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A parameterization of parahermitian matrix functions and its application to a state-space solution for μ-analysis

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ABSTRACT

In this paper, an economic parameterization for positive parahermitian matrix functions is introduced and applied to the μ-analysis framework wherein we propose a new state-space optimization problem for finding the required D-scales. Among the four state-space matrices to be used to realize the optimal D-scale, A and B are chosen via a Laguerre parameterization whereas the other two state-space matrices, C and D are obtained by spectral factorization after solving a convex optimization problem formulated in an LMI framework. The obtained D-scale satisfies the commuting property with the uncertainty structure. The proposed economic parameterization yields advantages in terms of less computational time and less number of decision variables and also, the proposed state-space optimization framework gives a frequency independent solution algorithm in state-space variables for the required D-scales. Two numerical examples are used to demonstrate the effectiveness of the proposed algorithm.

1. Introduction

Positive parahermitian matrix functions often occur when aiming to parameterize positive frequency response functions $D(j\omega)^*D(j\omega) > 0 \ \forall \omega \in \mathbb{R} \cup \{\infty\}$ (where D is a unit in $\mathcal{RH}_\infty$) via state-space optimization schemes involving the Kalman–Yakubovich–Popov (KYP) lemma [1–5]. By definition, a parahermitian matrix function $\Gamma(s)$ is a matrix function that satisfies $\Gamma(s) = \Gamma^{\sim}(s)$ and is given by

$$\Gamma(s) = \left[\begin{array}{cc|c} A & 0 & B \\ -P & -A^T & -S \\ \hline S^T & B^T & R \end{array}\right]$$
$$= \begin{bmatrix} (-sI - A)^{-1}B \\ I \end{bmatrix}^T \begin{bmatrix} P & S \\ S^T & R \end{bmatrix} \begin{bmatrix} (sI - A)^{-1}B \\ I \end{bmatrix}, \tag{1}$$

where the real constant matrices A, B, S, P and R are of compatible dimensions.

This paper introduces an economic parameterization of parahermitian matrix functions and highlights how such an economic parameterization saves computational time by having a considerable smaller number of decision variables when implemented within an optimization problem. In this paper, we also propose a new state-space D-scale optimization problem of μ-analysis and the advantages of the proposed economic parameterization are illustrated via a numerical example by finding the D-scale state-space solutions used in μ-analysis in the following optimization problem:

$$\sup_{\omega\in\mathbb{R}} \inf_{D(j\omega)\in\mathbf{D}} \bar{\sigma}[DMD^{-1}(j\omega)] < \gamma. \tag{2}$$

The D-scaling is often carried out by following two approaches—either in pointwise-in-frequency and subsequently using the curve-fitting technique [6], or using the M/F-iterative framework [7]. In [8], a dynamic D-scaling technique has been proposed, suitable for gain-scheduling synthesis, where a set of necessary and sufficient conditions for the existence of a robust stabilizing controller is presented in a non-convex form. In that paper, the D-scales are parameterized by using multipliers and the method avoids curve-fitting and loop transformation techniques. For solving the D-scales problem, in the literature, different versions of KYP lemma have been used ([9,10] and references therein). Note that a state-space solution for D-scales is beneficial because it avoids the cumbersome D-scale fitting of pointwise-in-frequency D-scale data typically required in μ-synthesis (see D–K iterations) [11–15, 7]. Furthermore, gridding the frequency axis ω can only give confidence that the supremum over frequency in (2) does not exceed the optimized level γ [11], unlike the absolute guarantee provided by a direct state-space solution for D-scales that avoids frequency gridding.

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In the present work, this D-scale optimization problem has been solved in state-space domain. It is achieved by reformulating the above optimization problem into an equivalent form involving the frequency function $\Gamma(j\omega) := D(j\omega)^*D(j\omega)$ and then, $\Gamma(j\omega)$ is replaced by the expression as given in (1) and all the design constraints are formulated into an LMI framework. If the A-matrix in $\Gamma(s)$ has large dimension, P also will be having a large dimension that causes a large number of decision variables and more computation time and memory. Hence, the optimization problem would benefit from a method which pins P to zero as we will do here. To achieve this objective, an economic parameterization of parahermitian matrix function is introduced within the solution algorithm for the posed problem.

Although the motivation of this work is along the same line of objective of [7,8], the present work carries some differences. In [7], authors introduced an optimization framework for avoiding the cumbersome curve-fitting technique using the multiplier theory and the structure of D-scale was considered as diagonal. In [7], no such state-space approach was explicitly suggested. However, in this present work the parahermitian matrix function is used for the D-scale optimization and the structure of the D-scale is considered as block diagonal which is more general compared to [7]. Furthermore, the proposed economic parameterization also facilitates the computational advantage that cannot be obtained in [7].

The contribution of this work is with twofold objectives: first, a new parameterization of parahermitian matrix function is proposed and secondly, a state-space solution is introduced for the D-scale problem and the computational advantage of the proposed optimization problem is enhanced by applying the economic parameterization. Although the speed is not a limiting factor in the D–K iteration, the introduced economic parameterization becomes effective when a large dimensional system matrix is dealt with.

The rest of the paper is organized as follows: Section 2 carries preliminary results; Section 3 proposes the economic parameterization of parahermitian matrix function and Section 4 proposes the D-scale optimization problem formulation in state-space framework; Section 5 presents the feasible solution algorithm; Section 6 illustrates two numerical examples and finally, Section 7 gives concluding remarks.

Notations. The following mathematical notations are used in this paper. Let $\mathbb{R}$ and $\mathbb{R}^{n\times m}$ denote the respective set of real numbers and real constant matrices with n rows and m columns. Let $\mathbb{C}$, $\mathbb{C}_+$, $\bar{\mathbb{C}}_+$ and $\mathbb{C}^{n\times m}$ denote the respective set of complex numbers, complex numbers with strictly positive real parts, complex numbers with non-negative real parts and complex constant matrices with n rows and m columns. Let $\lambda_i(A)$, $\rho(A)$ and $\bar{\sigma}(A)$ respectively denote the i-th eigenvalue, spectral radius and maximum singular value of an arbitrary constant matrix A. Let A^T and A^* respectively be the transpose and complex conjugate transpose of an arbitrary constant matrix A. Let $\det(A)$ denote the determinant of the arbitrary constant matrix A. Let $A\otimes B$ denote the Kronecker Product of the arbitrary constant matrices A and B. Let I_n denote an $n\times n$ unit matrix. Let $\mathcal{RH}_\infty^{n\times m}$ denote the set of real-rational stable $n\times m$ transfer function matrices that are analytic and bounded in $\mathbb{C}_+$. Let $\min_{D\in\mathbf{D}}$ and $\max_{D\in\mathbf{D}}$ denote the respective minimum and maximum taken over a set $\mathbf{D}$. Let $\inf_{D\in\mathbf{D}}$ and $\sup_{D\in\mathbf{D}}$ denote the respective infimum and supremum taking over a set $\mathbf{D}$. Let $\mathrm{diag}(\triangle_1,\dots,\triangle_n)$ denote a block diagonal matrix with the blocks $\triangle_1,\dots,\triangle_n$ on its main diagonal. Let $G^\sim(s)$ denote $G(-s)^T$ for an arbitrary transfer function matrix $G(s)$. Let $\left[\begin{array}{c|c} A & B \\ \hline C & D\end{array}\right]$ be shorthand for the state-space realization $C(sI-A)^{-1}B+D$.

2. Preliminaries

Some definitions and preliminary results are presented in this section which will be required to develop the main results of this paper.

Definition 1 ([14]). Let $\Delta\subset\mathbb{C}^{m\times m}$ be an allowable set of structured uncertainty given by

$$\Delta := \{\mathrm{diag}(\delta_1 I_{r_1},\dots,\delta_S I_{r_S},\triangle_1,\dots,\triangle_F):\ \delta_i\in\mathbb{C},\ \triangle_k\in\mathbb{C}^{p_k\times p_k},\ i=1,\dots,S,\ k=1,\dots,F\},$$

where $\Sigma_{i=1}^S r_i + \Sigma_{k=1}^F p_k = m$.

Definition 2 ([16]). For $M\in\mathbb{C}^{m\times m}$, the structured singular value $\mu_\Delta(M)$ is defined as

$$\mu_\Delta(M)^{-1} := \min_{\triangle\in\Delta}\{\bar{\sigma}(\triangle):\ \det(I-M\triangle)=0\}$$

unless there is no $\triangle\in\Delta$ that makes $(I-M\triangle)$ singular, in which case $\mu_\Delta(M) := 0$.

