

Robust stability and performance analysis for uncertain linear systems—The distance measure approach

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SUMMARY

This paper presents readily applicable distance measures, robust stability margins and associated robust stability and robust performance theorems for three commonly used uncertainty structures (additive, input/output multiplicative, output/input inverse multiplicative). Besides providing robust stability results for a larger uncertainty class than previously reported (\mathcal{RL}_∞ instead of \mathcal{RH}_∞), this paper also states robust performance theorems for the above uncertainty structures. In contrast to previous methods for robust performance analysis, they only require the computation of two infinity norms for every uncertain plant considered. The theorems in this paper enable practising engineers to choose the most suitable uncertainty structure for a family of uncertain plants, as illustrated through a physically motivated numerical example. Copyright © 2011 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Based on the small-gain theorem [1], many of the initial stability analysis results in the area of \mathcal{H}_∞ control considered intuitive uncertainty structures like additive or multiplicative uncertainty, using stable uncertainty blocks [2], [3, Table 9.1]. For the additive uncertainty structure only, Glover [4] extended the space of allowable systems from \mathcal{RH}_∞ to \mathcal{RL}_∞ . Robust performance analysis for these uncertainty structures can be carried out e.g. using μ analysis [3], which requires for the computation of an upper bound (i.e. a sufficient condition) the infimum of a singular value condition to be found at each point in the frequency spectrum.

Following a different approach, the gap metric [5], graph metric [6] and later the ν -gap metric [7, 8] introduced the notion of distance between systems. Instead of focussing on one nominal system and a norm-bounded uncertainty structure, the emphasis here is on obtaining uncertainty blocks that explain closed-loop discrepancies between systems specified by their open-loop transfer functions. This allows quantification of robust stability and robust performance for entire families of uncertain plants (e.g. obtained via a range of system identification methodologies), and sidesteps the difficulties associated with specifying the size of dynamic uncertainties around a nominal model. The distance measures of [5–8] assumed a normalized coprime factor uncertainty structure with uncertainty blocks in \mathcal{RH}_∞ in the gap metric and \mathcal{RL}_∞ in the ν -gap. Recently, Lanzon

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and Papageorgiou [9] proposed distance measures (and associated robust stability and performance theorems) for a generic four-block uncertainty structure with the potential of capturing many different uncertainty structures via the choice of submatrices of a generalized plant matrix, which specify the inputs and outputs of the uncertainty block relative to the nominal plant, using an approach inspired by robust model validation techniques (see e.g. [10–15]).

The importance of analysing robustness problems for different uncertainty structures is emphasized in model validation literature, yet specific results are scarce. With a view to applying model validation techniques to robust identification problems [14] provides a computational algorithm for obtaining a distance-like uncertainty block in a generic structured singular value framework (with the associated numerical difficulties). Explicit results for particular uncertainty structures are not provided. In [15], robust stability and performance results for single-input, single-output systems with additive, normalized coprime factor and Youla-type uncertainty are derived, with a focus on how these uncertainty structures can be made equivalent via the choice of a different nominal plant. For the Youla controller structure only, these results have been extended to the multivariable case [16]. Model embedding in different uncertainty structures is also important in iterative identification and control redesign methods (see e.g. [17–19] and the references therein), where the controller is synthesized using a nominal plant model and an uncertainty set around this nominal plant model obtained during the identification stage. These two stages of identification and control synthesis are iterated until no further improvement in performance can be achieved. Quantitative measures that enable model embedding in small uncertainty sets can hence assist in obtaining less conservative results.

Outside of robust control, different discrepancy structures are also used in model order reduction [20], in model approximation and in system identification [21]. Most authors use an additive discrepancy criterion due to its simplicity and ease of intuition. However, recent results on model approximation techniques invoke different discrepancy criteria between original model and data and the approximated model (see [22] for model order reduction and system identification using multiplicative discrepancy criteria and [23] for model order reduction using a normalized coprime factor discrepancy criterion). In order to differentiate these methods quantitatively rather than only qualitatively, and to link the resulting discrepancy to degradation in closed-loop stability margins and robust performance, the distance measure approach is indispensable.

This paper is concerned with the robust stability and robust performance analysis for multiple-input, multiple-output systems with additive, input/output multiplicative or output/input inverse multiplicative uncertainty structures via the generic distance measures of [9]. The results for the additive case mainly serve a didactic purpose by illustrating in an intuitive manner how distance is measured, and how it affects the stability and performance of the perturbed system. The analysis for the multiplicative cases is much more intricate ([24] presented an abbreviated version of these results). For both input/output multiplicative and output/input inverse multiplicative structures, the first difficulty lies in finding a minimal-size uncertainty that explains the difference between a nominal and a perturbed system, e.g. in the input multiplicative case a $\Delta \in \mathcal{RL}_\infty$ that fulfills

$$P_\Delta = (I - \Delta)P$$

for given P , $P_\Delta \in \mathcal{RL}_\infty$. First, a parameterization of all solutions Δ must be found, and then the smallest of these solutions must be chosen. The parameterization depends on the dimension of the plant, i.e. is different for square, tall and fat plants. For output/input inverse multiplicative uncertainty, a second difficulty arises in ensuring that the perturbed plant description is well-posed. From this requirement, additional constraints on the set containing all possible solutions Δ are derived. By giving analytic expressions of all minimal-size solutions Δ for input/output multiplicative and output/input inverse multiplicative uncertainty structures, we are able to give concise formulations of the distance measures and of robust stability and robust performance theorems for these structures.

Taken together, the readily applicable theorems of this paper on robust stability and performance for additive and various multiplicative structures form a valuable tool for the practising engineer to determine the most suitable uncertainty structure for a given family of uncertain plant models.

In fact, engineers have been using similar approaches without rigorous theoretical underpinning for many years (see e.g. how a minimal-size multiplicative uncertainty is constructed in [25]). In contrast to the structured singular value approach, the robust performance theorems in this paper require only two infinity norms to be computed for each uncertain plant for which performance guarantees are sought. For the structures considered in this paper, the additional winding number condition reduces to a simple count of right half plane poles or zeros.

The choice of uncertainty structure greatly influences the resulting stability and performance guarantees. This fact is illustrated through a physically motivated example, in which uncertainty arising from neglected actuator dynamics is analysed using multiplicative distance measures. The results are compared with the v -gap theory analysis for the same plant, and it is shown that for this case, multiplicative distance measures enable less conservative robust stability and performance guarantees.

The remainder of the paper is structured as follows: Section 2 summarizes the theory of generic distance measures. In Section 3, we introduce the uncertainty structures studied in this paper, for which the robust stability and performance theorems are derived in Sections 4 and 5, respectively. The example follows in Section 6, while Section 7 concludes the paper.

