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A "mixed" small gain and passivity theorem in the frequency domain☆

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Abstract

We show that the negative feedback interconnection of two causal, stable, linear time-invariant systems, with a "mixed" small gain and passivity property, is guaranteed to be finite-gain stable. This "mixed" small gain and passivity property refers to the characteristic that, at a particular frequency, systems in the feedback interconnection are either both "input and output strictly passive"; or both have "gain less than one"; or are both "input and output strictly passive" and simultaneously both have "gain less than one". The "mixed" small gain and passivity property is described mathematically using the notion of dissipativity of systems, and finite-gain stability of the interconnection is proven via a stability result for dissipative interconnected systems.

Keywords: Linear systems; Stability; Dissipative systems; Finite gain; Passivity

1. Introduction

The small gain and passivity theorems are two of the most important results in the theory of stability of input–output systems. The small gain theorem states that if the product of the gains of two stable systems is less than one then the feedback interconnection of the two systems is stable [3,9,4,16]. The passivity theorem guarantees stability of a feedback interconnection of two stable systems if, for instance, both of the systems are passive, and one of them is input strictly passive with finite gain [3,9,4,14]. Of course, there exist many situations where stability cannot be guaranteed by use of the small gain or passivity theorems because the classes of systems under consideration are not compatible.

For instance, it has been observed that high frequency dynamics can frequently destroy the passivity property of an otherwise passive system. A celebrated controversy in adaptive control [13] depended on the observation that passivity conditions normally forming part of the hypotheses of the proofs of convergence of certain adaptive control algorithms should not be assumed to be valid in practice (because high frequency dynamics often neglected for modeling purposes will always be present in a real system). Failure of the passivity condition invalidated the applicability of the associated theorem on the algorithm convergence to most real-life applications, and left a cloud hanging over the real-life use of the algorithm. Simulations of [13] confirmed that adverse behavior could occur when high frequency dynamics were explicitly taken into account.

The book [2] (see also [8,1]) described tools for establishing stability of adaptive systems of the type examined in [13]; that is, where passivity properties hold only for low frequency signals (in a sense made precise later in this paper). Stability is established if additionally (and in a rough manner of speaking) gains are small at high frequencies, i.e. a small gain property

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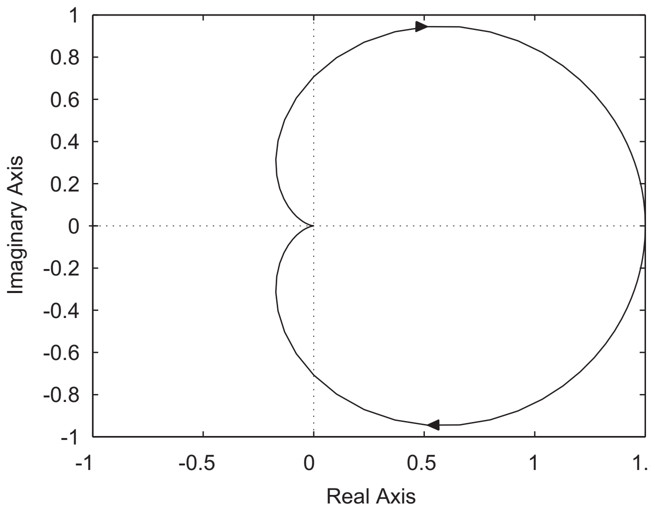

Fig. 1. Nyquist diagram of $m_1(s)$.

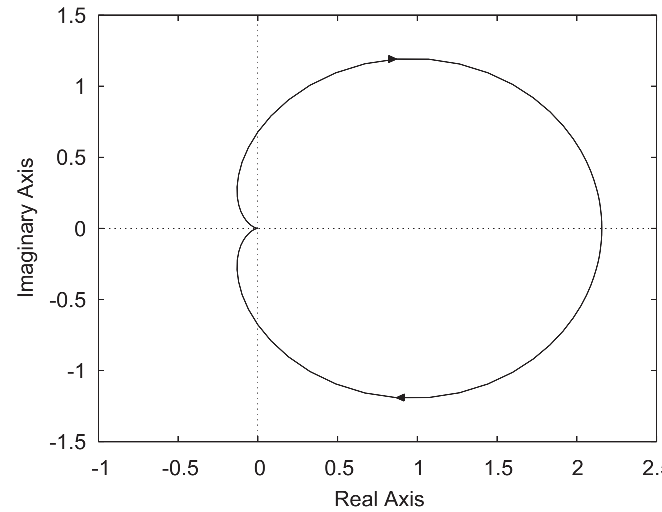

Fig. 2. Nyquist diagram of $m_2(s)$.