The set of block diagonal and stable uncertainty transfer functions is defined below.

Definition 3 ([14]). Let $\mathcal{M}(\Delta)$ denote the set of all block diagonal and stable transfer functions such that

$$\mathcal{M}(\Delta) := \{\triangle(\cdot)\in\mathcal{RH}_\infty:\ \triangle(s_o)\in\Delta\ \forall s_o\in\bar{\mathbb{C}}_+\}.$$

The structured singular value $\mu_\Delta(M)$ is bounded by two mathematical quantities as presented in Theorem 1 below.

Theorem 1 ([17,18]). $\max_{U\in\mathbf{U}}\rho(MU)\le\mu_\Delta(M)\le\inf_{D\in\mathbf{D}}\bar{\sigma}(DMD^{-1})$ *where* $\mathbf{D} := \{D=\mathrm{diag}(D_1,\dots,D_S,d_1I_{p_1},\dots,d_FI_{p_F}):\ \det(D_i)\ne 0, D_i=D_i^*\in\mathbb{C}^{r_i\times r_i}, d_k\ne 0, k=1,\dots,F, i=1,\dots,S\}$, *$D$ commutes with* $\triangle\in\Delta$ *and* $\mathbf{U} := \{U\in\Delta:\ UU^*=I_m\}$.

Furthermore, for $2S+F\le 3$, the computation of $\mu_\Delta(M)$ is reduced to [19]

$$\mu_\Delta(M) = \inf_{D\in\mathbf{D}}\bar{\sigma}(DMD^{-1}).$$

Otherwise, $\inf_{D\in\mathbf{D}}\bar{\sigma}(DMD^{-1})$ is an upper bound of $\mu_\Delta(M)$. It is to be noted that in solving the above optimal D-scale problem, one encounters the frequency dependent positive definite frequency function $D(j\omega)^*D(j\omega)$ and hence a parahermitian matrix function can be used to parameterize $D(j\omega)^*D(j\omega)$. Upon parameterizing $D(j\omega)^*D(j\omega)$ using (1), both the state matrix A_D and input matrix B_D of the transfer function $D(s) := \left[\begin{array}{c|c} A_D & B_D \\ \hline C_D & D_D\end{array}\right]$ can then be predetermined using a stable basis function that yields a uniform approximation of $D(s)$ such that the remaining parameters P, S and R can then be obtained as decision variables of an LMI problem. A method of obtaining this stable basis that yields a uniform approximation is the Laguerre parameterization which is explained below for completeness.

It is known that any transfer function matrix $D(s) := \left[\begin{array}{c|c} A_D & B_D \\ \hline C_D & D_D\end{array}\right]\in\mathcal{RH}_\infty^{p\times q}$ can be uniformly approximated by parameterizing a subspace of $\mathcal{RH}_\infty$. This parameterization can be obtained via for example a Laguerre parameterization [20,21], and it is given as follows:

$D(s) = \check{\mathcal{Q}}\mathcal{B}(s)$ in which

$$\begin{cases}\check{\mathcal{Q}} := \begin{bmatrix}\check{\mathcal{Q}}_0 & \check{\mathcal{Q}}_1 & \check{\mathcal{Q}}_2 & \cdots & \check{\mathcal{Q}}_N\end{bmatrix}\in\mathbb{R}^{p\times(N+1)q}\\ \mathcal{B}(s) := \begin{bmatrix} I_q & \left(\dfrac{\frac{2}{\tau}-s}{\frac{2}{\tau}+s}\right)I_q & \left(\dfrac{\frac{2}{\tau}-s}{\frac{2}{\tau}+s}\right)^2 I_q & \cdots & \left(\dfrac{\frac{2}{\tau}-s}{\frac{2}{\tau}+s}\right)^N I_q\end{bmatrix}^T.\end{cases} \quad (3)$$

A sufficiently small positive real value of τ and a sufficiently large positive integer value of N help to achieve a more accurate transfer function model approximation. In this paper, for a given pair (N, τ), (3) will be invoked to obtain a controllable basis pair (A_D, B_D). Note that, the smaller τ (keeping N fixed) gives quick convergence of the algorithm as it approximates the transfer function more accurately.

3. Economic parameterization of parahermitian matrix function

A parahermitian matrix function $\Gamma(s)$ can be re-written into an equivalent form by replacing the (1, 1)-block of $\begin{bmatrix} P & S \\ S^T & R \end{bmatrix}$ in (1) with an arbitrary constant matrix P_e with compatible dimension as given in Lemma 1 below. The equivalent *'economized'* form is obtained by setting $P_e = 0$.

Lemma 1 ([21,22]). *Let A_D, B_D, P, S, R be real matrices of compatible dimensions such that $P = P^T \in \mathbb{R}^{n\times n}$, $R = R^T \in \mathbb{R}^{m\times m}$, $S \in \mathbb{R}^{n\times m}$ and $\lambda_i(A_D) \neq -\lambda_k(A_D^T)$ $\forall i, k$. Define a parahermitian rational matrix function $\Gamma(s)$ by*

$$\Gamma(s) = \begin{bmatrix} B_D^T(-sI - A_D^T)^{-1} & I \end{bmatrix} \begin{bmatrix} P & S \\ S^T & R \end{bmatrix} \begin{bmatrix} (sI - A_D)^{-1}B_D \\ I \end{bmatrix}. \qquad (4)$$

Given any arbitrary real matrix $P_e = P_e^T \in \mathbb{R}^{n\times n}$ then $\exists S_e \in \mathbb{R}^{n\times m}$ such that

$$\Gamma(s) = \begin{bmatrix} B_D^T(-sI - A_D^T)^{-1} & I \end{bmatrix} \begin{bmatrix} P_e & S_e \\ S_e^T & R \end{bmatrix} \begin{bmatrix} (sI - A_D)^{-1}B_D \\ I \end{bmatrix}. \qquad (5)$$

Furthermore, S_e is given by $S_e = S + XB_D$, where $X = X^T \in \mathbb{R}^{n\times n}$ is the unique solution to the Lyapunov equation

$$A_D^T X + XA_D + P = P_e. \qquad (6)$$

Proof. The Lyapunov equation $A_D^T X + XA_D + P = P_e$ has a unique solution since $\lambda_i(A_D) + \lambda_k(A_D^T) \neq 0$ $\forall i, k$ (see Lemma 2.1 in [23]). Furthermore, since A_D is real and $(P_e - P)$ is real and symmetric, then such a solution X is real and symmetric. Since (4) can be written as $\Gamma(s) = \left[\begin{array}{cc|c} A_D & 0 & B_D \\ -P & -A_D^T & -S \\ \hline S^T & B_D^T & R \end{array}\right]$, applying the similarity transformation $\Upsilon = \begin{bmatrix} I & 0 \\ X & I \end{bmatrix}$ to the above state-space realization gives

$$\Gamma(s) = \left[\begin{array}{cc|c} A_D & 0 & B_D \\ -(A_D^T X + XA_D + P) & -A_D^T & -(XB_D + S) \\ \hline (B_D^T X + S^T) & B_D^T & R \end{array}\right]. \qquad (7)$$

From (6) and simple algebra, we have that (7) is identical to (5). □

An economized parameterization of the frequency function $D(j\omega)^* D(j\omega)$ (where D is a unit in $\mathcal{RH}_\infty$) is given in Lemma 2. This is obtained by letting $P_e = 0$ be the arbitrary (1, 1)-block of $\begin{bmatrix} P & S \\ S^T & R \end{bmatrix}$ as given in Lemma 1. Letting $P_e = 0$ reduces the number of decision variables in an optimization problem involving (5). Note that the dimension of the P matrix is the same as that of A_D, hence for a matrix A_D with significantly large dimension, P also will have a huge number of decision variables within it.

