ■

III. MAIN RESULT

The key result in this note is Theorem 5. It is an analysis theorem that states that, provided one system belongs to class \mathcal{C} and the other system belongs to class \mathcal{C}_s , then a necessary and sufficient condition² for internal stability of a positive feedback interconnection of these two systems is to check that the dc loop gain (i.e., the loop gain at zero frequency) is less than unity.

Theorem 5: Given $M(s) \in \mathcal{C}$ and $N(s) \in \mathcal{C}_s$ that also satisfy $M(\infty)N(\infty) = 0$ and $N(\infty) \geq 0$. Then,

$$[M(s), N(s)] \text{ is internally stable} \Leftrightarrow \bar{\lambda}(M(0)N(0)) < 1.$$

Proof: Let

$$M(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$$

and

$$N(s) = \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array} \right]$$

be minimal realizations. Then, by the suppositions of this theorem and Lemma 1, A is Hurwitz, $D = D^*$, \bar{A} is Hurwitz, $\bar{D} = \bar{D}^* \geq 0$, $D\bar{D} = 0$, and there exists real matrices $Y > 0$ and $\bar{Y} > 0$ such that

$$AY + YA^* \leq 0 \quad \text{and} \quad B = -AYC^*, \quad (4)$$

$$\bar{A}\bar{Y} + \bar{Y}\bar{A}^* \leq 0 \quad \text{and} \quad \bar{B} = -\bar{A}\bar{Y}\bar{C}^*. \quad (5)$$

Now, define

$$\Phi := \left[\begin{array}{cc|c} AY & 0 & \\ \hline 0 & \bar{A}\bar{Y} & \end{array} \right] \quad \text{and} \quad T := \left[\begin{array}{cc|c} Y^{-1} - C^*\bar{D}C & -C^*\bar{C} & \\ \hline -\bar{C}^*C & \bar{Y}^{-1} - \bar{C}^*D\bar{C} & \end{array} \right]$$

and note that

$[M(s), N(s)]$ is internally stable

$$\Leftrightarrow (I - M(s)N(s))^{-1} =$$

$$\left[\begin{array}{cc|c} \left(\begin{array}{cc} A & B\bar{C} \\ 0 & \bar{A} \end{array} \right) + \left(\begin{array}{c} B\bar{D} \\ \bar{B} \end{array} \right) (C \quad D\bar{C}) & \left| \begin{array}{c} B\bar{D} \\ \bar{B} \end{array} \right. \\ \hline C & D\bar{C} & \left| \begin{array}{c} \\ I \end{array} \right. \end{array} \right] \in \mathcal{RH}_\infty$$

$$\Leftrightarrow \mathcal{A} := \left[\begin{array}{cc} A & B\bar{C} \\ 0 & \bar{A} \end{array} \right] + \left[\begin{array}{c} B\bar{D} \\ \bar{B} \end{array} \right] [C \quad D\bar{C}] = \Phi T \text{ is Hurwitz}$$

(as the previous realization is stabilizable and detectable).

$$\Leftrightarrow T > 0$$

\Leftrightarrow [\Rightarrow] Since \mathcal{A} is Hurwitz and Φ is nonsingular, T is nonsingular. Since $\Phi + \Phi^* \leq 0$, it follows that $T\mathcal{A} + \mathcal{A}^*T \leq 0$. Consequently, \mathcal{A} is Hurwitz implies $T \geq 0$. But T is also nonsingular; therefore, $T > 0$.

\Leftrightarrow [\Leftarrow] Since $\Phi + \Phi^* \leq 0$, it follows that $T\mathcal{A} + \mathcal{A}^*T \leq 0$. Consequently, $T > 0$ implies $\Re(\lambda_i(\mathcal{A})) \leq 0 \forall i$. But $T > 0$ and Φ nonsingular also imply \mathcal{A} has no eigenvalue at the origin. We now invoke Lemma 4 and use the fact that $M(s) \in \mathcal{C}$ and $N(s) \in \mathcal{C}_s$ to conclude that $\det(I - M(j\omega)N(j\omega)) \neq 0 \forall \omega \in (0, \infty)$, which in turn is equivalent to \mathcal{A} having no eigenvalue at $j\omega$ for all $\omega \in (0, \infty)$.]