1.1. Notation

Notation is standard. Let \mathcal{R} denote the set of proper real-rational transfer functions. Also, let $P^*(s)$ denote the adjoint of $P(s) \in \mathcal{R}$ defined by $P^*(s) = P(-s)^T$. Let \mathcal{RL}_∞ denote the space of proper real-rational functions bounded on $j\mathbb{R}$ including ∞ , and \mathcal{RH}_∞ denote the space of proper real-rational functions bounded and analytic in the open right half complex plane. Denote the space of functions that are units in \mathcal{RH}_∞ by \mathcal{GH}_∞ (that is, $f \in \mathcal{GH}_\infty \Leftrightarrow f, f^{-1} \in \mathcal{RH}_\infty$). Let $\mathcal{F}_l(\cdot, \cdot)$ (resp. $\mathcal{F}_u(\cdot, \cdot)$) denote a lower (resp. upper) linear fractional transformation. For a scalar $p(s) \in \mathcal{R}$, its winding number $\text{wno } p(s)$ is defined as the number of encirclements of the origin made by $p(s)$ as s follows the standard Nyquist D-contour, indented into the right half plane around any imaginary axis poles or zeros of $p(s)$. Furthermore, let $\eta(P)$ denote the number of open right half plane poles of $P \in \mathcal{R}$. For a plant $P \in \mathcal{R}$ and a controller $C \in \mathcal{R}$, let $[P, C]$ denote the nominal feedback interconnection displayed in Figure 2. Notation for coprime factors and graph symbols follows [8], see also [9].

2. ROBUST STABILITY AND PERFORMANCE VIA GENERIC DISTANCE MEASURES

Different measurement or modelling methods may result in situations where the same physical system is described by various models, leading to uncertainty in the model description. Let us consider a nominal plant P and one of its perturbed versions P_Δ . Often a controller is designed based on the nominal plant but is supposed to work well with a family of perturbed plants. In such a situation, the control engineer will have to answer two questions: Keeping the controller unchanged, will the system remain stable when P is replaced by P_Δ ? And how will the performance of the system be affected by the model change? To answer these, the engineer needs to quantify how different P_Δ is from P in a closed-loop sense. This difference is called the distance between P and P_Δ . Of course, there are numerous ways to measure distance in a function space [5, 6, 8, 26, 27], which depend on the underlying metric topology (or allowable uncertainty structure). The distance notion used in this paper is generic in the sense that it can capture many different uncertainty structures, and therefore allows an easy comparison of distances under various different uncertainty structures. In this section, we will give an introduction and engineering motivation of the distance measure and related concepts and theorems first defined in [9].

In robust control, families of uncertain systems are often described via an LFT [3, Chapter 10] as shown in Figure 1, where

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \in \mathcal{R}$$

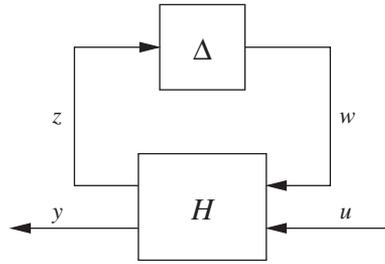


Figure 1. An uncertain plant family parameterized by the perturbation Δ .

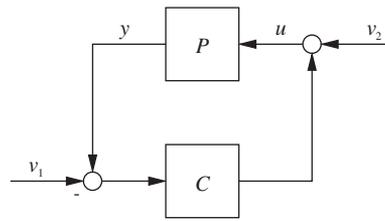


Figure 2. The nominal closed-loop system.

is the generalized plant and $\Delta \in \mathcal{R}$ is an uncertainty. If the element H_{22} describes the nominal plant $P \in \mathcal{R}^{p \times q}$, the size of $\Delta \in \mathcal{R}$ corresponds to the ‘radius’ of an uncertain region around P . The shape of this region depends on the other elements of H , which describe the uncertainty structure, i.e. the allowable way in which the uncertainty can alter the plant model. For any given Δ , it is easy to compute the resulting perturbed plant $P_\Delta \in \mathcal{R}^{p \times q}$ (under the assumption that $(I - H_{11}\Delta)^{-1} \in \mathcal{R}$):

$$P_\Delta = \mathcal{F}_u(H, \Delta) = H_{22} + H_{21}\Delta(I - H_{11}\Delta)^{-1}H_{12}. \tag{1}$$

When $\Delta=0$, it is easy to see that $\mathcal{F}_u(H, 0) = P$, corresponding to the centre of this uncertain family. We arrive at the question: Which Δ yields a given perturbed plant P_Δ , if connected with H as shown in Figure 1? From Equation (1), we can deduce two conditions for the solutions Δ as follows: They must satisfy a well-posedness condition ($(I - H_{11}\Delta)^{-1} \in \mathcal{R}$), and also the consistency equation ($P_\Delta = \mathcal{F}_u(H, \Delta)$). In the general case, there is a set of solutions $\Delta \subset \mathcal{R}$ for Equation (1) which may have multiple elements, or none at all, or exactly one. We allow perturbations $\Delta \in \mathcal{RL}_\infty$, the minimal restriction for which the size of Δ can still be measured with the infinity norm. Hence, $\|\Delta\|_\infty$ can be used to measure the distance between P and P_Δ . This distance measurement is unambiguous if there is just a single solution $\Delta \in \mathcal{RL}_\infty$ satisfying Equation (1), or no solution at all (in which case the distance is infinite). If the set Δ contains several elements Δ that satisfy the consistency equation relating P and P_Δ and also the well-posedness condition, we choose the smallest $\|\Delta\|_\infty$ among the possible solutions that explain the change from P to P_Δ . We can now formally define the distance measure.

Definition 1

Given a plant $P \in \mathcal{R}^{p \times q}$, a generalized plant $H \in \mathcal{R}$ with $H_{22} = P$, and a perturbed plant $P_\Delta \in \mathcal{R}^{p \times q}$. Let the set of all admissible perturbations be given by $\Delta = \{\Delta \in \mathcal{RL}_\infty : (I - H_{11}\Delta)^{-1} \in \mathcal{R}, P_\Delta = \mathcal{F}_u(H, \Delta)\}$. Define the distance measure $d^H(P, P_\Delta)$ between plants P and P_Δ for the uncertainty structure implied by H as:

$$d^H(P, P_\Delta) := \begin{cases} \inf_{\Delta \in \Delta} \|\Delta\|_\infty & \text{if } \Delta \neq \emptyset \\ \infty, & \text{otherwise.} \end{cases}$$

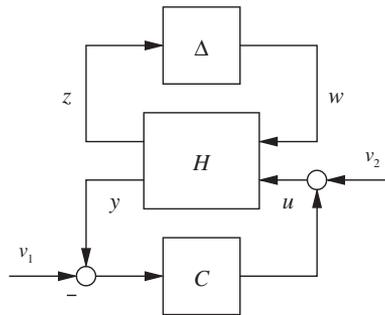


Figure 3. The closed-loop system when a perturbation Δ is present.

Note that if $\Delta \neq \emptyset$ then the infimum always exists in \mathbb{R} as $\Delta \in \mathcal{RL}_\infty$. If multiple solutions Δ exist, it may be that several of them have the same, smallest size. Hence, the following definition is introduced for ease of notation:

Definition 2

Given a plant $P \in \mathcal{R}^{p \times q}$, a generalized plant $H \in \mathcal{R}$ with $H_{22} = P$ and a perturbed plant $P_\Delta \in \mathcal{R}^{p \times q}$. Define

$$\Delta^{\min} := \{\Delta \in \Delta : \|\Delta\|_\infty = d^H(P, P_\Delta)\}.$$

We can now begin to address the question of robust stability. The nominal closed-loop system is shown in Figure 2. If an uncertainty is present in the system, the plant P is replaced by P_Δ , which corresponds to an LFT structure as depicted in Figure 3. The classical robust stability analysis for uncertainties in \mathcal{RH}_∞ relies on the standard small-gain theorem [3, p. 218] as follows: The loop is broken into two parts, Δ and $\mathcal{F}_l(H, C)$, and the loop gain is computed by multiplying the infinity norms of both of these parts. A loop gain strictly smaller than unity implies internal stability. Following this line of thought, define a generic stability margin notion indicating the size of the smallest uncertainty (measured by the infinity norm) that violates the classical small-gain stability condition.