Lemma 2 ([21,22]). *Given $A_D \in \mathbb{R}^{n\times n}$ and $B_D \in \mathbb{R}^{n\times m}$ with A_D Hurwitz.*

I. *For every $C_D \in \mathbb{R}^{m\times n}$ and $D_D \in \mathbb{R}^{m\times m}$ such that $D(s) := \left[\begin{array}{c|c} A_D & B_D \\ \hline C_D & D_D \end{array}\right] \in \mathcal{RH}_\infty$ satisfies $D^{-1}(s) \in \mathcal{RH}_\infty$, $\exists S_e \in \mathbb{R}^{n\times m}$ and $R = R^T \in \mathbb{R}^{m\times m}$ such that*

$$D(j\omega)^* D(j\omega) = \begin{bmatrix} (j\omega I - A_D)^{-1}B_D \\ I \end{bmatrix}^* \begin{bmatrix} 0 & S_e \\ S_e^T & R \end{bmatrix} \times \begin{bmatrix} (j\omega I - A_D)^{-1}B_D \\ I \end{bmatrix} > 0 \quad \forall \omega \in \mathbb{R} \cup \{\infty\}.$$

II. *For every $S_e \in \mathbb{R}^{n\times m}$ and $R = R^T \in \mathbb{R}^{m\times m}$ such that*

$$\begin{bmatrix} (j\omega I - A_D)^{-1}B_D \\ I \end{bmatrix}^* \begin{bmatrix} 0 & S_e \\ S_e^T & R \end{bmatrix} \times \begin{bmatrix} (j\omega I - A_D)^{-1}B_D \\ I \end{bmatrix} > 0 \quad \forall \omega \in \mathbb{R} \cup \{\infty\},$$

$\exists C_D \in \mathbb{R}^{m\times n}$ and $D_D \in \mathbb{R}^{m\times m}$ such that $D(s) := \left[\begin{array}{c|c} A_D & B_D \\ \hline C_D & D_D \end{array}\right] \in \mathcal{RH}_\infty$ satisfies $D^{-1}(s) \in \mathcal{RH}_\infty$ and

$$D(j\omega)^* D(j\omega) := \begin{bmatrix} (j\omega I - A_D)^{-1}B_D \\ I \end{bmatrix}^* \begin{bmatrix} 0 & S_e \\ S_e^T & R \end{bmatrix} \times \begin{bmatrix} (j\omega I - A_D)^{-1}B_D \\ I \end{bmatrix} \quad \forall \omega \in \mathbb{R} \cup \{\infty\}.$$

Proof. I. For any $C_D \in \mathbb{R}^{m\times n}$ and $D_D \in \mathbb{R}^{m\times m}$ such that $D(s) := \left[\begin{array}{c|c} A_D & B_D \\ \hline C_D & D_D \end{array}\right] \in \mathcal{RH}_\infty$ satisfies $D^{-1}(s) \in \mathcal{RH}_\infty$, we have

$$D(j\omega)^* D(j\omega) = \begin{bmatrix} (j\omega I - A_D)^{-1}B_D \\ I \end{bmatrix}^* \begin{bmatrix} C_D^T C_D & C_D^T D_D \\ D_D^T C_D & D_D^T D_D \end{bmatrix} \times \begin{bmatrix} (j\omega I - A_D)^{-1}B_D \\ I \end{bmatrix} > 0 \quad \forall \omega \in \mathbb{R} \cup \{\infty\}.$$

From Lemma 1 with $P_e = 0$, $\exists S_e \in \mathbb{R}^{n\times m}$ and $R = R^T \in \mathbb{R}^{m\times m}$ ($R = D_D^T D_D$) such that the required result holds.

II. For any $S_e \in \mathbb{R}^{n\times m}$ and $R = R^T \in \mathbb{R}^{m\times m}$ such that

$$\begin{bmatrix} (j\omega I - A_D)^{-1}B_D \\ I \end{bmatrix}^* \begin{bmatrix} 0 & S_e \\ S_e^T & R \end{bmatrix} \begin{bmatrix} (j\omega I - A_D)^{-1}B_D \\ I \end{bmatrix} > 0 \quad \forall \omega \in \mathbb{R} \cup \{\infty\},$$

it follows from spectral factorization results [14] that $\exists C_D \in \mathbb{R}^{m\times n}$ and $D_D \in \mathbb{R}^{m\times m}$ such that $D(s) := \left[\begin{array}{c|c} A_D & B_D \\ \hline C_D & D_D \end{array}\right] \in \mathcal{RH}_\infty$ satisfies $D^{-1}(s) \in \mathcal{RH}_\infty$ and

$$D(j\omega)^* D(j\omega) = \begin{bmatrix} (j\omega I - A_D)^{-1}B_D \\ I \end{bmatrix}^* \begin{bmatrix} 0 & S_e \\ S_e^T & R \end{bmatrix} \begin{bmatrix} (j\omega I - A_D)^{-1}B_D \\ I \end{bmatrix} \quad \forall \omega \in \mathbb{R} \cup \{\infty\}. \quad □$$

The economic parameterization result given in Lemma 2, indicates the reduction of decision variable count in a convex optimization problem that solves for $D(j\omega)$, $D(j\omega)^{-1} \in \mathcal{RH}_\infty$ where $D(j\omega)^* D(j\omega) > 0$ $\forall \omega \in \mathbb{R} \cup \{\infty\}$.

4. State-space solution for the *D*-scale optimization problem in μ-analysis

In this section, the D-scale optimization problem of μ-analysis is formulated in the state-space framework. The parahermitian matrix function and the KYP lemma are invoked to formulate the design constraints in the state-space convex optimization problem. The state-space matrices A_D and B_D are obtained from the Laguerre parameterization, and the commuting property between the uncertainty block Δ and the block diagonal D-scale is fulfilled by using the copying operator. The economic parameterization proposed in the previous section is then invoked to reduce the number of variables that further enhances the computational advantage of the algorithm.

Given $M(s) \in \mathcal{RH}_\infty^{m\times m}$, we wish to find a state-space D-scale (in μ-analysis) of a specific structure that satisfies

$$\inf_{D(j\omega)\in\mathbf{D}} \bar{\sigma}\left[DMD^{-1}(j\omega)\right] < \gamma \quad \forall\omega \in \mathbb{R}\cup\{\infty\}. \tag{8}$$

The inequality (8) can equivalently be written as

$$M(j\omega)^*D(j\omega)^*D(j\omega)M(j\omega) < \gamma^2 D(j\omega)^*D(j\omega) \quad \forall\omega \in \mathbb{R}\cup\{\infty\}. \tag{9}$$

Let a positive parahermitian matrix function $\Gamma(j\omega) = \varphi(j\omega)^* \begin{bmatrix} P & S \\ S^T & R \end{bmatrix} \varphi(j\omega) > 0\ \forall\omega \in \mathbb{R}\cup\{\infty\}$ replace $D(j\omega)^*D(j\omega)$ in (9), where $\varphi(j\omega) = \begin{bmatrix} (j\omega I - A_D)^{-1}B_D \\ I \end{bmatrix}$, $D(s) := \left[\begin{array}{c|c} A_D & B_D \\ \hline C_D & D_D \end{array}\right]$, $P = P^T$, S and $R = R^T$ are decision variable matrices that we wish to find. Via Lemma 2, once P, S and R are found, spectral factorization [14] of $\Gamma(j\omega)$ would yield C_D and D_D which then give the complete D-scale $D(s)$ in its state-space data. The problem in (9) can then be reformulated as

$$M(j\omega)^*\varphi(j\omega)^* \begin{bmatrix} P & S \\ S^T & R \end{bmatrix} \varphi(j\omega)M(j\omega) < \gamma^2 \left(\varphi(j\omega)^* \begin{bmatrix} P & S \\ S^T & R \end{bmatrix} \varphi(j\omega)\right) \quad \forall\omega \in \mathbb{R}\cup\{\infty\},$$

where S, $P = P^T$ and $R = R^T$ are real constant matrices of compatible dimensions and the real constant matrices A_D, B_D will be pre-assigned as basis matrices via a Laguerre parameterization. In this work, for a pre-defined uncertainty structure, we will use a Laguerre basis (as in (3)) to choose A_D and B_D matrices of suitable structure.