(in the sense of the small gain theorem) holds in the frequency band where the passivity condition fails (again, more precise discussion of what is meant by small gain just at high frequencies appears later). The fact that there is an important class of applications in which passivity and small gain ideas have to be blended is a motivation for this paper.

Indeed, the idea of a "merging" of the passivity and small gain theorems to provide stability results of feedback interconnections of a class of systems, broader than those dealt with by the small gain and passivity theorems, respectively, would be extremely useful. Consider two open-loop causal, stable, single-input single-output (SISO) systems with linear time-invariant (LTI) transfer functions; say

$$m_1(s) = \frac{3}{(s+1)(s+2)}$$

and

$$m_2(s) = \frac{4.314}{(s+0.5)(s+4)},$$

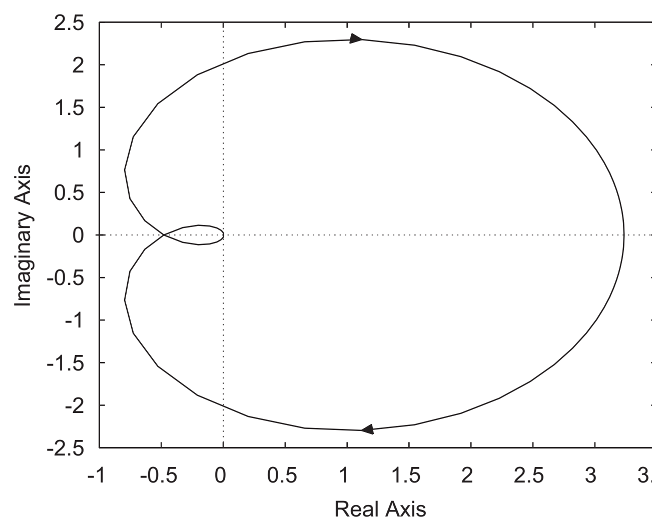

Fig. 3. Nyquist diagram of $m_1(s)m_2(s)$.

with Nyquist diagrams shown in Figs. 1 and 2. It is clear that, if in some frequency range $[0, \Omega]$ the systems are passive (i.e. the real part of each of the transfer functions is positive), and if in the frequency range $[\Omega, \infty)$ the product of the amplitudes of the transfer functions is less than one, then there is no way that the Nyquist diagram of the cascade would encircle the point $-1 + \mathrm{j}0$. Accordingly, the closed-loop would be stable. For example, Fig. 3 shows that the Nyquist diagram of $m_1(s)m_2(s)$ does not encircle $-1 + \mathrm{j}0$.

Continuing with the example, note that one could not simply determine stability by scaling one of the systems with transfer functions $m_1(s)$ or $m_2(s)$ to have gain less than one, as this would result in an increase in the other system's gain. That is, absolute feedback loop gain is constant. Similarly, multipliers cannot always be used to transform the feedback loop such that both systems are passive and one is strictly passive with finite gain. There exist transformations in the literature that transform a passive system to a system with gain less than one; and vice versa [3,9]. As an extension to this, one could consider doing a stability-preserving loop transformation on the feedback loop such that the product of the gains of the transformed systems is less than one say, and hence the small gain theorem could be applied to determine stability. Initial investigations hint that such a successful loop transformation may be difficult to find.

The use of integral quadratic constraints (IQCs) to describe systems in feedback interconnections was introduced in [10] as a powerful method of determining closed-loop stability. The result assumes that one of the systems in the feedback interconnection is described by an LTI operator, while the other system represents the "trouble-making" (nonlinear, time-varying or uncertain) components of the feedback loop. The stability theorem [10, Theorem 1] captures the classical small gain and passivity/dissipativity theorems under the proviso that one of the two cascaded systems in the loop is linear and time-invariant.

In this paper, we develop the idea of merging the passivity and small gain theorems for multi-input multi-output (MIMO), causal, stable, LTI systems which have associated with them the following "mixed" small gain and passivity property (illustrated by the above example). We consider the frequency range $-\infty < \omega < \infty$ which can be divided into intervals for which two systems to be considered in an interconnection are both either: (a) "input and output strictly passive"; (b) "input and output strictly passive" and have "gain less than one"; or (c) have "gain less than one". The notion of dissipativity, initiated by [15] and used by [7,12,11,6] to produce stability results for interconnected systems, is exploited to mathematically describe this "mixed" property of systems. This escription is given in Section 2. The main result of the paper shows that finite-gain stability of two such "mixed" MIMO systems connected in a negative feedback loop is guaranteed. The feedback interconnection is described in Section 3 and the main result is given in Section 4. Section 5 contains the conclusions and outlines intended future development of the ideas presented in this paper. We proceed with this approach (as opposed to an IQC approach, which would seem to be readily possible) to "merge" the passivity and small gain theorems when both systems are linear, as we intend for our methodology to pave the way for similar results when both systems are nonlinear (which have been submitted for publication).