Definition 3

Given a plant $P \in \mathcal{R}^{p \times q}$, a generalized plant $H \in \mathcal{R}$ with $H_{22} = P$ and a controller $C \in \mathcal{R}^{q \times p}$. Define the stability margin $b^H(P, C)$ of the feedback interconnection $\langle H, C \rangle$ as:

$$b^H(P, C) := \begin{cases} \|\mathcal{F}_l(H, C)\|_\infty^{-1} & \text{if } 0 \neq \mathcal{F}_l(H, C) \in \mathcal{RL}_\infty \text{ and } [P, C] \text{ is internally stable,} \\ 0 & \text{otherwise.} \end{cases}$$

Notice how this stability margin is related to the distance measure of Definition 1: The distance measure allows us to make a statement on the minimal size of a perturbation that yields a plant P_Δ when connected with a nominal plant P . Given P and P_Δ , we therefore know the smallest size of a Δ which explains the difference. In conjunction with the stability margin of Definition 3 we can then compute the loop gain of the closed-loop system of Figure 3 when $d^H(P, P_\Delta)$ is finite and $b^H(P, C) \neq 0$. A small gain-type condition using $\Delta \in \Delta^{\min}$ can be stated as follows:

$$\|\Delta\|_\infty \|\mathcal{F}_l(H, C)\|_\infty < 1 \Leftrightarrow d^H(P, P_\Delta) < b^H(P, C).$$

The above classical small-gain condition is valid only for stable perturbations. As mentioned earlier, we do not restrict our perturbations Δ to \mathcal{RH}_∞ , and hence need a more powerful stability theorem. The small-gain theorem can be extended to systems in \mathcal{RL}_∞ [8, 28], and this comes at the cost of introducing an additional condition on the winding numbers similar to generalized forms of the Nyquist stability theorem. Simply ensuring that the loop gain of the system is smaller than one is not sufficient when systems are assumed to belong to \mathcal{RL}_∞ . The following theorem uses

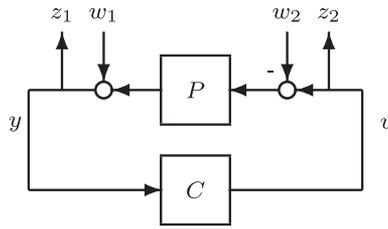


Figure 4. The left four-block interconnection.

the generic distance measure and stability margin defined above to give necessary and sufficient stability conditions for systems in \mathcal{RL}_∞ .

Theorem 1 (Robust Stability, Lanzon and Papageorgiou [9, Section III])

Given a plant $P \in \mathcal{R}^{p \times q}$, a stabilizable generalized plant $H \in \mathcal{R}$ with $H_{22} = P$, a perturbed plant $P_\Delta \in \mathcal{R}^{p \times q}$ and a controller $C \in \mathcal{R}^{q \times p}$ such that $d^H(P, P_\Delta) < b^H(P, C)$ and $\Delta^{\min} \neq \emptyset$, then the following statements are equivalent:

- (a) $[P_\Delta, C]$ is internally stable;
 - (b) $\forall \Delta \in \Delta^{\min}, \eta(P_\Delta) = \eta(P) + \text{wno det}(I - H_{11}\Delta)$;
 - (c) $\exists \Delta \in \Delta^{\min} : \eta(P_\Delta) = \eta(P) + \text{wno det}(I - H_{11}\Delta)$.
- (2)

A very powerful implication of (c) \Rightarrow (b), or equivalently its contrapositive

$$\begin{aligned} &\exists \Delta \in \Delta^{\min} : \eta(P_\Delta) \neq \eta(P) + \text{wno det}(I - H_{11}\Delta) \\ &\Rightarrow \eta(P_\Delta) \neq \eta(P) + \text{wno det}(I - H_{11}\Delta) \quad \forall \Delta \in \Delta^{\min} \end{aligned}$$

is that winding number condition (2) is true or false for all $\Delta \in \Delta^{\min}$. Therefore, to check whether or not $[P_\Delta, C]$ is internally stable it suffices to compute the winding number in (2) for a single arbitrary $\Delta \in \Delta^{\min}$. From this theorem, we are now in a position to check stability for any perturbed plant P_Δ which can be obtained from a nominal plant P through a perturbation $\Delta \in \mathcal{RL}_\infty$ (in an LFT fashion). However, we need to proceed further than just robust stability. The robust performance of an uncertain family of plants is often of similar or even greater importance as robust stability itself. In the control of manned aircraft for example, the delay due to pilot reaction places strict robust performance limitations on the non-human parts of the control loop in order to achieve overall robust stability. Hence, we need to characterize the worst-case performance degradation when the nominal plant P is replaced by P_Δ . To this end, we specify the structure of the generalized plant H more specifically. Figure 4 shows a left four-block structure, which will be used to express different generalized plants throughout this paper. It can be specialized to many common uncertainty structures through the suitable choice of two selection/filtering matrices $S_w, S_z \in \mathcal{R}$ (see Section 3 for specific values of S_w and S_z and the resulting uncertainty structures), where the connection between the signals w and z of Figure 3 and w_1, w_2, z_1 and z_2 of Figure 4 is given by

$$z = S_z \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \quad \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = S_w w.$$

In this setting, H takes the following form:

$$H = \begin{bmatrix} S_z & & \\ & & I \end{bmatrix} \begin{bmatrix} I & -P & P \\ 0 & 0 & I \\ I & -P & P \end{bmatrix} \begin{bmatrix} S_w & & \\ & & I \end{bmatrix}. \tag{3}$$

Having established the structure of the generalized plant H , we can now come to our desired objective of defining a unified framework for the analysis of robust performance and possible degradation of the robust stability margin of uncertain systems. The following theorem answers the question: How will the worst-case closed-loop performance of the system be affected if P is replaced by P_Δ while the controller C remains unchanged? This question has been asked by [6, 8, 26, 27] before, but these authors answered it only in a normalized coprime factor setting. The following theorem is more general because it captures normalized coprime factor uncertainty (or four-block uncertainty) when S_w and S_z are simply the identity matrices [29], but it can also be reduced to additive, input/output multiplicative, input/output inverse multiplicative, feedback uncertainty (also known as inverse additive uncertainty), non-normalized coprime factors [9], etc., which are all cases of interest to the practising engineer. In order for the performance measures to be well-defined, we require stability of the perturbed closed-loop system. Hence, we assume that Theorem 1 holds. Using the distance measure and the robust stability margin defined above, we can then quantify the change in robust stability margin, and the worst-case robust performance.