Via algebra (see Appendix A), (9) can be reformulated as follows:

$$\begin{bmatrix} (j\omega I - \hat{A}_e)^{-1}\hat{B}_e \\ I_m \end{bmatrix}^* \left[\begin{array}{ccc|c} P & SC_g & 0 & SD_g \\ C_g^TS^T & C_g^TRC_g & 0 & C_g^TRD_g \\ 0 & 0 & -\gamma^2P & -\gamma^2S \\ \hline D_g^TS^T & D_g^TRC_g & -\gamma^2S^T & D_g^TRD_g - \gamma^2R \end{array}\right] \times \begin{bmatrix} (j\omega I - \hat{A}_e)^{-1}\hat{B}_e \\ I_m \end{bmatrix} < 0 \quad \forall\omega \in \mathbb{R}\cup\{\infty\}, \tag{10}$$

where $M(s) := \left[\begin{array}{c|c} A_g & B_g \\ \hline C_g & D_g \end{array}\right] \in \mathcal{RH}_\infty$, $\hat{B}_e := \begin{bmatrix} B_DD_g \\ B_g \\ B_D \end{bmatrix} \in \mathbb{R}^{(r+2n)\times m}$, $\hat{A}_e := \begin{bmatrix} A_D & B_DC_g & 0 \\ 0 & A_g & 0 \\ 0 & 0 & A_D \end{bmatrix} \in \mathbb{R}^{(r+2n)\times(r+2n)}$, $0 < \gamma \in \mathbb{R}$, $P = P^T \in \mathbb{R}^{n\times n}$, $0 < R = R^T \in \mathbb{R}^{m\times m}$ and $S \in \mathbb{R}^{n\times m}$. Invoking KYP Lemma on (10) yields an equivalent LMI condition as follows: $\exists Z = Z^T \in \mathbb{R}^{(r+2n)\times(r+2n)}$ such that

$$\left[\begin{array}{ccc|c} P & SC_g & 0 & SD_g \\ C_g^TS^T & C_g^TRC_g & 0 & C_g^TRD_g \\ 0 & 0 & -\gamma^2P & -\gamma^2S \\ \hline D_g^TS^T & D_g^TRC_g & -\gamma^2S^T & D_g^TRD_g - \gamma^2R \end{array}\right] + \begin{bmatrix} \hat{A}_e^TZ + Z\hat{A}_e & Z\hat{B}_e \\ \hat{B}_e^TZ & 0 \end{bmatrix} < 0. \tag{11}$$

Also, the frequency condition $\Gamma(j\omega) := D(j\omega)^*D(j\omega) > 0\ \forall\omega \in \mathbb{R}\cup\{\infty\}$ is formulated in LMI form as follows. Invoking KYP lemma we have, $\Gamma(j\omega) > 0\ \forall\omega \in \mathbb{R}\cup\{\infty\}$ if and only if there exists a real matrix $Y = Y^T$ of compatible dimension such that

$$\begin{bmatrix} A_D^TY + YA_D - P & YB_D - S \\ B_D^TY - S^T & -R \end{bmatrix} < 0, \tag{12}$$

where $P = P^T$, S and $R = R^T$ remain as earlier defined in (11).

The frequency condition (9) is reformulated in LMI form as shown in (11), while (12) indicates the equivalent LMI formulation for $\Gamma(j\omega) = D(j\omega)^*D(j\omega) > 0\ \forall\omega \in \mathbb{R}\cup\{\infty\}$.

Since the structure of $D(j\omega)$ is not arbitrary but depends on the structure of the uncertainty $\triangle \in \Delta$, it means additional constraints reflecting the structure of $D(j\omega)$ are required in the D-scale problem. Consequently, the structure and commuting property of $D(s)$ and $\Gamma(s)$ are handled next.

Definition 4 ([21]). Let the scaling transfer function set be defined as:

$$\mathbf{D}^{TF} := \left\{D \in \mathcal{RH}_\infty : D^{-1} \in \mathcal{RH}_\infty, D(s_o) \in \mathbf{D}, \forall s_o \in \bar{\mathbb{C}}_+\right\}.$$

For an arbitrary realization $D(s) := \left[\begin{array}{c|c} A_D & B_D \\ \hline C_D & D_D \end{array}\right] \in \mathbf{D}^{TF}$ with a Hurwitz $A_D \in \mathbb{R}^{n\times n}$, since $D^*D = \varphi(j\omega)^*\breve{D}\varphi(j\omega)$ where $\breve{D} = \begin{bmatrix} P & S \\ S^T & R \end{bmatrix}$ and $\varphi(j\omega) = \begin{bmatrix} (j\omega I - A_D)^{-1}B_D \\ I \end{bmatrix}$, it is required by the problem formulation that D^*D commutes with Δ. This commuting condition is handled via the following definitions.

Definition 5 ([24]). Let

$$\hat{\Delta} := \left\{\hat{\triangle} := \mathrm{diag}(\,\mathrm{diag}_{i=1}^F(I_{\eta_i} \otimes \triangle_i), \mathrm{diag}_{j=1}^S(I_{\zeta_j} \otimes \delta_jI_{r_j})\,), \quad \triangle_i \in \mathbb{C}^{\beta_i\times\beta_i}, \delta_j \in \mathbb{C}\right\}$$

where $\sum_{j=1}^S \zeta_jr_j + \sum_{i=1}^F \eta_i\beta_i = n$ and η_i, ζ_j are the number of times $\triangle_i$ and $\delta_jI_{r_j}$ are repeated respectively. Note that n is obtainable from the dimension of the $n\times n$ matrix variable P.

Definition 6 ([21,24]). Let the copying operator $\mathcal{C}_{\mathcal{R}}$ be defined as:

$$\mathcal{C}_{\mathcal{R}}: \triangle \in \Delta \longmapsto \hat{\triangle} \in \hat{\Delta}.$$

As given in [21,24], $D^*D = \varphi(j\omega)^*\breve{D}\varphi(j\omega)$ commutes with Δ if $\forall\hat{\triangle} = \mathcal{C}_{\mathcal{R}}(\triangle)$ and $\triangle \in \Delta$,

$$\left\{\begin{array}{c} \begin{bmatrix} \hat{\triangle} & 0 \\ 0 & \triangle \end{bmatrix}\breve{D} = \breve{D}\begin{bmatrix} \hat{\triangle} & 0 \\ 0 & \triangle \end{bmatrix} \\ \text{and} \\ \begin{bmatrix} \hat{\triangle} & 0 \\ 0 & \triangle \end{bmatrix}\varphi(j\omega) = \varphi(j\omega)\triangle \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{c} \hat{\triangle}P = P\hat{\triangle},\ \hat{\triangle}S = S\triangle, \\ \triangle R = R\triangle,\ \hat{\triangle}A_D = A_D\hat{\triangle} \\ \text{and} \\ \hat{\triangle}B_D = B_D\triangle \end{array}\right\}. \tag{13}$$

The sets of $\breve{D}$ and (A_D, B_D) having properties desired for usage in the problem formulation are defined as follows:

Definition 7 ([21,24]). Let

$$\Xi_D := \left\{\begin{array}{c} \breve{D} = \begin{bmatrix} P & S \\ S^T & R \end{bmatrix} : P = P^T \in \mathbb{R}^{n\times n}, \\ S \in \mathbb{R}^{n\times m}, R = R^T \in \mathbb{R}^{m\times m}, \\ \begin{bmatrix} \hat{\triangle} & 0 \\ 0 & \triangle \end{bmatrix}\breve{D} = \breve{D}\begin{bmatrix} \hat{\triangle} & 0 \\ 0 & \triangle \end{bmatrix}, \\ \hat{\triangle} = \mathcal{C}_{\mathcal{R}}(\triangle) \quad \text{and} \quad \triangle \in \Delta \end{array}\right\},$$

$$\Xi_{(A_D,B_D)} := \left\{\begin{array}{c} (A_D, B_D) : A_D \in \mathbb{R}^{n\times n}, A_D \text{ is Hurwitz}, \\ B_D \in \mathbb{R}^{n\times m},\ \hat{\triangle}A_D = A_D\hat{\triangle}, \\ \hat{\triangle}B_D = B_D\triangle,\ \hat{\triangle} = \mathcal{C}_{\mathcal{R}}(\triangle) \quad \text{and} \quad \triangle \in \Delta \end{array}\right\}.$$

Subsequently, via a combination of (11) and (12), and the D-scale structure and commuting property definitions above, the final optimization problem is given as follows.