1.1. Notation

The results of this paper are concerned with the frequency domain. We consider frequency domain signals $f \in \mathscr{L}_2(\mathrm{j}\mathbb{R})$, where $\mathscr{L}_2(\mathrm{j}\mathbb{R})$ denotes the real frequency domain Lebesgue space in which

$$\|f\| = \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} f^*(\mathrm{j}\omega) f(\mathrm{j}\omega)\, \mathrm{d}\omega \right\}^{1/2}$$

and the superscript $(\cdot)^*$ denotes the complex conjugate transpose. $\mathscr{L}_2(\mathrm{j}\mathbb{R})$ is a Hilbert space under the inner product

$$\langle f, g\rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} g^*(\mathrm{j}\omega) f(\mathrm{j}\omega)\, \mathrm{d}\omega.$$

$\mathscr{R}$ denotes the set of proper real rational transfer function matrices. For a transfer function matrix $G \in \mathscr{R}$, $G^*(s)$ is defined to mean $G(-s)^{\mathrm{T}}$. $\mathscr{L}_\infty$ is a Banach space of matrix- (or scalar-) valued functions that are essentially bounded on $\mathrm{j}\mathbb{R}$. The Hardy space, $\mathscr{H}_\infty$, is the closed subspace of $\mathscr{L}_\infty$ with functions that are analytic and bounded in the open right-half plane (RHP), with norm denoted $\|\cdot\|_\infty$. In other words, $\mathscr{H}_\infty$ is the space of transfer functions of stable, LTI, continuous-time systems. $\mathscr{RH}_\infty$ denotes the subspace of $\mathscr{H}_\infty$ whose transfer function matrices are proper and real rational.

2. Mathematical description of systems

In this section, a mathematical description for a causal LTI system with transfer function matrix $M \in \mathscr{RH}_\infty$ and with the following frequency domain property is formulated. Consider the frequency range $-\infty < \omega < \infty$ and divide this range into intervals for which system M is either: (a) "input and output strictly passive"; (b) "input and output strictly passive and has gain less than one"; or (c) "has gain less than one". (This property will be referred to throughout the paper as the "mixed" small gain and passivity frequency domain property of a system M.) What is meant by a system being input and output strictly passive on a frequency interval, or having finite gain on a frequency interval, is defined below. The standard notions of input and output strict passivity and finite gain which refer to the full $\mathrm{j}\omega$-axis are also provided.

Definition 1 (*Desoer and Vidyasagar [3]; Teel et al. [14]*). Consider a causal system with transfer function matrix $M \in \mathscr{RH}_\infty$. This system is input and output strictly passive if $\exists \varepsilon > 0,\ \delta > 0$ such that

$$\langle Mx, x\rangle \geqslant \varepsilon \|x\|^2 + \delta \|Mx\|^2$$

$\forall x \in \mathscr{L}_2(\mathrm{j}\mathbb{R})$.

In [11,6,5], input and output strict passivity is referred to as very strong passivity (VSP).

Definition 2. Consider a causal system with transfer function matrix $M \in \mathscr{RH}_\infty$ and consider frequencies in the interval $[a, b]$. Call the system input and output strictly passive on the frequency interval $[a, b]$ if $\exists \varepsilon > 0,\ \delta > 0$ such that

$$\langle Mx, x\rangle_{[a,b]} \geqslant \varepsilon \|x\|^2_{[a,b]} + \delta \|Mx\|^2_{[a,b]}$$

$\forall x \in \mathscr{L}_2(\mathrm{j}\mathbb{R})$, where, given $x, y \in \mathscr{L}_2(\mathrm{j}\mathbb{R})$,

$$\langle y, x\rangle_{[a,b]} := \frac{1}{2\pi} \int_a^b x^*(\mathrm{j}\omega) y(\mathrm{j}\omega)\, \mathrm{d}\omega \tag{1}$$

and

$$\|(\cdot)\|^2_{[a,b]} := \langle (\cdot), (\cdot)\rangle_{[a,b]}. \tag{2}$$

If $x \in \mathscr{L}_2(\mathrm{j}\mathbb{R})$ is the Fourier transform of a real-valued signal, it follows that $\langle Mx, x\rangle_{[-b,-a]} \geqslant \varepsilon \|x\|^2_{[-b,-a]} + \delta \|Mx\|^2_{[-b,-a]}\ \forall x \in \mathscr{L}_2(\mathrm{j}\mathbb{R})$.

Definition 3 (*Desoer and Vidyasagar [3]; Marquez [9]*). Consider a causal system with transfer function matrix $M \in \mathscr{RH}_\infty$. This system is said to have finite gain if $\exists k < \infty$ such that

$$\|Mx\| \leqslant k\|x\|$$

$\forall x \in \mathscr{L}_2(\mathrm{j}\mathbb{R})$.