Theorem 2 (Robust Performance, Lanzon and Papageorgiou [9, Section V])

Given a nominal plant $P \in \mathcal{R}^{p \times q}$, a stabilizable generalized plant

$$H = \left[\begin{array}{c|c} S_z & \\ \hline & I \end{array} \right] \left[\begin{array}{cc|c} I & -P & P \\ \hline 0 & 0 & I \\ \hline I & -P & P \end{array} \right] \left[\begin{array}{c|c} S_w & \\ \hline & I \end{array} \right].$$

where $S_w, S_z \in \mathcal{R}$, a perturbed plant $P_\Delta \in \mathcal{R}^{p \times q}$ and a controller $C \in \mathcal{R}^{q \times p}$ such that $d^H(P, P_\Delta) < b^H(P, C)$ and $\Delta^{\min} \neq \emptyset$. Assume furthermore that there exists a $\Delta \in \Delta^{\min}$ that satisfies $\eta(P_\Delta) = \eta(P) + \text{wn} \det(I - H_{11}\Delta)$, where the winding number is evaluated on a contour indented to the right around any imaginary axis poles of P and P_Δ . Furthermore, suppose that $S_{w\Delta} = S_w(I - k\Delta S_z S_w)^{-1} \in \mathcal{R}$ for a given $k \in \{0, 1\}$, $S = (1 - k)S_z S_w$ and

$$H_\Delta = \left[\begin{array}{c|c} S_z & \\ \hline & I \end{array} \right] \left[\begin{array}{cc|c} I & -P_\Delta & P_\Delta \\ \hline 0 & 0 & I \\ \hline I & -P_\Delta & P_\Delta \end{array} \right] \left[\begin{array}{c|c} S_{w\Delta} & \\ \hline & I \end{array} \right].$$

Then the following results hold when $S \in \mathcal{RL}_\infty$ and $(I - \Delta S)^{-1} \in \mathcal{R}$:

- (a) $0 \neq \mathcal{F}_l(H_\Delta, C) \in \mathcal{RL}_\infty$ and $[P_\Delta, C]$ is internally stable;
- (b) $|b^{H_\Delta}(P_\Delta, C) - b^H(P, C)| \leq \| \mathcal{F}_l(H_\Delta, C) - S \|_\infty b^{H_\Delta}(P_\Delta, C) d^H(P, P_\Delta)$; and
- (c) $\| \mathcal{F}_l(H_\Delta, C) - \mathcal{F}_l(H, C) \|_\infty \leq \frac{\| \mathcal{F}_l(H_\Delta, C) - S \|_\infty d^H(P, P_\Delta)}{b^H(P, C)}$.

Result (b) gives an upper bound on the variation between $b^H(P, C)$ and $b^{H_\Delta}(P_\Delta, C)$. Also result (c) gives an upper bound on the worst-case discrepancy between the transfer functions of interest when P changes to P_Δ . Whenever $\| \mathcal{F}_l(H_\Delta, C) - S \|_\infty b^{H_\Delta}(P_\Delta, C) \leq 1$ (as is the case for additive uncertainty, normalized coprime factor uncertainty and four-block uncertainty [9]), result (b) reduces to the familiar $|b^{H_\Delta}(P_\Delta, C) - b^H(P, C)| \leq d^H(P, P_\Delta)$, and result (c) reduces to $\| \mathcal{F}_l(H_\Delta, C) - \mathcal{F}_l(H, C) \|_\infty \leq d^H(P, P_\Delta) / (b^H(P, C) b^{H_\Delta}(P_\Delta, C))$, which have been derived only for the normalized coprime factor case in the gap and v -gap theory. From results (b) and (c), intuitively, a small distance between plants will generally result in tight bounds on the robust performance variation.

Table I. The basic uncertainty structures considered in this paper.

Uncertainty structure	$P_\Delta =$	$S_w =$	$S_z =$	$H =$	$\mathcal{F}_l(H, C) =$
Additive	$P + \Delta$	$\begin{bmatrix} I \\ 0 \end{bmatrix}$	$[0 \ I]$	$\begin{bmatrix} 0 & I \\ I & P \end{bmatrix}$	$C(I - PC)^{-1}$
Input multiplicative	$P(I - \Delta)$	$\begin{bmatrix} 0 \\ I \end{bmatrix}$	$[0 \ I]$	$\begin{bmatrix} 0 & I \\ -P & P \end{bmatrix}$	$-C(I - PC)^{-1}P$
Output inverse multiplicative	$(I - \Delta)^{-1}P$	$\begin{bmatrix} I \\ 0 \end{bmatrix}$	$[I \ 0]$	$\begin{bmatrix} I & P \\ I & P \end{bmatrix}$	$(I - PC)^{-1}$

3. SPECIFIC UNCERTAINTY STRUCTURES

In the following sections, Theorems 1 and 2 will be specialized for the cases of additive, input/output multiplicative and output inverse multiplicative uncertainty structures. These three cases are of practical importance for design problems, and also display some interesting properties of the general theory. Previous papers describe specializations for four-block/normalized coprime factor [29] and non-normalized coprime factor [9] uncertainty structures. The generalized plant given in Equation (3) can capture all of these uncertainty structures through the choice of the selection matrices S_w and S_z . Table I lists the structures explored in this paper with their consistency equations, their corresponding S_w and S_z matrices, the resulting generalized plant H as given in Equation (3), and the input–output transfer function $\mathcal{F}_l(H, C)$ as required for computation of the stability margin given in Definition 3. Using these basic results, much of the work will be to express the distance measure and the winding number condition described in Theorem 1 independent of Δ for each specific uncertainty characterization.

4. ADDITIVE UNCERTAINTY CHARACTERIZATION WITH AN ADDITIVE PERFORMANCE MEASURE

In the case of additive uncertainty, the analysis is uniquely straightforward. Plugging $\mathcal{F}_l(H, C)$ from Table I into Definition 3, we see that the stability margin $b_a(P, C)$ for an additive uncertainty characterization is given by:

$$b_a(P, C) := \begin{cases} \|C(I - PC)^{-1}\|_\infty^{-1} & \text{if } [P, C] \text{ is internally stable,} \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

The uncertainty Δ that yields P_Δ when connected to P is also easily characterized as $\Delta = P_\Delta - P$. Thus, for this specific case, a necessary and sufficient condition for the existence of a $\Delta \in \mathcal{RL}_\infty$ that satisfies the consistency equation $\Delta = P_\Delta - P$ is $P_\Delta - P \in \mathcal{RL}_\infty$, with an obvious simple sufficient condition being $P, P_\Delta \in \mathcal{RL}_\infty$. Also, given any P, P_Δ pair that satisfy $P_\Delta - P \in \mathcal{RL}_\infty$, there only exists a unique solution for $\Delta \in \mathcal{RL}_\infty$ given by $\Delta = P_\Delta - P$. Note that the LFT $\mathcal{F}_u(H, \Delta)$ is always well-posed (i.e. $\det(I - H_{11}\Delta) \neq 0$) for any Δ as $H_{11} = 0$ in this simple case as seen from Table I. Consequently, as shown above,

$$P_\Delta - P = \Delta \iff P_\Delta = \mathcal{F}_u(H, \Delta).$$

Hence, given a nominal plant P and a perturbed plant P_Δ , we have shown above that $P_\Delta - P \in \mathcal{RL}_\infty$ is a necessary and sufficient condition for there to exist a $\Delta \in \mathcal{RL}_\infty$ satisfying $P_\Delta = \mathcal{F}_u(H, \Delta)$.