D-scale optimization problem:

Given $\gamma > 0$ and a state-space realization of $M = \left[\begin{array}{c|c} A_g & B_g \\ \hline C_g & D_g \end{array}\right] \in \mathcal{RH}_\infty^{m\times m}$ with $A_g \in \mathbb{R}^{r\times r}$, $B_g \in \mathbb{R}^{r\times m}$, $C_g \in \mathbb{R}^{m\times r}$ and $D_g \in \mathbb{R}^{m\times m}$. Let $A_D \in \mathbb{R}^{n\times n}$ and $B_D \in \mathbb{R}^{n\times m}$ be obtained via (3) and satisfy $(A_D, B_D) \in \Xi_{(A_D,B_D)}$. Find $P = P^T \in \mathbb{R}^{n\times n}$, $S \in \mathbb{R}^{n\times m}$ and $R = R^T \in \mathbb{R}^{m\times m}$ so that $\begin{bmatrix} P & S \\ S^T & R \end{bmatrix} \in \Xi_D$ and there exists $Z = Z^T \in \mathbb{R}^{(r+2n)\times(r+2n)}$ and $Y = Y^T \in \mathbb{R}^{n\times n}$ satisfying

$$\left[\begin{array}{ccc|c} P & SC_g & 0 & SD_g \\ C_g^TS^T & C_g^TRC_g & 0 & C_g^TRD_g \\ 0 & 0 & -\gamma^2P & -\gamma^2S \\ \hline D_g^TS^T & D_g^TRC_g & -\gamma^2S^T & D_g^TRD_g - \gamma^2R \end{array}\right] + \begin{bmatrix} \hat{A}_e^TZ + Z\hat{A}_e & Z\hat{B}_e \\ \hat{B}_e^TZ & 0 \end{bmatrix} < 0,$$

$$\begin{bmatrix} A_D^TY + YA_D - P & YB_D - S \\ B_D^TY - S^T & -R \end{bmatrix} < 0,$$

where $\hat{B}_e := \begin{bmatrix} B_DD_g \\ B_g \\ B_D \end{bmatrix} \in \mathbb{R}^{(r+2n)\times m}$, $\hat{A}_e := \begin{bmatrix} A_D & B_DC_g & 0 \\ 0 & A_g & 0 \\ 0 & 0 & A_D \end{bmatrix} \in \mathbb{R}^{(r+2n)\times(r+2n)}$.

Even though the above LMI problem in P, S, R, Z and Y (with A_D and B_D pre-defined and fixed using (3)) yields an entirely legitimate solution via convex optimization, nevertheless, we can considerably reduce the number of decision variables in such an optimization problem. To economize it, using Lemmas 1 and 2 all the decision variables in P can be pinned down to zero. This consideration is highly important since the size of P is the same as that of A_D obtained from a Laguerre basis (noting that A_D can have very high dimension). The next section shows the solution algorithm to be used to implement the D-scale optimization problem. Furthermore, once P, S and R are obtained, a state-space D-scale can be obtained via simple spectral factorization of $\Gamma(s)$.

5. Feasible solution algorithm

This section presents the algorithm to be used to implement the D-scale optimization problem given in Section 4. For the sake of comparison, the solution algorithm is detailed below in a way that includes the corresponding steps required when using full or economized parameterization of the parahermitian matrix function. In reality, the only difference in the implementation of the solution algorithm for the two scenarios is that $P = 0$ under the economic parameterization case.

Inputs to algorithm

- A stable system $M(s) \in \mathcal{RH}_\infty^{m\times m}$ with state-space realization $\left[\begin{array}{c|c} A_g & B_g \\ \hline C_g & D_g \end{array}\right]$.
- A controllable pair A_D and B_D obtained via (3) and belonging to $\Xi_{(A_D,B_D)}$.
- An uncertainty block with specific structure Δ is assumed to be interconnected with $M(s)$.
- An initial value for $\gamma > 0$.

Solution algorithm

Step 1. Using the initial test value γ, check if the formulated problem is feasible. If it is feasible, then iteratively decrease γ (using a *Bisection algorithm* [25,23] for example). If it is not feasible, then iteratively increase γ. Stop when the obtained γ is within a tolerance away from the minimal value for which feasibility is obtained.

Step 2. With feasibility established, extract the sets $\{P, S, R\}$ or $\{S_e, R_e\}$ (when pinning down $P = 0$ via Lemma 1) from the overall vector of decision variables under the full and economic parameterization scenarios respectively.

Step 3. Using the set of values $\{P, S, R, A_D, B_D\}$ (under the full parameterization scenario) or $\{S_e, R_e, A_D, B_D\}$ (under the economic parameterization scenario), obtain the corresponding Riccati stabilizing solutions (Theorem 13.9 in [14]) X_f and X_e associated with the full and economic parameterization scenarios respectively. A Riccati stabilizing solution will always exist as Step 1 ensures feasibility of the problem and fulfillment of the suppositions of Theorem 13.9 in [14].

Step 4. Invoke spectral factorization results (Corollary 13.20 in [14]) using the results obtained in Step 3, construct the corresponding transfer functions $D_{full} \in \mathcal{RH}_\infty$ (under the full parameterization scenario) and $D_{econ} \in \mathcal{RH}_\infty$ (under the economic parameterization scenario) using the respective set of values $\{P, S, R, A_D, B_D, X_f\}$ and $\{S_e, R_e, A_D, B_D, X_e\}$ resulting from Step 3 and the previous steps.

Outputs from algorithm

- $D_{full} \in \mathcal{RH}_\infty$ and $D_{econ} \in \mathcal{RH}_\infty$, i.e. the full and economized D-scaling transfer functions.

6. Illustrative examples

In this section, two numerical examples are demonstrated—the first example shows a comparative study with the frequency gridding method for finding the structured singular value bound and illustrates how the state-space method is useful compared to the frequency gridding technique; and in the second example, the usefulness of the proposed economic parameterization is demonstrated. Note that for frequency gridding method, the Matlab command 'mu' is used to find the structured singular value bound for the given system and uncertainty structure. The proposed solution algorithm is implemented in Matlab® R2007b running on a 2.33 GHz Intel™ Core2Duo Desktop PC with 2.00 GB of memory and 32 bit Windows XP Professional Service Pack 3 operating system.

6.1. Example 1

Let a stable plant $M(s) = \left[\begin{array}{c|c} A_g & B_g \\ \hline C_g & D_g \end{array}\right]$ is given where

$$A_g = \begin{bmatrix} -2 & -400 & 0.1 & 0.2 \\ 10 & 0 & 0.5 & 0 \\ 0 & 2 & -3 & -8 \\ 0 & 0 & 50 & 0 \end{bmatrix}; \qquad B_g = \begin{bmatrix} 2 & 0.8 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix};$$

$$C_g = \begin{bmatrix} 1.5 & 0 & 1 & 0 \\ 0 & 1 & 2 & 2 \end{bmatrix}; \qquad D_g = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Also, let the uncertainty structure Δ be defined as:

$$\Delta := \left\{\mathrm{diag}(\delta_1, \delta_2) : \delta_1, \delta_2 \in \mathbb{C}\right\} \tag{14}$$

and the corresponding D-scale structure $\mathbf{D}$ is given as:

$$\mathbf{D} := \{\mathrm{diag}(D_1, D_2) : D_1 \in \mathbb{C}, D_2 \in \mathbb{C}, D_i > 0, 1 \le i \le 2\}. \tag{15}$$

In Table 1, the results have been shown where the structured singular value bound is calculated using the Matlab command 'mu', which is based on the frequency gridding and curve-fitting technique. It is apparent that the number of frequency grid points and the chosen frequency range affect the accuracy of the result. Whereas the state-space method is free from these disadvantages,

Table 1
Using 'mu' command of Matlab.

Frequency grid points	Upper μ bound when grid points in between 10^{-2} and 10^2 rad/s	Computation time (s)	Upper μ bound grid points in between 10^0 and 10^2 rad/s	Computation time (s)
50	1.3834	0.1563	1.8840	0.1713
100	1.7651	0.2969	1.8949	0.3594
200	1.9215	0.6094	1.9215	0.7188
500	1.9167	1.7344	1.9274	1.7656
1000	1.9276	3.2969	1.9276	3.4375

and for the given system and uncertainty structure, the proposed algorithm of this paper is used to find the structured singular value bound. For $N = 3$ and $\tau = 0.001$, we found $\mu = 1.9279$ which is very close to the bound that we obtain from the curve-fitting technique with higher frequency grid points. The values 10, 10×10^{-10} and 0.001 were chosen as the respective upper limit on γ, lower limit on γ and tolerance value within the bisection algorithm. The computation time is 6.25 s for full parameterization.

In the following example, we will show the usefulness of the economic parameterization when the number of decision variables are large.