It is worth noting that input and output strict passivity is equivalent to input strict passivity with finite gain [11,14,5].

Definition 4. Consider a causal system with transfer function matrix $M \in \mathscr{RH}_\infty$ and consider frequencies in the finite

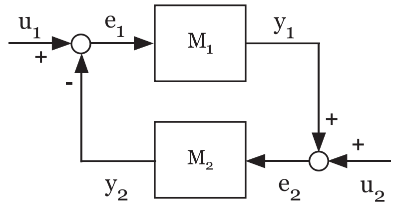

Fig. 4. Feedback interconnection of M_1 and M_2.

interval $[a, b]$. Say that the system has finite gain on the frequency interval $[a, b]$ if $\exists k < \infty$ such that

$$\|Mx\|_{[a,b]} \leqslant k\|x\|_{[a,b]}$$

$\forall x \in \mathscr{L}_2(\mathrm{j}\mathbb{R})$, where $\|(\cdot)\|_{[a,b]}$ is defined by (2). If $x \in \mathscr{L}_2(\mathrm{j}\mathbb{R})$ is the Fourier transform of a real-valued signal, it follows that $\|Mx\|_{[-b,-a]} \leqslant k\|x\|_{[-b,-a]}$ $\forall x \in \mathscr{L}_2(\mathrm{j}\mathbb{R})$.

Finite frequency intervals $[a, b]$ are considered in the above definitions of input and output strict passivity on a frequency interval and finite gain on a frequency interval. However, infinite frequency intervals $[a, b)$, $(a, b]$ or (a, b), where a or b may be equal to $\pm\infty$, may be considered by taking improper integrals in (1) and (2) where appropriate.

Later, we will be interested in forming a feedback interconnection of two systems M_1 and M_2 as shown in Fig. 4, which both have the same frequency domain property from those of type (a), (b) or (c) above, at a particular frequency. That is, at some frequency ω, systems M_1 and M_2 are either both input and output strictly passive on a frequency interval; or both have finite gain less than one on a frequency interval; or are both input and output strictly passive on a frequency interval and simultaneously have finite gain less than one on a frequency interval. For example, the systems m_1 and m_2 described in Section 1 have finite gain less than one on the frequency intervals $(-\infty, -1.414]$ and $[1.414, \infty)$; are input and output strictly passive on the frequency interval $[-0.924, 0.924]$; and are both input and output strictly passive and have finite gain less than one on the frequency intervals $(-1.414, -0.924)$ and $(0.924, 1.414)$. Furthermore, relaxation of the input and output strict passivity on a frequency interval requirement of one of the systems in the feedback interconnection, analogous to the passivity theorem's requirement that one system is passive while the other is input strictly passive with finite gain for instance, will be discussed at the end of Section 4.

The "mixed" small gain and passivity frequency domain property of a system M can be described mathematically using the notion of dissipativity of systems as follows. First we give a definition of a dissipative system.

Definition 5. Consider a causal system with transfer function matrix $M \in \mathscr{RH}_\infty$. Denote the system's input and output signals, $e \in \mathscr{L}_2(\mathrm{j}\mathbb{R})$ and $y \in \mathscr{L}_2(\mathrm{j}\mathbb{R})$, respectively. The system is said to be dissipative with respect to the triple $(Q(\omega), S(\omega), R(\omega))$ if

$$\langle y, Q(\omega)y\rangle + 2\langle y, S(\omega)e\rangle + \langle e, R(\omega)e\rangle \geqslant 0$$

$\forall e \in \mathscr{L}_2(\mathrm{j}\mathbb{R})$, where $Q(\omega)$ and $R(\omega)$ are self-adjoint at every ω (i.e. $Q(\omega)^{\mathrm{T}} = Q(\omega)$ and $R(\omega)^{\mathrm{T}} = R(\omega)$) and $Q(\omega)$ is also negative semi-definite at every ω.

Define a real continuous (even) function of frequency that is equal to one on frequency intervals of type (a) above, i.e. at those frequencies for which M is input and output strictly passive on a frequency interval; equal to zero on intervals of type (c) above, i.e. at those frequencies for which M has finite gain less than one on a frequency interval; and is strictly greater than zero and strictly less than one on intervals of type (b) above, i.e. at those frequencies for which M has both properties. Denote this function $\alpha(\omega)$. Then the "mixed" small gain and passivity frequency domain property of system M can be described by letting

$$\begin{aligned}
Q_m(\omega) &:= Q(\omega) = -(\varepsilon\alpha(\omega) + 1 - \alpha(\omega))I,\\
S_m(\omega) &:= S(\omega) = \alpha(\omega)I,\\
R_m(\omega) &:= R(\omega) = (k^2(1 - \alpha(\omega)) - \delta\alpha(\omega))I
\end{aligned}$$

in Definition 5, where $k < 1$, $\varepsilon > 0$ and $\delta > 0$. That is, the statement that system M is dissipative with respect to the triple $(Q_m(\omega), S_m(\omega), R_m(\omega))$ means that

$$\langle y, Q_m y\rangle + 2\langle y, S_m e\rangle + \langle e, R_m e\rangle \geqslant 0 \tag{3}$$

$\forall e \in \mathscr{L}_2(\mathrm{j}\mathbb{R})$.