Also, when this condition is satisfied, there only exists a unique solution $\Delta = P_\Delta - P \in \mathcal{RL}_\infty$ for $P_\Delta = \mathcal{F}_u(H, \Delta)$. Straight from Definition 1, the solution set Δ reduces to

$$\Delta = \begin{cases} \{\Delta = P_\Delta - P\} & \text{when } (P_\Delta - P) \in \mathcal{RL}_\infty, \\ \emptyset & \text{otherwise.} \end{cases}$$

Note that here Δ is either empty or contains only one element. It easily follows straight from Definition 1 that the distance measure $d_a(P, P_\Delta)$ for additive uncertainty characterizations is given by:

$$d_a(P, P_\Delta) := \begin{cases} \|P_\Delta - P\|_\infty & \text{when } (P_\Delta - P) \in \mathcal{RL}_\infty, \\ \infty & \text{otherwise.} \end{cases} \quad (5)$$

It is clear that $d_a(P, P_\Delta)$ is a metric on \mathcal{RL}_∞ in this specific case.[‡] Now, we need to rewrite winding number condition (2) independently of Δ . For additive uncertainty characterizations, this process is trivial since $H_{11} = 0$. Consequently, winding number condition (2) reduces to $\eta(P_\Delta) = \eta(P)$, which is easily computable as it is independent of Δ . All these computations now enable us to give specific versions of the generic robust stability Theorem 1 and the generic robust performance Theorem 2 for additive uncertainty.

Theorem 3 (Robust Stability—Additive)

Given a plant $P \in \mathcal{R}^{p \times q}$, a perturbed plant $P_\Delta \in \mathcal{R}^{p \times q}$ and a controller $C \in \mathcal{R}^{q \times p}$. Define a stability margin $b_a(P, C)$ as in (4) and a distance measure $d_a(P, P_\Delta)$ as in (5). Furthermore, suppose $d_a(P, P_\Delta) < b_a(P, C)$. Then

$$[P_\Delta, C] \text{ is internally stable} \iff \eta(P_\Delta) = \eta(P).$$

Proof

This theorem specializes Theorem 1 using formulae derived in the above section. Note that the supposition that ‘ H is stabilizable’ is automatically fulfilled in this specific design case, and hence does not need to be independently enforced. Also, the supposition $d_a(P, P_\Delta) < b_a(P, C)$ guarantees that $d_a(P, P_\Delta) < \infty$ which then implies via (Definition 5) that $(P_\Delta - P) \in \mathcal{RL}_\infty$. \square

A similar extension of the small-gain theorem for systems in \mathcal{RL}_∞ was first described by Glover [4, 28] for additive uncertainty only. Robust performance guarantees, however, were not given at the time, and are here expressed for additive uncertainty for the first time in the following theorem:

Theorem 4 (Robust Performance—Additive)

Given the suppositions of Theorem 3 and furthermore assuming $\eta(P_\Delta) = \eta(P)$. Then

$$|b_a(P_\Delta, C) - b_a(P, C)| \leq d_a(P, P_\Delta) \quad (6)$$

and

$$\|\mathcal{F}_l(H_\Delta, C) - \mathcal{F}_l(H, C)\|_\infty \leq \frac{d_a(P, P_\Delta)}{b_a(P, C)b_a(P_\Delta, C)}, \quad (7)$$

where

$$H = \begin{bmatrix} 0 & I \\ I & P \end{bmatrix}$$

[‡]This allows the use of mathematical results on metric spaces for further analysis. Note that the multiplicative distances in the following section generally violate the symmetry property of metrics, but can be turned into metrics by taking the maximum over both directed distances, as is done for the gap metric [8, Chapter 7.1].

and

$$H_{\Delta} = \begin{bmatrix} 0 & I \\ I & P_{\Delta} \end{bmatrix}.$$

Proof

This theorem specializes Theorem 2 using formulae derived in the above section. Specifically, choosing $k = 0$ and noting that $S = S_z S_w = 0 \in \mathcal{RL}_{\infty}$ gives $\|\mathcal{F}_l(H_{\Delta}, C) - S\|_{\infty} = 1/b_a(P_{\Delta}, C)$. This equality follows from Equation (4) for $b_a(P_{\Delta}, C)$ on noting that $[P_{\Delta}, C]$ is internally stable via Theorem 3. □

5. MULTIPLICATIVE UNCERTAINTY CHARACTERIZATION WITH A MULTIPLICATIVE PERFORMANCE MEASURE

In this section, we will derive the robust stability and robust performance results for the input multiplicative and output inverse multiplicative uncertainty characterization, and then show how these results can be modified to get dual results for the output multiplicative and input inverse multiplicative cases.

5.1. Input multiplicative

- (1) Define the stability margin $b_{im}(P, C)$. Using Table I, we see straight from Definition 3 that the stability margin $b_{im}(P, C)$ for an input multiplicative uncertainty characterization is given by:

$$b_{im}(P, C) := \begin{cases} \|C(I - PC)^{-1}P\|_{\infty}^{-1} & \text{if } [P, C] \text{ is internally stable,} \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

- (2) Solve consistency equation for all $\Delta \in \mathcal{RL}_{\infty}$. Using the structure of H defined in Equation (3) we can reformulate the consistency equation $P_{\Delta} = \mathcal{F}_u(H, \Delta)$ as follows:

$$P_{\Delta} = \mathcal{F}_u(H, \Delta) \iff P_{\Delta} - P = [I \quad -P] S_w \Delta S_z \begin{bmatrix} P_{\Delta} \\ I \end{bmatrix} \tag{9}$$

whenever well-posedness of $\mathcal{F}_u(H, \Delta)$ is assumed (i.e. $\det(I - H_{11}\Delta)(\infty) \neq 0$). Equation (9) further reduces to

$$P_{\Delta} - P = -P\Delta \tag{10}$$

on substituting $S_w = [0 \ I]^T$ and $S_z = [0 \ I]$ from Table I. We shall assume in this section that $P(\infty)$ has full rank, which is imposed for mathematical convenience.[§] To proceed, we have to split up our analysis into three cases—i.e. for square plants, fat plants and tall plants—since one of these cases has a unique solution, one case can possibly have no solution and one has multiple solutions.

Square plants:

Assume (in this square plants case) that $P, P_{\Delta} \in \mathcal{R}^{p \times q}$ is such that $p = q$ and $P(\infty)$ has full rank. Then it is easy to see that Equation (10) can be equivalently rearranged into

$$\Delta = P^{-1}(P - P_{\Delta}). \tag{11}$$

Consequently, for this case, a necessary and sufficient condition for there to exist a $\Delta \in \mathcal{RL}_{\infty}$ that satisfies consistency of equations is $P^{-1}P_{\Delta} \in \mathcal{RL}_{\infty}$, with an obvious simple sufficient

[§]If P does not satisfy this assumption, one can always negligibly perturb P at infinite frequency to satisfy this assumption.

condition being $P^{-1}, P_{\Delta} \in \mathcal{RL}_{\infty}$. Also, given any P, P_{Δ} pair that satisfy $P^{-1}P_{\Delta} \in \mathcal{RL}_{\infty}$, there only exists a unique solution for $\Delta \in \mathcal{RL}_{\infty}$ given by Equation (11).