6.2. Example 2

This numerical example shows the effectiveness of the economic parameterization. The implementation is carried out using the stable system $M(s) := \left[\begin{array}{c|c} A_g & B_g \\ \hline C_g & D_g \end{array}\right]$ with

$$A_g = \begin{bmatrix} -19.48 & -0.91 & -3.15 & 7.94 & -4.57 & -4.64 \\ -1.02 & -10.59 & -0.64 & 23.82 & -13.71 & -13.92 \\ -3.24 & -0.70 & -30.99 & 15.42 & -8.88 & -9.01 \\ 0 & -0.15 & -0.12 & -5.29 & -6.24 & -7.00 \\ 0 & 2.14 & 1.72 & 14.97 & -24.21 & -14.59 \\ 0 & 2.70 & 2.17 & 10.21 & -11.55 & -39.76 \end{bmatrix};$$

$$B_g = \begin{bmatrix} 1.25 & -0.38 & -1.27 & 0 & -0.02 & 0.76 \\ -0.64 & -0.19 & 0.98 & 0 & 0.62 & 0 \\ 0.58 & -1.38 & -0.04 & 0.86 & 0.47 & 0 \\ 0 & 0.03 & 0 & 0 & -0.05 & 0 \\ 0 & -0.36 & 0 & 0 & 0.71 & 0 \\ 0 & -0.45 & 0 & 0 & 0.89 & 0 \end{bmatrix};$$

$$C_g = \begin{bmatrix} -23.77 & 8.66 & 4.30 & 13.44 & -7.73 & -7.85 \\ -2.74 & 0.24 & 0.19 & 17.37 & -10.00 & -10.15 \\ 0 & -9.23 & -24.44 & 0 & 0 & 0 \\ 0 & -0.54 & -10.85 & 11.91 & -6.85 & -6.96 \\ -5.11 & 1.44 & 11.20 & -2.70 & 1.55 & 1.58 \\ -0.02 & 0.16 & 5.95 & 11.73 & -6.75 & -6.86 \end{bmatrix};$$

$$D_g = \begin{bmatrix} 4.14 & 8.55 & 0 & 9.24 & 0.61 & -5.50 \\ 0 & -0.40 & 3.52 & 0 & 0.79 & 0.40 \\ 0 & -7.26 & 11.33 & 0 & 0 & 0 \\ 3.18 & -4.72 & 0 & 0 & 0.54 & 0 \\ 0 & -6.07 & 0 & 10.28 & -0.12 & 0 \\ 0 & -0.27 & -0.52 & 3.95 & 0.53 & 11.53 \end{bmatrix}.$$

For ease of illustration, the considered uncertainty structure Δ is:

$$\Delta := \{\mathrm{diag}(\delta_1 I_2, \delta_2 I_4) : \delta_1, \delta_2 \in \mathbb{C}\} \tag{16}$$

and the corresponding D-scale structure $\mathbf{D}$ is given as:

$$\mathbf{D} := \Big\{\mathrm{diag}(D_1, D_2) : D_1 \in \mathbb{C}^{2\times 2}, D_2 \in \mathbb{C}^{4\times 4}, \\ D_i = D_i^* > 0, 1 \le i \le 2\Big\}. \tag{17}$$

The uncertainty structure given in (16) means that for this illustration, $\mu_\Delta(M)$ is strictly less than $\inf_{D\in\mathbf{D}} \bar{\sigma}(DMD^{-1})$ (see [19] for further details). The structure and values of A_D and B_D are determined by considering the structure of $D(s) \in \mathcal{RH}_\infty^{6\times 6}$ in (17) via (3) as follows:

$$D(s) := \mathcal{QB}(s) = \begin{bmatrix} \mathcal{Q}_1\mathcal{B}_1(s) & 0 \\ 0 & \mathcal{Q}_2\mathcal{B}_2(s) \end{bmatrix} = \begin{bmatrix} \mathcal{Q}_1 & 0 \\ 0 & \mathcal{Q}_2 \end{bmatrix}\begin{bmatrix} \mathcal{B}_1(s) & 0 \\ 0 & \mathcal{B}_2(s) \end{bmatrix}$$

where $\mathcal{B}_1(s) := \left[\begin{array}{c|c} A_{D_1} & B_{D_1} \\ \hline C_{D_1} & D_{D_1} \end{array}\right] \in \mathcal{RH}_\infty^{(N+1)2\times 2}$, $\mathcal{B}_2(s) := \left[\begin{array}{c|c} A_{D_2} & B_{D_2} \\ \hline C_{D_2} & D_{D_2} \end{array}\right] \in \mathcal{RH}_\infty^{(N+1)4\times 4}$, $\mathcal{Q}_1 \in \mathbb{R}^{2\times(N+1)2}$ and $\mathcal{Q}_1 \in \mathbb{R}^{4\times(N+1)4}$. Hence suitable structures (that also satisfy Definition 7) for A_D and B_D are given by $A_D := \begin{bmatrix} A_{D_1} & 0 \\ 0 & A_{D_2} \end{bmatrix} \in \mathbb{R}^{r\times r}$ and $B_D := \begin{bmatrix} B_{D_1} & 0 \\ 0 & B_{D_2} \end{bmatrix} \in \mathbb{R}^{r\times 6}$, where $A_{D_1} \in \mathbb{R}^{l_1\times l_1}$, $A_{D_2} \in \mathbb{R}^{l_2\times l_2}$, $B_{D_1} \in \mathbb{R}^{l_1\times 2}$, $B_{D_2} \in \mathbb{R}^{l_2\times 4}$ are full constant matrices, and $r = l_1 + l_2$.

Correspondingly, $\hat{\Delta} := \big\{\mathrm{diag}(\delta_1 I_{l_1}, \delta_2 I_{l_2}) : \delta_1, \delta_2 \in \mathbb{C}\big\}$, hence $P := \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$, $R := \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix}$ and $S := \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}$, where $P_1 = P_1^T \in \mathbb{R}^{l_1\times l_1}$, $P_2 = P_2^T \in \mathbb{R}^{l_2\times l_2}$, $R_1 = R_1^T \in \mathbb{R}^{2\times 2}$, $R_2 = R_2^T \in \mathbb{R}^{4\times 4}$, $S_1 \in \mathbb{R}^{l_1\times 2}$ and $S_2 \in \mathbb{R}^{l_2\times 4}$ are full constant matrices, and $r = l_1 + l_2$.

The pairs (N, τ): (2, 0.001), (2, 0.01) and (2, 0.1) are chosen as test inputs to the Laguerre parameterization, so as to obtain the corresponding pairs (A_D, B_D) required as inputs to the solution algorithm. The initial test value of γ is chosen via a *Bisection algorithm* [25,23]. The values 35, 2.2204×10^{-16} and 0.001 were chosen as the respective upper limit on γ, lower limit on γ and tolerance value within the bisection algorithm. Step 1 of the solution algorithm is implemented using bisection algorithm.

Considering the pair $(N, \tau) = (2, 0.001)$, the corresponding $A_D \in \mathbb{R}^{12\times 12}$ and $B_D \in \mathbb{R}^{12\times 6}$ are obtained as given in Appendix B (such that $l_1 = 4$ and $l_2 = 8$) and these are used as input to the solution algorithm. Using the Matlab® LMI solver '*feasp*' (see [6]), it is observed that the smallest γ for which feasibility is achieved is $\gamma = 13.4241$ in both the full and economized parameterization cases. With feasibility established, the sets $\{P, S, R\}$ (for the full parameterization case) and $\{S_e, R\}$ (for the economic parameterization case) are extracted from the overall vector of decision variables which are given in Appendix C. It is observed that there is a computational time reduction from 2.69 min (when full parahermitian parameterization is used) to 1.01 min (when economic parahermitian parameterization is used). Likewise, the decision variable count is also reduced from 642 variables (for full parameterization) to 596 variables (for economic parameterization).

Upon obtaining the corresponding Riccati stabilizing solutions X_f and X_e, spectral factorization results (see Corollary 13.20 in [14]) are invoked to obtain $D_{full} := \left[\begin{array}{c|c} A_D & B_D \\ \hline C_D & D_D \end{array}\right] \in \mathcal{RH}_\infty$ and $D_{econ} := \left[\begin{array}{c|c} A_D & B_D \\ \hline C_{D_e} & D_{D_e} \end{array}\right] \in \mathcal{RH}_\infty$. The state-space matrices C_D, C_{D_e}, D_D and D_{D_e} are given in Appendix C. The comparison of the decision variable count and computation time taken to implement the solution algorithm for the different pairs $(N, \tau) = (2, 0.001)$, (2, 0.01) and (2, 0.1) are given in Table 2. It is seen that there is considerable

Table 2
Results for the chosen (N, τ) pairs.

τ	$N = 2$		
		Computation time (min)	Decision variables count
0.001	Full	2.69	642
	Economized	1.01	596
0.01	Full	3.94	642
	Economized	1.37	596
0.1	Full	4.03	642
	Economized	1.82	596

reduction in the computation time among the full and economic parameterization cases. For all the pairs, the optimization problem becomes feasible at $\gamma = 13.4241$. In order to test the validity of this '*feasible*' value of γ (obtained via the solution algorithm), the 'mu' command of Matlab® is used to obtain the pointwise frequency values of the upper and lower bound on $\mu_\Delta(M)$ for the uncertainty structure Δ as given in (16). The maximum pointwise frequency value of the upper bound is obtained as 13.4187.