To see that the desired "mixed" property of a system M is accurately described by using the notion of dissipativity as above, note that the left-hand side (LHS) of (3) is equal to

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{\infty} Q_m(\omega)e^*(\mathrm{j}\omega)M^*(\mathrm{j}\omega)M(\mathrm{j}\omega)e(\mathrm{j}\omega)\,\mathrm{d}\omega\\
&\quad+\frac{1}{2\pi}\int_{-\infty}^{\infty} S_m(\omega)e^*(\mathrm{j}\omega)[M^*(\mathrm{j}\omega) + M(\mathrm{j}\omega)]e(\mathrm{j}\omega)\,\mathrm{d}\omega\\
&\quad+\frac{1}{2\pi}\int_{-\infty}^{\infty} R_m(\omega)e^*(\mathrm{j}\omega)e(\mathrm{j}\omega)\,\mathrm{d}\omega.
\end{aligned} \tag{4}$$

We continue by illustrating with a simple example. Suppose that system M has finite gain less than one on the frequency intervals $(-\infty, -\omega_b]$ and $[\omega_b, \infty)$; is input and output strictly passive and has finite gain less than one on the frequency intervals $(-\omega_b, -\omega_a)$ and (ω_a, ω_b); and is input and output strictly passive on the frequency interval $[-\omega_a, \omega_a]$. For the example systems m_1 and m_2 from Section 1, $\omega_a = 0.924$ and $\omega_b = 1.414$.

Breaking the integrals from $-\infty$ to ∞ of (4) into integrals from $-\infty$ to $-\omega_b$, $-\omega_b$ to $-\omega_a$, $-\omega_a$ to ω_a, ω_a to ω_b and ω_b to ∞; grouping the integrals from each respective frequency range together and adding the integrands; and substituting into the integrands values of $\alpha(\omega) = 1$ for the integrals from $-\omega_a$ to ω_a, and $\alpha(\omega) = 0$ for the integrals from $-\infty$ to $-\omega_b$ and ω_b

to ∞, gives

$$\frac{1}{2\pi}\int_{\omega_b}^{\infty} e^*(k^2 I - M^*M)e\,\mathrm{d}\omega \tag{5}$$

$$+\frac{1}{2\pi}\int_{\omega_a}^{\omega_b} e^*(Q_m M^*M + S_m(M^*+M) + R_m)e\,\mathrm{d}\omega \tag{6}$$

$$+\frac{1}{2\pi}\int_{-\omega_a}^{\omega_a} e^*(M^*+M-\varepsilon M^*M-\delta I)e\,\mathrm{d}\omega \tag{7}$$

$$+\frac{1}{2\pi}\int_{-\omega_b}^{-\omega_a} e^*(Q_m M^*M + S_m(M^*+M) + R_m)e\,\mathrm{d}\omega \tag{8}$$

$$+\frac{1}{2\pi}\int_{-\infty}^{-\omega_b} e^*(k^2 I - M^*M)e\,\mathrm{d}\omega. \tag{9}$$

Now integrals (5) and (9) are greater than or equal to zero since M has finite gain less than one on the frequency intervals $(-\infty, -\omega_b]$ and $[\omega_b, \infty)$. Integral (7) is greater than or equal to zero since M is input and output strictly passive on the frequency interval $[-\omega_a, \omega_a]$. It remains to show that integrals (6) and (8) are greater than or equal to zero. Note that integral (6) is equal to

$$\frac{1}{2\pi}\int_{\omega_a}^{\omega_b} \alpha e^*(M^*+M-\varepsilon M^*M-\delta I)e\,\mathrm{d}\omega + \frac{1}{2\pi}\int_{\omega_a}^{\omega_b}(1-\alpha)e^*(k^2 I - M^*M)e\,\mathrm{d}\omega,$$

which is greater than or equal to zero because $0 \leqslant \alpha(\omega) \leqslant 1$ and M is both input and output strictly passive and has finite gain less than one on the frequency interval (ω_a, ω_b). Similarly, integral (8) is greater than or equal to zero.