²Under some assumptions on the gains of the systems at infinite frequency.

$$\begin{aligned}
 &\Leftrightarrow \bar{Y}^{-1} - \bar{C}^* \bar{D} \bar{C} > 0 \quad \text{and} \\
 &\quad (Y^{-1} - C^* \bar{D} C) - C^* \bar{C} (\bar{Y}^{-1} - \bar{C}^* \bar{D} \bar{C})^{-1} \bar{C}^* C > 0 \\
 &\Leftrightarrow \bar{\lambda} [\bar{Y}^{1/2} \bar{C}^* \bar{D} \bar{C} \bar{Y}^{1/2}] < 1 \quad \text{and} \\
 &\quad Y^{-1} - C^* \bar{D} C - C^* (I - \bar{C} \bar{Y} \bar{C}^* D)^{-1} \bar{C} \bar{Y} \bar{C}^* C > 0 \\
 &\Leftrightarrow \bar{\lambda} [D \bar{C} \bar{Y} \bar{C}^*] < 1 \quad \text{and} \\
 &\quad Y^{-1} - C^* \bar{D} C - C^* (I - N(0)D)^{-1} (N(0) - \bar{D}) C > 0 \\
 &\quad [\text{as } N(0) - \bar{D} = \bar{C} \bar{Y} \bar{C}^* \text{ via (3) and } \bar{D} D = 0] \\
 &\Leftrightarrow \bar{\lambda} [DN(0)] < 1 \quad \text{and} \\
 &\quad Y^{-1} - C^* (I - N(0)D)^{-1} [\bar{D} + (N(0) - \bar{D})] C > 0 \\
 &\quad [\text{as } N(0) - \bar{D} = \bar{C} \bar{Y} \bar{C}^* \text{ via (3) and } D \bar{D} = 0] \\
 &\Leftrightarrow N(0)^{-1} - D > 0 \quad \text{and} \\
 &\quad Y^{-1} - C^* (N(0)^{-1} - D)^{-1} C > 0 \\
 &\quad [\text{as } N(0) > \bar{D} \text{ via Lemma 2 and } \bar{D} \geq 0] \\
 &\Leftrightarrow \begin{bmatrix} Y^{-1} & C^* \\ C & N(0)^{-1} - D \end{bmatrix} > 0 \\
 &\Leftrightarrow N(0)^{-1} - D - CYC^* > 0 \\
 &\Leftrightarrow N(0)^{-1} - M(0) > 0 \\
 &\quad [\text{as } M(0) = D + CYC^* \text{ via (3)}] \\
 &\Leftrightarrow \bar{\lambda}(M(0)N(0)) < 1.
 \end{aligned}$$

One may wonder whether integral quadratic constraint (IQC) theory [9] captures the sufficiency part of Theorem 5 or not. This question is subtle and its answer is nontrivial. However, in short, the answer is “no, it does not.” The subtlety of the question arises from the fact that IQC theory deals with the full frequency range $\omega \in \mathbb{R}$ whereas systems $N(s) \in \mathcal{C}_s$ satisfy the frequency-domain inequality $j[N(j\omega) - N(j\omega)^*] > 0$ only on an open frequency interval $\omega \in (0, \infty)$. This strict inequality cannot be satisfied at $\omega = 0$ because via Lemma 2, we know that $N(0) = N(0)^*$. This fact causes the main theorem in [9, Th. 1], which underpins IQC theory, to be inapplicable by violation of its suppositions.

The following corollary is a weaker restatement of the main theorem, written in the same form as the small-gain theorem or the passivity theorem.

Corollary 6:

- 1) Given $\gamma > 0$ and $M(s) \in \mathcal{C}_s$ with $M(\infty) \geq 0$. Then, $[\Delta(s), M(s)]$ is internally stable for all $\Delta(s) \in \mathcal{C}$ satisfying $\Delta(\infty)M(\infty) = 0$ and $\bar{\lambda}(\Delta(0)) < \gamma$ (respectively $\leq \gamma$) if and only if $\bar{\lambda}(M(0)) \leq 1/\gamma$ (respectively $< 1/\gamma$).
- 2) Given $\gamma > 0$ and $M(s) \in \mathcal{C}$. Then $[\Delta(s), M(s)]$ is internally stable for all $\Delta(s) \in \mathcal{C}_s$ satisfying $\Delta(\infty) \geq 0$, $\Delta(\infty)M(\infty) = 0$ and $\bar{\lambda}(\Delta(0)) < \gamma$ (respectively $\leq \gamma$) if and only if $\bar{\lambda}(M(0)) \leq 1/\gamma$ (respectively $< 1/\gamma$).