Fat plants:

Assume (in this fat plants case) that $P, P_{\Delta} \in \mathcal{R}^{p \times q}$ is such that $p < q$ and $P(\infty)$ has full rank. Let P have the state-space realization

$$P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$$

with D having full row rank, and define

$$\bar{P} = \left[\begin{array}{c|c} A - BD^{\dagger}C & -BD_{\perp}^{*} \\ \hline D^{\dagger}C & D_{\perp}^{*} \end{array} \right] \in \mathcal{R}^{q \times (q-p)},$$

where D^{\dagger} is the Moore–Penrose inverse of D and D_{\perp} satisfies

$$\begin{bmatrix} D \\ D_{\perp} \end{bmatrix} [D^{\dagger} \quad D_{\perp}^{*}] = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

Find an $X \in \mathcal{R}^{p \times p}$ satisfying

$$XX^{*} = PP^{*} \tag{12}$$

and a $Y \in \mathcal{R}^{(q-p) \times (q-p)}$ satisfying $Y^{*}Y = \bar{P}^{*}\bar{P}$. Note that since X (resp. Y) is square and DD^{*} (resp. $D_{\perp}D_{\perp}^{*}$) is nonsingular, it follows that $XX^{*} = PP^{*}$ (resp. $Y^{*}Y = \bar{P}^{*}\bar{P}$) implicitly implies that $X^{-1} \in \mathcal{R}^{p \times p}$ (resp. $Y^{-1} \in \mathcal{R}^{(q-p) \times (q-p)}$). Define

$$\Phi = [P^{*}X^{-*} \quad \bar{P}Y^{-1}] \in \mathcal{R}^{q \times q} \tag{13}$$

and note that $\Phi^{*}\Phi = I$ since $P\bar{P} = 0$. Since Φ is also square, we have $\Phi^{-1} = \Phi^{*}$. Now it is easy to see that Equation (10) can be equivalently rearranged into

$$\begin{aligned} P - P_{\Delta} &= P\Delta \\ \iff (P - P_{\Delta}) &= [X \quad 0]\Phi^{-1}\Delta \\ \iff \Delta &= \Phi \begin{bmatrix} X^{-1}(P - P_{\Delta}) \\ Q \end{bmatrix} \text{ for any } Q \in \mathcal{R}^{(q-p) \times q}. \end{aligned}$$

Consequently, for this specific case, since Φ is a unit in \mathcal{RL}_{∞} , a necessary and sufficient condition for there to exist a $\Delta \in \mathcal{RL}_{\infty}$ that satisfies consistency of equations is $X^{-1}P_{\Delta} \in \mathcal{RL}_{\infty}$, with an obvious simple sufficient condition being P having no transmission zeros on $j(\mathbb{R} \cup \{\infty\})$ and $P_{\Delta} \in \mathcal{RL}_{\infty}$. Then, given any P, P_{Δ} pair that satisfy $X^{-1}P_{\Delta} \in \mathcal{RL}_{\infty}$, there always exist multiple solutions for $\Delta \in \mathcal{RL}_{\infty}$ given by

$$\Delta = \Phi \begin{bmatrix} X^{-1}(P - P_{\Delta}) \\ Q \end{bmatrix} \text{ for any } Q \in \mathcal{RL}_{\infty}^{(q-p) \times q}. \tag{14}$$

Tall plants:

Assume (in this tall plants case) that $P, P_{\Delta} \in \mathcal{R}^{p \times q}$ is such that $p > q$ and $P(\infty)$ has full rank. Let P have the state-space realization

$$P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$$

with D having full column rank, and define

$$\check{P} = \left[\begin{array}{c|c} A - BD^{\dagger}C & -BD^{\dagger} \\ \hline D_{\perp}^{*}C & D_{\perp}^{*} \end{array} \right] \in \mathcal{R}^{(p-q) \times p}, \tag{15}$$

where D^\dagger is the Moore–Penrose inverse of D and D_\perp satisfies

$$\begin{bmatrix} D^\dagger \\ D_\perp^* \end{bmatrix} [D \ D_\perp] = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

Find a $U \in \mathcal{R}^{q \times q}$ satisfying $U^*U = P^*P$ and a $V \in \mathcal{R}^{(p-q) \times (p-q)}$ satisfying $VV^* = \check{P}\check{P}^*$. Note that since U (resp. V) is square and D^*D (resp. $D_\perp^*D_\perp$) is nonsingular, it follows that $U^*U = P^*P$ (resp. $VV^* = \check{P}\check{P}^*$) implicitly implies that $U^{-1} \in \mathcal{R}^{q \times q}$ (resp. $V^{-1} \in \mathcal{R}^{(p-q) \times (p-q)}$). Define $\Psi = \begin{bmatrix} U^{-*}P^* \\ V^{-1}\check{P} \end{bmatrix} \in \mathcal{R}^{p \times p}$ and note that $\Psi\Psi^* = I$ since $\check{P}P = 0$. Since Ψ is also square, we have $\Psi^{-1} = \Psi^*$. Now it is easy to see that Equation (10) can be equivalently rearranged into

$$\begin{aligned} P - P_\Delta = P\Delta &\iff \Psi(P - P_\Delta) = \Psi P\Delta \\ &\iff \begin{bmatrix} U^{-1}U^{-*}P^*(P - P_\Delta) \\ \check{P}P_\Delta \end{bmatrix} = \begin{bmatrix} \Delta \\ 0 \end{bmatrix}. \end{aligned}$$

Consequently, for this specific case, necessary and sufficient conditions for there to exist a $\Delta \in \mathcal{RL}_\infty$ that satisfies consistency of equations are

$$(P^*P)^{-1}P^*P_\Delta \in \mathcal{RL}_\infty \quad \text{and} \quad \check{P}P_\Delta = 0 \tag{16}$$

with an obvious simple sufficient condition being P having no transmission zeros on $j(\mathbb{R} \cup \{\infty\})$, $P_\Delta \in \mathcal{RL}_\infty$ and $\check{P}P_\Delta = 0$. Then, given any P, P_Δ pair that satisfy condition (16), there exists only a unique solution for $\Delta \in \mathcal{RL}_\infty$ given by

$$\Delta = (P^*P)^{-1}P^*(P - P_\Delta). \tag{17}$$

It is worth pointing out at this stage that it is well known in the engineering community that input multiplicative uncertainty characterizations for strictly tall plants have limited authority to alter the nominal plant dynamics. This engineering insight is captured above by the fact that the condition $\check{P}P_\Delta = 0$ is not easy to satisfy in practice, hence making this form of uncertainty modelling of limited engineering relevance.

- (3) *Derive conditions to guarantee well-posedness of $\mathcal{F}_u(H, \Delta)$.* Similar to the additive case, since $H_{11} = 0$, no extra conditions need to be imposed on the data P, P_Δ for well-posedness of the linear fractional transformation $\mathcal{F}_u(H, \Delta)$. Consequently, given a nominal plant P and a perturbed plant P_Δ , we have shown above that one of the following three conditions is a necessary and sufficient condition for there to exist a $\Delta \in \mathcal{RL}_\infty$ satisfying $P_\Delta = \mathcal{F}_u(H, \Delta)$:

- *Condition I* means $P, P_\Delta \in \mathcal{R}^{p \times q}$ with $p = q$ satisfying $P(\infty)$ having full rank and $P^{-1}P_\Delta \in \mathcal{RL}_\infty$.
- *Condition II* means $P, P_\Delta \in \mathcal{R}^{p \times q}$ with $p < q$ satisfying $P(\infty)$ having full rank and $X^{-1}P_\Delta \in \mathcal{RL}_\infty$ (where $X \in \mathcal{R}^{p \times p}$ satisfies $XX^* = PP^*$).
- *Condition III* means $P, P_\Delta \in \mathcal{R}^{p \times q}$ with $p > q$ satisfying $P(\infty)$ having full rank, $(P^*P)^{-1}P^*P_\Delta \in \mathcal{RL}_\infty$ and $\check{P}P_\Delta = 0$ (where \check{P} is defined in Equation (15)).