7. Conclusions

In this paper, an economic parameterization of parahermitian matrix functions is exploited to solve the D-scale optimization problem of μ-analysis directly in state-space variables for the D-scales, thereby avoiding pointwise-in-frequency gridding and subsequent fitting to form D-scales suitable for μ-synthesis via D–K iterations. To reduce the number of decision variables in the optimization problem, the economic parameterization results (Lemmas 1 and 2) are invoked, this in turn reduces the computational time of the solution algorithm for the optimization problem. The derived constraints are posed in the LMI framework that can easily be solved using available LMI toolbox. To illustrate the theory and results given in this paper a numerical example is given.

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Appendix A

Let M and D have compatible dimensions and property satisfying (9). From (9) we have that

$$\begin{bmatrix} M(j\omega) \\ I_m \end{bmatrix}^* \begin{bmatrix} D(j\omega)^*D(j\omega) & 0 \\ 0 & -\gamma^2 D(j\omega)^*D(j\omega) \end{bmatrix} \times \begin{bmatrix} M(j\omega) \\ I_m \end{bmatrix} < 0 \quad \forall\omega \in \mathbb{R}\cup\{\infty\}. \tag{18}$$

With $\begin{bmatrix} M(j\omega) \\ I_m \end{bmatrix} := \begin{bmatrix} C_g & D_g & 0 \\ 0 & 0 & I_m \end{bmatrix} \begin{bmatrix} (j\omega I - A_g)^{-1}B_g \\ I_m \\ I_m \end{bmatrix}$ and $\Gamma(j\omega) = D(j\omega)^*D(j\omega)$, from (18), we have

$$\begin{bmatrix} (j\omega I - A_g)^{-1}B_g \\ I_m \\ I_m \end{bmatrix}^* \begin{bmatrix} C_g & D_g & 0 \\ 0 & 0 & I_m \end{bmatrix}^* \times \begin{bmatrix} (j\omega I - A_D)^{-1}B_D & 0 \\ I_m & 0 \\ 0 & (j\omega I - A_D)^{-1}B_D \\ 0 & I_m \end{bmatrix}^* \times \begin{bmatrix} P & S & 0 & 0 \\ S^T & R & 0 & 0 \\ 0 & 0 & -\gamma^2 P & -\gamma^2 S \\ 0 & 0 & -\gamma^2 S^T & -\gamma^2 R \end{bmatrix} \times \begin{bmatrix} (j\omega I - A_D)^{-1}B_D & 0 \\ I_m & 0 \\ 0 & (j\omega I - A_D)^{-1}B_D \\ 0 & I_m \end{bmatrix} \times \begin{bmatrix} C_g & D_g & 0 \\ 0 & 0 & I_m \end{bmatrix} \begin{bmatrix} (j\omega I - A_g)^{-1}B_g \\ I_m \\ I_m \end{bmatrix} < 0$$
$$\forall\omega \in \mathbb{R}\cup\{\infty\}. \tag{19}$$

Simplifying we have

$$\begin{bmatrix} \begin{bmatrix} (j\omega I - A_D)^{-1}B_D \\ I_m \end{bmatrix} \begin{bmatrix} C_g & D_g \end{bmatrix} \begin{bmatrix} (j\omega I - A_g)^{-1}B_g \\ I_m \end{bmatrix} \\ \begin{bmatrix} (j\omega I - A_D)^{-1}B_D \\ I_m \end{bmatrix} \end{bmatrix}^* \times \begin{bmatrix} P & S & 0 & 0 \\ S^T & R & 0 & 0 \\ 0 & 0 & -\gamma^2 P & -\gamma^2 S \\ 0 & 0 & -\gamma^2 S^T & -\gamma^2 R \end{bmatrix} \times \begin{bmatrix} \begin{bmatrix} (j\omega I - A_D)^{-1}B_D \\ I_m \end{bmatrix} \begin{bmatrix} C_g & D_g \end{bmatrix} \\ \begin{bmatrix} (j\omega I - A_g)^{-1}B_g \\ I_m \end{bmatrix} \\ \begin{bmatrix} (j\omega I - A_D)^{-1}B_D \\ I_m \end{bmatrix} \end{bmatrix} < 0$$
$$\forall\omega \in \mathbb{R}\cup\{\infty\}. \tag{20}$$

Let

$$\begin{bmatrix} \begin{bmatrix} (j\omega I - A_D)^{-1}B_D \\ I_m \end{bmatrix} \begin{bmatrix} C_g & D_g \end{bmatrix} \begin{bmatrix} (j\omega I - A_g)^{-1}B_g \\ I_m \end{bmatrix} \\ \begin{bmatrix} (j\omega I - A_D)^{-1}B_D \\ I_m \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] \\ \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right] \end{bmatrix}$$

and

$$\left[\begin{array}{c|c} \hat{A}_e & \hat{B}_e \\ \hline \hat{C}_e & \hat{D}_e \end{array}\right] = \begin{bmatrix} \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] \\ \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right] \end{bmatrix}. \tag{21}$$

From operations on state-space systems, we have that

$$\begin{bmatrix} \begin{bmatrix} (j\omega I - A_D)^{-1}B_D \\ I_m \end{bmatrix} \begin{bmatrix} C_g & D_g \end{bmatrix} \begin{bmatrix} (j\omega I - A_g)^{-1}B_g \\ I_m \end{bmatrix} \\ \begin{bmatrix} (j\omega I - A_D)^{-1}B_D \\ I_m \end{bmatrix} \end{bmatrix} \equiv \left[\begin{array}{ccc|c} A_D & B_DC_g & 0 & B_DD_g \\ 0 & A_g & 0 & B_g \\ 0 & 0 & A_D & B_D \\ \hline I_n & 0 & 0 & 0 \\ 0 & C_g & 0 & D_g \\ 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & I_m \end{array}\right]. \tag{22}$$

$$A_D = \begin{bmatrix} -4000 & -4000000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4000 & -4000000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4000 & -4000000 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -4000 & -4000000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4000 & -4000000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4000 & -4000000 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\text{and} \quad B_D = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Box I.

For ease of illustration, the validity of (22) will only be shown going from right to left. Considering (21), the right hand side of (22) and the state-space system definitions given in Section 3.4 of [23], it is straightforward to see that $\left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] = \left[\begin{array}{cc|c} A_D & B_D C_g & B_D D_g \\ 0 & A_g & B_g \\ \hline I_n & 0 & 0 \\ 0 & C_g & D_g \end{array}\right]$ and $\left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right] = \left[\begin{array}{c|c} A_D & B_D \\ \hline I_n & 0 \\ 0 & I_m \end{array}\right]$, satisfies the expression given in (22). Thus, having now established that

$$\left[\begin{array}{c|c} \hat{A}_e & \hat{B}_e \\ \hline \hat{C}_e & \hat{D}_e \end{array}\right] := \left[\begin{array}{ccc|c} A_D & B_D C_g & 0 & B_D D_g \\ 0 & A_g & 0 & B_g \\ 0 & 0 & A_D & B_D \\ \hline I_n & 0 & 0 & 0 \\ 0 & C_g & 0 & D_g \\ 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & I_m \end{array}\right], \tag{23}$$

(20) can be re-written as

$$\begin{bmatrix} (j\omega I - \hat{A}_e)^{-1}\hat{B}_e \\ I_m \end{bmatrix}^* \begin{bmatrix} \hat{C}_e & \hat{D}_e \end{bmatrix}^* \times \begin{bmatrix} P & S & 0 & 0 \\ S^T & R & 0 & 0 \\ 0 & 0 & -\gamma^2 P & -\gamma^2 S \\ 0 & 0 & -\gamma^2 S^T & -\gamma^2 R \end{bmatrix} \times \begin{bmatrix} \hat{C}_e & \hat{D}_e \end{bmatrix} \begin{bmatrix} (j\omega I - \hat{A}_e)^{-1}\hat{B}_e \\ I_m \end{bmatrix} < 0$$

$$\forall \omega \in \mathbb{R} \cup \{\infty\}, \tag{24}$$

where $\hat{A}_e$, $\hat{B}_e$, $\hat{C}_e$ and $\hat{D}_e$ are as defined in (23). Now, $\begin{bmatrix} \hat{C}_e & \hat{D}_e \end{bmatrix}^* \begin{bmatrix} P & S & 0 & 0 \\ S^T & R & 0 & 0 \\ 0 & 0 & -\gamma^2 P & -\gamma^2 S \\ 0 & 0 & -\gamma^2 S^T & -\gamma^2 R \end{bmatrix} \begin{bmatrix} \hat{C}_e & \hat{D}_e \end{bmatrix}$ is equal to

$$\left[\begin{array}{ccc|c} P & SC_g & 0 & SD_g \\ C_g^T S^T & C_g^T R C_g & 0 & C_g^T R D_g \\ 0 & 0 & -\gamma^2 P & -\gamma^2 S \\ \hline D_g^T S^T & D_g^T R C_g & -\gamma^2 S^T & D_g^T R D_g - \gamma^2 R \end{array}\right], \tag{25}$$

hence (24) is equal to (10).