Proof: Sufficiency of the two statements follows on noting that $\bar{\lambda}(M(0))\bar{\lambda}(\Delta(0)) < 1$ implies $\bar{\lambda}(\Delta(0)M(0)) < 1$. Necessity can be proved via a contrapositive argument on choosing $\Delta(s) = 1/[\bar{\lambda}(M(0))(s+1)]I$ as the destabilizing $\Delta(s)$.

IV. ILLUSTRATIVE EXAMPLE

Consider the lightly damped mechanical plant depicted in Fig. 2, which consists of two unit masses constrained to slide rectilinearly

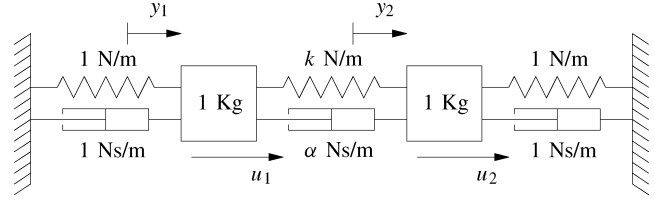


Fig. 2. Lightly damped uncertain mechanical plant.

on a frictionless table. Each mass is attached to a fixed wall via a spring of known unit stiffness and via a damper of known unit viscous resistance. Furthermore, the two unit masses are coupled together via a spring of uncertain stiffness k (in Newton per meter) and via a damper of uncertain viscous resistance α (in Newton second per meter). A force is applied to each mass (denoted by u_1 and u_2 , respectively) and the displacement of each mass is measured (denoted by y_1 and y_2 , respectively).

Although, this is not an extremely difficult design problem, it does illustrate a number of important points arising from the results in this note, as it contains key features such as an uncertain MIMO system with uncertainty that has negative imaginary frequency response. Similar examples have been considered in the literature as benchmark problems by a number of authors, including [1] and [10] to mention a few.

For shorthand, let us define some commonly appearing transfer functions and matrices. Let

$$p(s) := \frac{1}{s^2 + s + 1}, \quad \delta(s) := \frac{1}{s^2 + (2\alpha + 1)s + (2k + 1)}$$

and $\Psi := \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

Then, elementary mechanical modeling reveals that the transfer function matrix for the plant depicted in Fig. 2 from force input vector

$$u := \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

to displacement output measurements

$$y := \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

is given by $y(s) = P_\Delta(s)u(s)$ where

$$P_\Delta := p(s)\delta(s) \times \begin{bmatrix} (s^2 + (\alpha + 1)s + (k + 1)) & (\alpha s + k) \\ (\alpha s + k) & (s^2 + (\alpha + 1)s + (k + 1)) \end{bmatrix}.$$

It is clear that $P_\Delta(s)$ is uncertain because k and α are unknown.

For the purpose of control system design, we now choose to split the uncertain plant $P_\Delta(s)$ as $P_\Delta(s) = P(s) + \Delta(s)$, where $P(s)$ is the nominal completely known plant model and $\Delta(s)$ is the uncertain remainder. Via partial fraction expansion, we see that $P(s) = \Psi \text{diag}(0.5p(s), 0)\Psi^*$ and $\Delta(s) = \Psi^{-1} \text{diag}(0.5\delta(s), 0)\Psi^*$. It is then a simple computation to check that $\Delta(s) \in \mathcal{C}$ for all $\alpha > 0$ and $k > 0$.

Now, let us consider the controlled closed-loop system given in Fig. 3, and let $C(s)$ be chosen as $C(s) := \Psi^* \text{diag}(-2(s^2 + s + 1)/(2s^3 + 4s^2 + 4s + 3), -1/(s + 1))\Psi^{-1}$. Then, define $M(s) := -C(s)(I + P(s)C(s))^{-1}$ to be the transfer

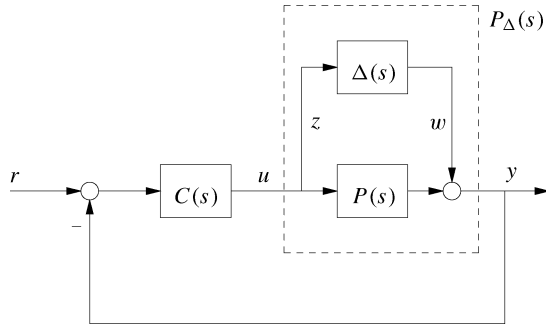


Fig. 3. Controlled closed-loop system.