Also, for the equation $P_\Delta = \mathcal{F}_u(H, \Delta)$, when:

- *Condition I* is satisfied (square plant case), there only exists a unique solution $\Delta \in \mathcal{RL}_\infty$ given by Equation (11);
- *Condition II* is satisfied (fat plant case), there always exist multiple solutions $\Delta \in \mathcal{RL}_\infty$ given by Equation (14);
- *Condition III* is satisfied (tall plant case), there only exists a unique solution $\Delta \in \mathcal{RL}_\infty$ given by Equation (17).

- (4) Define the solution set Δ and distance measure $d_{\text{im}}(P, P_\Delta)$. Straight from Definition 1, the solution set Δ reduces to

$$\Delta = \begin{cases} \{P^{-1}(P - P_\Delta)\} & \text{when Condition I holds,} \\ \left\{ \Phi \begin{bmatrix} X^{-1}(P - P_\Delta) \\ Q \end{bmatrix} : Q \in \mathcal{RL}_\infty^{(q-p) \times q} \right\} & \text{when Condition II holds,} \\ \{(P^*P)^{-1}P^*(P - P_\Delta)\} & \text{when Condition III holds,} \\ \emptyset & \text{otherwise} \end{cases}$$

in this specific case. The objects X, Φ in the definition of set Δ above are given by Equations (12) and (13). The definition of the distance measure $d_{\text{im}}(P, P_\Delta)$ when Condition I or Condition III holds is trivial since there is only one element in the set Δ . When Condition II holds, and since Φ is allpass, $\inf_{\Delta \in \Delta} \|\Delta\|_\infty = \|X^{-1}(P - P_\Delta)\|_\infty$. Then, it easily follows straight from Definition 1 that—for an $X \in \mathcal{R}^{p \times p}$ that satisfies $XX^* = PP^*$ —the distance measure $d_{\text{im}}(P, P_\Delta)$ for input multiplicative uncertainty characterizations is given by[‡]:

$$d_{\text{im}}(P, P_\Delta) := \begin{cases} \|X^{-1}(P - P_\Delta)\|_\infty & \text{when Condition I or II holds,} \\ \|(P^*P)^{-1}P^*(P - P_\Delta)\|_\infty & \text{when Condition III holds,} \\ \infty & \text{otherwise.} \end{cases} \quad (18)$$

- (5) Write the winding number condition independent of Δ . For input multiplicative uncertainty characterizations as considered here, this process is trivial since $H_{11} = 0$. Winding number condition Equation (2) reduces to $\eta(P_\Delta) = \eta(P)$, which is easily computable as it is independent of Δ .
- (6) State robust stability and robust performance theorems.

Theorem 5 (Robust Stability—Input Multiplicative)

Given a plant $P \in \mathcal{R}^{p \times q}$, a perturbed plant $P_\Delta \in \mathcal{R}^{p \times q}$ and a controller $C \in \mathcal{R}^{q \times p}$. Define a stability margin $b_{\text{im}}(P, C)$ as in (8) and a distance measure $d_{\text{im}}(P, P_\Delta)$ as in (18). Furthermore, suppose $d_{\text{im}}(P, P_\Delta) < b_{\text{im}}(P, C)$. Then

$$[P_\Delta, C] \text{ is internally stable } \iff \eta(P_\Delta) = \eta(P).$$

Proof

This theorem specializes Theorem 1 using formulae derived in the above subsection. Similar to the additive case, the supposition that ‘ H is stabilizable’ is automatically fulfilled because of the special form of H . □

Theorem 6 (Robust Performance—Input Multiplicative)

Given the suppositions of Theorem 5 and furthermore assuming $\eta(P_\Delta) = \eta(P)$. Then

$$\left| 1 - \frac{b_{\text{im}}(P, C)}{b_{\text{im}}(P_\Delta, C)} \right| \leq \| (I - CP_\Delta)^{-1} \|_\infty d_{\text{im}}(P, P_\Delta) \quad (19)$$

and

$$\frac{\|\mathcal{F}_l(H_\Delta, C) - \mathcal{F}_l(H, C)\|_\infty}{\|\mathcal{F}_l(H, C)\|_\infty} \leq \| (I - CP_\Delta)^{-1} \|_\infty d_{\text{im}}(P, P_\Delta), \quad (20)$$

where

$$H = \begin{bmatrix} 0 & I \\ -P & P \end{bmatrix}$$

[‡]Note that in the case of square plants (Condition I), $X^{-1} = P^{-1}$.

and

$$H_{\Delta} = \left[\begin{array}{c|c} 0 & I \\ \hline -P_{\Delta} & P_{\Delta} \end{array} \right].$$

Proof

This theorem specializes Theorem 2 using formulae derived in the above subsection. The result follows on choosing $k=0$ and noting that $S = S_z S_w = I \in \mathcal{RL}_{\infty}$ thereby giving $\|\mathcal{F}_l(H_{\Delta}, C) - S\|_{\infty} = \|(I - CP_{\Delta})^{-1}\|_{\infty}$. The result given in inequality (19) follows from that of inequality (b) in Theorem 2 on dividing both sides with $b_{im}(P_{\Delta}, C)$, since this is guaranteed to be strictly greater than zero via Theorem 5. The result given in inequality (20) follows from that of inequality (c) in Theorem 2 on multiplying both sides with $b_{im}(P, C) = 1/\|\mathcal{F}_l(H, C)\|_{\infty}$, since this too is guaranteed to be strictly greater than zero via the suppositions of this theorem. \square

It is worth pointing out that the discrepancy between the nominal and perturbed stability margin and also the discrepancy between the nominal and perturbed closed-loop in this theorem are naturally written in multiplicative form since we are handling multiplicative uncertainty characterizations and performance measures in this section. Also, note that the object $\|(I - CP_{\Delta})^{-1}\|_{\infty}$ corrupts the right side of inequalities (19) and (20). This quantity is typically very small in the pass-band, close to unity in the stop-band and not too big around crossover for a good design. Hence, this factor assists in tightening the inequalities in the pass-band.

5.2. Output multiplicative

It should be noted that output multiplicative uncertainty, i.e. $P_{\Delta} = (I - \Delta)P$, is simply a dual to input multiplicative uncertainty. The results for this dual case can easily be obtained by the substitutions $P \rightarrow P^T$, $P_{\Delta} \rightarrow P_{\Delta}^T$ and $\Delta \rightarrow \Delta^T$ in the relevant definitions, equations and theorems in Section 5.1.

5.3. Output inverse multiplicative

For inverse multiplicative uncertainty, it is suitable to begin with the output inverse case, as this exhibits some degree of similarity with the input multiplicative case. However, the problem of output inverse multiplicative uncertainty characterization with output inverse multiplicative performance measure is *not* a simple dual problem to that derived above in Section 5.1.

- (1) *Define the stability margin $b_{oim}(P, C)$.* Using Table I, we see straight from Definition 3 that the stability margin $b_{oim}(P, C)$ for an output inverse multiplicative uncertainty characterization is given by:

$$b_{oim}(P, C) := \begin{cases} \|(I - PC)^{-1}\|_{\infty}^{-1} & \text{if } [P, C] \text{ is internally stable,} \\ 0 & \text{otherwise.} \end{cases} \tag{21}$$

- (2) *Solve consistency equation for all $\Delta \in \mathcal{RL}_{\infty}$.* In this specific case, Equation (9) reduces to

$$P_{\Delta} - P = \Delta P_{\Delta} \tag{22}$$

on substituting $S_w = [I \ 0]^T$ and $S_z = [I \ 0]$. We shall assume in this section that $P_{\Delta}(\infty)$ has full rank, which is imposed for mathematical convenience. If the perturbed plant P_{Δ} does not satisfy this assumption, one can always negligibly perturb P_{Δ} at infinite frequency so as to satisfy this assumption. Luckily, to answer the question of solving consistency equation (22) for all $\Delta \in \mathcal{RL}_{\infty}$, we do not need to repeat the whole derivation process of Section V-A2 as quick comparison of Equation (22) with Equation (10) reveals that the required results follow by performing the substitutions $P \rightarrow P_{\Delta}^T$, $P_{\Delta} \rightarrow P^T$ and $\Delta \rightarrow \Delta^T$ in the derivation of Section V-A2.