3. Interconnection of systems

Now consider the feedback interconnection of two systems M_1 and M_2 as shown in Fig. 4, which are dissipative in the sense of Definition 5. Let the $(Q(\omega), S(\omega), R(\omega))$ triple associated with system M_i, $i = 1, 2$, be given by

$$Q_i(\omega) = -(\varepsilon_i\alpha(\omega) + 1 - \alpha(\omega))I, \tag{10}$$

$$S_i(\omega) = \alpha(\omega)I, \tag{11}$$

$$R_i(\omega) = (k_i^2(1-\alpha(\omega)) - \delta_i\alpha(\omega))I, \tag{12}$$

where $\alpha(\omega)$ is as described in the preceding section. In the spirit of [7,12,11,6], where constant (Q_i, S_i, R_i) triples are considered as opposed to frequency-dependent triples, we show that the interconnected system is also dissipative in a sense to be described. This description of dissipativity of the closed-loop provides us with a tool to prove finite-gain stability of the interconnection of systems M_1 and M_2, which is done in the next section.

Denote the interconnection of systems M_1 and M_2 by M_{sys}. So M_1 and M_2 are interconnected via

$$e_1 = u_1 - y_2, \tag{13}$$

$$e_2 = u_2 + y_1, \tag{14}$$

as indicated in Fig. 4. The input and output signal space for M_{sys} is the product space $\mathscr{L}_2(\mathrm{j}\mathbb{R}) \times \mathscr{L}_2(\mathrm{j}\mathbb{R})$, and the elements of the input and output signal space are $u := \binom{u_1}{u_2}$ and $y := \binom{y_1}{y_2}$, respectively. Note that inner products in these spaces are derived by summing inner products in the component spaces.

Assume that the system M_{sys} is well posed in the sense of [16]. Write (13) and (14) in the compact form

$$e = u - Hy, \tag{15}$$

where $H := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. Define

$$\tilde{Q} := \begin{pmatrix} Q_1(\omega) & 0 \\ 0 & Q_2(\omega) \end{pmatrix},$$

$$\tilde{S} := \begin{pmatrix} S_1(\omega) & 0 \\ 0 & S_2(\omega) \end{pmatrix},$$

$$\tilde{R} := \begin{pmatrix} R_1(\omega) & 0 \\ 0 & R_2(\omega) \end{pmatrix}.$$

Then similarly to [7,12,11,6], it can be shown that M_{sys} is $(\bar{Q}, \bar{S}, \tilde{R})$ dissipative, where

$$\bar{Q} := \tilde{Q} + H^{\mathrm{T}}\tilde{R}H - \tilde{S}H - H^{\mathrm{T}}\tilde{S}^{\mathrm{T}} = \begin{pmatrix} -\bar{q}_1 I & 0 \\ 0 & -\bar{q}_2 I \end{pmatrix},$$

with $\bar{q}_1 := (1-k_2^2)(1-\alpha(\omega)) + (\varepsilon_1+\delta_2)\alpha(\omega) > 0$, $\bar{q}_2 := (1-k_1^2)(1-\alpha(\omega)) + (\varepsilon_2+\delta_1)\alpha(\omega) > 0$ and

$$\bar{S} := \tilde{S} - H^{\mathrm{T}}\tilde{R} = \begin{pmatrix} \alpha(\omega)I & \bar{s}_1 I \\ -\bar{s}_2 I & \alpha(\omega)I \end{pmatrix},$$

with $\bar{s}_1 := k_2^2(1-\alpha(\omega)) - \delta_2\alpha(\omega)$, $\bar{s}_2 := k_1^2(1-\alpha(\omega)) - \delta_1\alpha(\omega)$, by adding inequalities

$$\langle y_i, Q_i y_i\rangle + 2\langle y_i, S_i e_i\rangle + \langle e_i, R_i e_i\rangle \geqslant 0,$$

with $i = 1, 2$ and substituting (15) in as follows:

$$\begin{aligned}
&\langle y_1, Q_1 y_1\rangle + \langle y_2, Q_2 y_2\rangle + 2\langle y_1, S_1 e_1\rangle + 2\langle y_2, S_2 e_2\rangle \\
&\quad + \langle e_1, R_1 e_1\rangle + \langle e_2, R_2 e_2\rangle \geqslant 0 \\
&\quad\Leftrightarrow \langle y, \tilde{Q}y\rangle + 2\langle y, \tilde{S}e\rangle + \langle e, \tilde{R}e\rangle \geqslant 0 \\
&\quad\Leftrightarrow \langle y, \tilde{Q}y\rangle + 2\langle y, \tilde{S}u - \tilde{S}Hy\rangle \\
&\qquad + \langle u - Hy, \tilde{R}u - \tilde{R}Hy\rangle \geqslant 0 \\
&\quad\Leftrightarrow \langle y, \tilde{Q}y\rangle + \langle y, H^{\mathrm{T}}\tilde{R}Hy\rangle + \langle y, -\tilde{S}Hy\rangle \\
&\qquad + \langle y, -H^{\mathrm{T}}\tilde{S}^{\mathrm{T}}y\rangle + 2\langle y, \tilde{S}u\rangle \\
&\qquad + 2\langle y, -H^{\mathrm{T}}\tilde{R}u\rangle + \langle u, \tilde{R}u\rangle \geqslant 0 \\
&\quad\Leftrightarrow \langle y, \bar{Q}y\rangle + 2\langle y, \bar{S}u\rangle + \langle u, \tilde{R}u\rangle \geqslant 0.
\end{aligned}$$