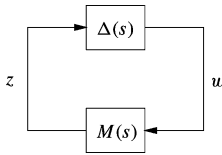


Fig. 4. Rearranged closed loop.

function matrix mapping w to z so that the closed-loop system in Fig. 3 can be rearranged into Fig. 4 for robust analysis.

Since $P(s) \in \mathcal{RH}_\infty^{2 \times 2}$, internal stability of the nominal feedback loop [i.e., pretending $\Delta(s) = 0$] is equivalent to $M(s) \in \mathcal{RH}_\infty^{2 \times 2}$. Furthermore, since $\Delta(s) \in \mathcal{C}$, we additionally require $M(s) \in \mathcal{C}_s$ to be able to apply the proposed results of Theorem 5 and Corollary 6 to conclude robust stability.

For our particular choice of $C(s)$, it is easy to see that $M(s) = 1/(s+1)\Psi^{-*}\Psi^{-1}$, which clearly belongs to \mathcal{C}_s , and furthermore, satisfies $M(\infty) = 0$. Since $M(0) = \Psi^{-*}\Psi^{-1}$ (which incidentally is strictly greater than $M(\infty)$ as stated in Lemma 2) and $\Delta(0) = \Psi^{-1}\text{diag}(0.5/(2k+1), 0)\Psi^{-*}$ [which incidentally is greater than or equal to $\Delta(\infty)$ as stated in Lemma 2], it follows that $\bar{\lambda}(\Delta(0)M(0)) = 5/[2(2k+1)]$, $\bar{\lambda}(M(0)) = (3 + \sqrt{5})/2$ and $\bar{\lambda}(\Delta(0)) = 1/(2k+1)$.

Consequently, Corollary 6 Part I states that the feedback interconnection given in Fig. 3 is robustly stable for all uncertainties $\Delta \in \mathcal{C}$ (not just those of the form $\Delta(s) = \Psi^{-1}\text{diag}(0.5\delta(s), 0)\Psi^{-*}$) satisfying $\bar{\lambda}(\Delta(0)) < 2/(3 + \sqrt{5})$ ($= 1/\bar{\lambda}(M(0))$). Additionally, Theorem 5 states that, for any given $\alpha > 0$, the physical system of Fig. 2 is robustly stabilized by the controller $C(s)$ defined earlier if and only if $k > 0.75$, obtained through the condition $\bar{\lambda}(\Delta(0)M(0)) < 1$. The former statement is powerful because it characterizes a huge class of systems (including spillover dynamics) for which the closed-loop system is robustly stable. The latter statement is powerful because, for a given uncertainty class, it tells us exactly in a necessary and sufficient manner the parameter boundary of robust stability.

V. CONCLUSION

SISO LTI systems with negative imaginary frequency response have been studied in the context of positive position feedback control of lightly damped structures and the analysis/synthesis methods there depended on graphically Nyquist plots. This note generalizes the key stability result of MIMO LTI systems with negative imaginary frequency response showing that, even in this case, a necessary and sufficient condition for the internal stability of such systems is that the dc loop gain (measured in a particular precise sense) is less than unity. We also gave in this note a complete state-space characterization of MIMO LTI systems with negative imaginary frequency response. This could

possibly be used in future work to assist with synthesizing systems with negative imaginary frequency response.

The next steps to extend applicability of this research are: 1) devise a controller synthesis procedure that generates systems that belong to either class \mathcal{C} or \mathcal{C}_s and 2) generalize the analysis result given in this note to allow one (or possibly both) systems to be nonlinear and/or time-varying. Focusing on the latter, we point out that a theory for SISO nonlinear systems has been derived in [11] with counterclockwise input–output dynamics that is closely related to this research. It is consequently interesting to see whether the ideas in [11] generalize to MIMO systems or not, or whether the use of dissipativity theory may lead to the required MIMO nonlinear generalizations.

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