Appendix B

Using (3), the controllable constant matrix pair (A_D, B_D) given in Box I is obtained for the test pair $(N, \tau) = (2, 0.001)$.

Appendix C

$$S = \begin{bmatrix} -0.25 & -0.01 & 0 & 0 & 0 & 0 \\ -0.09 & 0.18 & 0 & 0 & 0 & 0 \\ 0.13 & -0.17 & 0 & 0 & 0 & 0 \\ -0.01 & 0.32 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.12 & -2.05 & 0.90 & -0.22 \\ 0 & 0 & -1.75 & -2.86 & 5.62 & 1.01 \\ 0 & 0 & 0.03 & -0.91 & 0.30 & -0.06 \\ 0 & 0 & -0.58 & -0.75 & 1.69 & 0.30 \\ 0 & 0 & 0 & -0.09 & -0.15 & -0.05 \\ 0 & 0 & -0.14 & -0.47 & 0.87 & 0.21 \\ 0 & 0 & 0.01 & -0.77 & 0.41 & -0.27 \\ 0 & 0 & -0.47 & -0.76 & 2.04 & 0.34 \end{bmatrix},$$

$$S_e = \begin{bmatrix} -0.04 & -0.01 & 0 & 0 & 0 & 0 \\ -0.06 & 0.18 & 0 & 0 & 0 & 0 \\ 0.13 & 0.04 & 0 & 0 & 0 & 0 \\ -0.01 & 0.29 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.08 & -2.05 & 0.90 & -0.22 \\ 0 & 0 & -1.75 & -2.85 & 5.63 & 1.00 \\ 0 & 0 & 0.03 & -0.70 & 0.30 & -0.06 \\ 0 & 0 & -0.60 & -0.78 & 1.70 & 0.28 \\ 0 & 0 & 0 & -0.09 & 0.06 & -0.05 \\ 0 & 0 & -0.14 & -0.47 & 0.88 & 0.21 \\ 0 & 0 & 0.01 & -0.77 & 0.41 & -0.07 \\ 0 & 0 & -0.46 & -0.74 & 2.04 & 0.30 \end{bmatrix},$$

$$R = \begin{bmatrix} 71.73 & 19.70 & 0 & 0 & 0 & 0 \\ 19.70 & 189.73 & 0 & 0 & 0 & 0 \\ 0 & 0 & 102.29 & -42.04 & -36.45 & -12.21 \\ 0 & 0 & -42.04 & 511.35 & -91.18 & 5.87 \\ 0 & 0 & -36.45 & -91.18 & 252.97 & -54.14 \\ 0 & 0 & -12.21 & 5.87 & -54.14 & 333.04 \end{bmatrix},$$

$$P = \left[\begin{matrix} 1665.92 & -224.68 & 0 & 2.05 & 0 & 0 \\ -224.68 & 256567.35 & -2.02 & 0.03 & 0 & 0 \\ 0 & -2.02 & 1666.11 & -412.30 & 0 & 0 \\ 2.05 & 0.03 & -412.30 & -200708.29 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1667.32 & 306.86 \\ 0 & 0 & 0 & 0 & 306.86 & 797.00 \\ 0 & 0 & 0 & 0 & 0 & 59.02 \\ 0 & 0 & 0 & 0 & -59.49 & -0.01 \\ 0 & 0 & 0 & 0 & 0.01 & 18.41 \\ 0 & 0 & 0 & 0 & -19.00 & 0 \\ 0 & 0 & 0 & 0 & 0 & -9.45 \\ 0 & 0 & 0 & 0 & 11.90 & -0.08 \end{matrix}\right.$$

$$\left.\begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -59.49 & 0.01 & -19.00 & 0 & 11.90 \\ 59.02 & -0.01 & 18.41 & 0 & -9.45 & -0.08 \\ 1666.90 & -49.09 & 0 & -4.58 & 0 & 46.09 \\ -49.09 & -250258.39 & 4.51 & 0.07 & -46.61 & 0.01 \\ 0 & 4.51 & 1667.40 & 414.85 & 0 & 23.28 \\ -4.58 & 0.07 & 414.85 & 97892.96 & -23.79 & 0.01 \\ 0 & -46.61 & 0 & -23.79 & 1665.89 & -609.28 \\ 46.09 & 0.01 & 23.28 & 0.01 & -609.28 & -339487.76 \end{matrix}\right].$$

$$R_e = \begin{bmatrix} 71.73 & 19.70 & 0 & 0 & 0 & 0 \\ 19.70 & 189.73 & 0 & 0 & 0 & 0 \\ 0 & 0 & 102.29 & -42.04 & -36.45 & -12.21 \\ 0 & 0 & -42.04 & 511.35 & -91.18 & 5.87 \\ 0 & 0 & -36.45 & -91.18 & 252.97 & -54.14 \\ 0 & 0 & -12.21 & 5.87 & -54.14 & 333.04 \end{bmatrix},$$

$$D_D = \begin{bmatrix} 8.423 & 0.889 & 0 & 0 & 0 & 0 \\ 0.889 & 13.746 & 0 & 0 & 0 & 0 \\ 0 & 0 & 9.870 & -1.423 & -1.606 & -0.528 \\ 0 & 0 & -1.423 & 22.434 & -2.460 & 0.027 \\ 0 & 0 & -1.606 & -2.460 & 15.546 & -1.629 \\ 0 & 0 & -0.528 & 0.027 & -1.629 & 18.169 \end{bmatrix},$$

$$D_{D_e} = \begin{bmatrix} 8.423 & 0.889 & 0 & 0 & 0 & 0 \\ 0.889 & 13.746 & 0 & 0 & 0 & 0 \\ 0 & 0 & 9.870 & -1.423 & -1.606 & -0.528 \\ 0 & 0 & -1.423 & 22.434 & -2.460 & 0.027 \\ 0 & 0 & -1.606 & -2.460 & 15.546 & -1.629 \\ 0 & 0 & -0.528 & 0.027 & -1.629 & 18.169 \end{bmatrix},$$

$$C_D = \left[\begin{matrix} -0.364 & -0.307 & 1.132 & 0.199 & 0 & 0 & 0 \\ -0.176 & 2.418 & 0.705 & 4.010 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2.409 & -22.777 & 0.835 \\ 0 & 0 & 0 & 0 & -48.225 & -75.143 & -16.529 \\ 0 & 0 & 0 & 0 & 19.246 & 95.623 & 6.463 \\ 0 & 0 & 0 & 0 & -5.484 & 9.909 & -1.615 \end{matrix}\right.$$

$$\left.\begin{matrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -7.652 & 0.097 & -2.238 & 0.618 & -6.944 \\ -20.886 & -2.101 & -12.563 & -18.332 & -21.067 \\ 28.785 & 1.170 & 14.709 & 8.351 & 33.845 \\ 2.707 & -1.031 & 2.370 & -1.908 & 2.263 \end{matrix}\right] \quad \text{and}$$

$$C_{D_e} = \left[\begin{matrix} -0.364 & -0.307 & 1.132 & 0.199 & 0 & 0 & 0 \\ -0.176 & 2.418 & 0.705 & 4.010 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2.409 & -22.777 & 0.835 \\ 0 & 0 & 0 & 0 & -48.225 & -75.144 & -16.529 \\ 0 & 0 & 0 & 0 & 19.246 & 95.624 & 6.463 \\ 0 & 0 & 0 & 0 & -5.484 & 9.910 & -1.615 \end{matrix}\right.$$

$$\left.\begin{matrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -7.652 & 0.097 & -2.238 & 0.618 & -6.944 \\ -20.886 & -2.101 & -12.563 & -18.333 & -21.067 \\ 28.785 & 1.170 & 14.709 & 8.351 & 33.846 \\ 2.708 & -1.031 & 2.370 & -1.908 & 2.263 \end{matrix}\right].$$

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