4. Stability theorem

The following result shows that input–output stability of the interconnected system M_{sys}, as described in the previous section, is always guaranteed. This is the main contribution of the paper.

Theorem 1. *Consider two causal systems with transfer function matrices $M_1 \in \mathcal{RH}_\infty$ and $M_2 \in \mathcal{RH}_\infty$ which are interconnected as shown in Fig. 4. Furthermore, suppose that systems M_1 and M_2 are dissipative in the sense of Definition 5 with respect to the triples $(Q_i(\omega), S_i(\omega), R_i(\omega))$, $i = 1, 2$, given at the beginning of Section 3. Then the interconnection of the systems, denoted $M_{\rm sys}$, is finite-gain stable.*

Proof. Note that $\bar{\bar{Q}} := -\bar{Q}$ is positive definite. As in [7,12,11,6], but considering frequency-dependent (as opposed to constant) $\bar{\bar{Q}}$, it is shown that, since $\bar{\bar{Q}}$ is positive definite, $M_{\rm sys}$ is finite-gain stable.

From Definition 5, the statement that $M_{\rm sys}$ is $(\bar{Q}, \bar{S}, \tilde{R})$ dissipative means that

$$\langle y, \bar{\bar{Q}}y\rangle - 2\langle y, \bar{\bar{Q}}^{1/2}\bar{\bar{S}}u\rangle \leqslant \langle u, \tilde{R}u\rangle$$

$\forall u \in \mathcal{L}_2(\mathrm{j}\mathbb{R})$, where $\bar{\bar{S}} := \bar{\bar{Q}}^{-1/2}\bar{S}$. The matrix $\tilde{R} + \bar{\bar{S}}^{\mathrm{T}}\bar{\bar{S}}$ is a symmetric matrix, equal to

$$\begin{pmatrix} \left(\bar{s}_2 + \dfrac{\alpha^2}{\bar{q}_1} + \dfrac{\bar{s}_2^2}{\bar{q}_2}\right) I & \alpha(\omega)\left(\dfrac{\bar{s}_1}{\bar{q}_1} - \dfrac{\bar{s}_2}{\bar{q}_2}\right) I \\ \alpha(\omega)\left(\dfrac{\bar{s}_1}{\bar{q}_1} - \dfrac{\bar{s}_2}{\bar{q}_2}\right) I & \left(\bar{s}_1 + \dfrac{\bar{s}_1^2}{\bar{q}_1} + \dfrac{\alpha^2}{\bar{q}_2}\right) I \end{pmatrix}.$$

Then $\tilde{R} + \bar{\bar{S}}^{\mathrm{T}}\bar{\bar{S}}$ is orthogonally similar to a diagonal matrix, i.e.

$$\tilde{R}(\omega) + \bar{\bar{S}}(\omega)^{\mathrm{T}}\bar{\bar{S}}(\omega) = U(\omega)^{\mathrm{T}}D(\omega)U(\omega),$$

and so there always exists a finite scalar $\kappa > 0$ such that $\tilde{R} + \bar{\bar{S}}^{\mathrm{T}}\bar{\bar{S}} \leqslant \kappa^2 I$, i.e. $U(\omega)^{\mathrm{T}}D(\omega)U(\omega) \leqslant \kappa^2 I = \kappa^2 U(\omega)^{\mathrm{T}}U(\omega)$ and

$$U(\omega)^{\mathrm{T}}\begin{pmatrix} \lambda_1(\omega) - \kappa^2 & & 0 \\ & \ddots & \\ 0 & & \lambda_p(\omega) - \kappa^2 \end{pmatrix} U(\omega) \leqslant 0.$$

So $\exists\kappa > 0$ such that

$$\langle y, \bar{\bar{Q}}y\rangle - 2\langle y, \bar{\bar{Q}}^{1/2}\bar{\bar{S}}u\rangle \leqslant \kappa^2\langle u, u\rangle - \langle u, \bar{\bar{S}}^{\mathrm{T}}\bar{\bar{S}}u\rangle \tag{16}$$

$\forall u \in \mathcal{L}_2(\mathrm{j}\mathbb{R})$.

Inequality (16) is equivalent to

$$\begin{aligned} &\langle y, \bar{\bar{Q}}^{1/2}\bar{\bar{Q}}^{1/2}y\rangle - 2\langle y, \bar{\bar{Q}}^{1/2}\bar{\bar{S}}u\rangle + \langle u, \bar{\bar{S}}^{\mathrm{T}}\bar{\bar{S}}u\rangle \\ &\quad \leqslant \kappa^2\langle u, u\rangle \\ &\quad \Leftrightarrow \|\bar{\bar{Q}}^{1/2}y - \bar{\bar{S}}u\|^2 \leqslant \kappa^2\|u\|^2 \\ &\quad \Leftrightarrow \|\bar{\bar{Q}}^{1/2}y - \bar{\bar{S}}u\| \leqslant \kappa\|u\|. \end{aligned}$$

It follows easily that

$$\|\bar{\bar{Q}}^{1/2}y\| \leqslant (\kappa + \|\bar{\bar{S}}\|_\infty)\|u\|. \tag{17}$$

Finally, note that $y = (\bar{\bar{Q}}^{1/2})^{-1}\bar{\bar{Q}}^{1/2}y$ implies that $\|y\| \leqslant \|\bar{\bar{Q}}^{-1/2}\|_\infty\|\bar{\bar{Q}}^{1/2}y\|$, or $\|\bar{\bar{Q}}^{-1/2}\|_\infty^{-1}\|y\| \leqslant \|\bar{\bar{Q}}^{1/2}y\|$. Then from (17),

$$\begin{aligned} \|\bar{\bar{Q}}^{-1/2}\|_\infty^{-1}\|y\| &\leqslant (\kappa + \|\bar{\bar{S}}\|_\infty)\|u\| \\ &\Leftrightarrow \|y\| \leqslant \bar{k}\|u\|, \end{aligned}$$

where $\bar{k} := \|\bar{\bar{Q}}^{-1/2}\|_\infty(\kappa + \|\bar{\bar{S}}\|_\infty)$. $\square$

So by setting the $(Q(\omega), S(\omega), R(\omega))$ triples associated with systems M_1 and M_2 to be equal to the triples given by (10)–(12), a mathematical description in terms of dissipativity was given to describe the property that, at some frequency ω, M_1 and M_2 are both either input and output strictly passive on a frequency interval; or both have finite gain less than one on a frequency interval; or are both input and output strictly passive on a frequency interval and simultaneously both have finite gain less than one on a frequency interval. Given this dissipative property of systems M_1 and M_2, it was shown that the interconnected system $M_{\rm sys}$ is $(\bar{Q}, \bar{S}, \tilde{R})$ dissipative; and since $\bar{Q}$ is negative definite, then $M_{\rm sys}$ is finite-gain stable.

Now let ε_1 and ε_2 from (10) be equal to zero. (We could say that this corresponds to relaxing input and output strict passivity on a frequency interval to "input strict passivity on a frequency interval".) Note that $\bar{Q}$ remains negative definite and so finite-gain stability of $M_{\rm sys}$ is still guaranteed. Alternatively, let δ_1 and δ_2 from (12) be equal to zero (which we could say corresponds to relaxing input and output strict passivity on a frequency interval to "output strict passivity on a frequency interval"). In this case, $\bar{Q}$ also remains negative definite and so finite-gain stability of $M_{\rm sys}$ is still guaranteed. Alternatively still, let ε_1 and δ_1 (or ε_2 and δ_2) of (10) and (12) be equal to zero (which corresponds to relaxing input and output strict passivity on a frequency interval of system M_1 (or system M_2) to what we could call "passivity on a frequency interval"). The matrix $\bar{Q}$ remains negative definite, and so finite-gain stability of $M_{\rm sys}$ is guaranteed in this case also.

5. Conclusions and future work

It was shown that finite-gain stability is guaranteed for the feedback loop denoted $M_{\rm sys}$, which consists of two causal, stable, LTI systems M_1 and M_2 with a "mixed" small gain and passivity frequency domain property, which we described via the notion of dissipative systems. It is clear that, in the case of MIMO LTI systems, there already exist simple techniques to determine stability of a feedback interconnection. For example, one need only check that the transfer function matrix mapping signals u_1 and u_2 to e_1 and e_2 of Fig. 4 are in $\mathcal{RH}_\infty$. However, these simple techniques often fail in the time-varying and/or nonlinear case. It is expected that the technique of ensuring finite-gain stability presented in this paper will be able to be extended to the time-varying and/or nonlinear case by first reformulating the present results in the time domain. Preliminary investigations indicate that this is possible by considering frequency-dependent filters on the interconnection input and output signals. Work of this nature has been submitted for publication.

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