- (3) *Derive conditions to guarantee well-posedness of $\mathcal{F}_u(H, \Delta)$.* The derivations here depart from the similarity observed before. We now wish to make a connection between consistency

Equation (22) and the uncertainty characterization $P_\Delta = \mathcal{F}_u(H, \Delta)$. Since we define $P_\Delta = \mathcal{F}_u(H, \Delta)$ to be well-posed when $\det(I - H_{11}\Delta)(\infty) \neq 0$, we first need to express $\det(I - H_{11}\Delta)(\infty)$ independently of Δ . Since $H_{11} = I$ (a very important difference from the additive and input multiplicative cases), it follows (after some simple algebra) that

$$\det(I - H_{11}\Delta)(\infty) \neq 0 \iff \begin{cases} \det(P(\infty)) \neq 0 & \text{if } p = q, \\ \det[P \quad (\check{P}_\Delta^* V^{-*} - Q)](\infty) \neq 0 & \text{if } p > q, \\ \det(P P_\Delta^*)(\infty) \neq 0 & \text{if } p < q, \end{cases} \quad (23)$$

on using the results derived from Section V-A2 via the substitutions given in the preceding step of our procedure. In equivalence (23), $Q \in \mathcal{RL}_\infty^{p \times (p-q)}$ is arbitrary, $V \in \mathcal{R}^{(p-q) \times (p-q)}$ satisfies $VV^* = \check{P}_\Delta \check{P}_\Delta^*$ and

$$\check{P}_\Delta = \left[\begin{array}{c|c} A - BD^\dagger C & -BD^\dagger \\ \hline D_\perp^* C & D_\perp^* \end{array} \right] \in \mathcal{R}^{(p-q) \times p}$$

in which

$$P_\Delta = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right],$$

D^\dagger is the Moore–Penrose inverse of D and D_\perp satisfies

$$\begin{bmatrix} D^\dagger \\ D_\perp^* \end{bmatrix} [D \quad D_\perp] = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

The inequalities in (23) restrict the allowable $P(\infty)$, $P_\Delta(\infty)$ data and $Q(\infty)$ for well-posedness of the linear fractional transformation $\mathcal{F}_u(H, \Delta)$. The following technical lemma is needed to simplify condition (23).

Lemma 7

- (1) When $p = q$, condition (23) is equivalent to $P(\infty)$ having full rank.
- (2) When $p > q$, $\exists Q \in \mathcal{RL}_\infty$ so that condition (23) is fulfilled if and only if $P(\infty)$ has full rank.
- (3) When $p < q$, condition (23) is equivalent to $P(\infty)$ having full rank under the supposition $P P_\Delta = 0$.

Proof

- (1) Trivial.
- (2) Since $\check{P}_\Delta^* V^{-*} \in \mathcal{RL}_\infty$, $\exists Q \in \mathcal{RL}_\infty$ so that condition (23) is fulfilled if and only if $\exists \hat{Q} \in \mathcal{RL}_\infty$ so that $\det[P \quad \hat{Q}](\infty) \neq 0$ if and only if $P(\infty)$ has full rank.
- (3) Since $\text{rank}(P(\infty)) = \text{rank}(P(\infty)[P_\Delta(\infty)^* \check{P}_\Delta(\infty)]) = \text{rank}([(P P_\Delta^*)(\infty) \quad 0])$, it easily follows that $P(\infty)$ has full rank if and only if $\det(P P_\Delta^*)(\infty) \neq 0$. □

Under the restrictions imposed by (23), we can now make the connection

$$P_\Delta - P = \Delta P_\Delta \iff P_\Delta = \mathcal{F}_u(H, \Delta),$$

as shown above in Equation (9). Consequently, given a nominal plant P and a perturbed plant P_Δ , we have shown above that one of the following three conditions is a necessary and sufficient condition for there to exist a $\Delta \in \mathcal{RL}_\infty$ satisfying $P_\Delta = \mathcal{F}_u(H, \Delta)$:

- *Condition I* means $P, P_\Delta \in \mathcal{R}^{p \times q}$ with $p = q$ satisfying $P(\infty), P_\Delta(\infty)$ having full rank and $P P_\Delta^{-1} \in \mathcal{RL}_\infty$.
- *Condition II* means $P, P_\Delta \in \mathcal{R}^{p \times q}$ with $p > q$ satisfying $P(\infty), P_\Delta(\infty)$ having full rank and $P U^{-1} \in \mathcal{RL}_\infty$ (where $U \in \mathcal{R}^{q \times q}$ satisfies $U^* U = P_\Delta^* P_\Delta$).

- *Condition III* means $P, P_\Delta \in \mathcal{R}^{p \times q}$ with $p < q$ satisfying $P(\infty), P_\Delta(\infty)$ having full rank, $P P_\Delta^* (P_\Delta P_\Delta^*)^{-1} \in \mathcal{RL}_\infty$ and $P \check{P}_\Delta = 0$.

Also, for the equation $P_\Delta = \mathcal{F}_u(H, \Delta)$, when:

- *Condition I* is satisfied (square plant case), there only exists a unique solution $\Delta \in \mathcal{RL}_\infty$ given by

$$\Delta = (P_\Delta - P)P_\Delta^{-1}. \tag{24}$$

- *Condition II* is satisfied (tall plant case), there always exist multiple solutions $\Delta \in \mathcal{RL}_\infty$ given by

$$\Delta = [(P_\Delta - P)U^{-1} \quad Q]\Psi \tag{25}$$

for any $Q \in \mathcal{RL}_\infty^{p \times (p-q)}$ that satisfies (23) (where $\Psi = \begin{bmatrix} U^{-*} P^* \\ V^{-1} \check{P} \end{bmatrix} \in \mathcal{R}^{p \times p}$).

- *Condition III* is satisfied (fat plant case), there only exists a unique solution $\Delta \in \mathcal{RL}_\infty$ given by

$$\Delta = (P_\Delta - P)P_\Delta^* (P_\Delta P_\Delta^*)^{-1}. \tag{26}$$

(4) Define the solution set Δ and distance measure $d_{\text{oim}}(P, P_\Delta)$. From Definition 1, the solution set Δ is characterized as follows:

$$\Delta = \begin{cases} \{(P_\Delta - P)P_\Delta^{-1}\} & \text{when Condition I holds,} \\ \{[(P_\Delta - P)U^{-1} Q]\Psi : Q \in \mathcal{RL}_\infty^{p \times (p-q)}, \\ \det[P \quad (\check{P}_\Delta^* V^{-*} - Q)](\infty) \neq 0\} & \text{when Condition II holds,} \\ \{(P_\Delta - P)P_\Delta^* (P_\Delta P_\Delta^*)^{-1}\} & \text{when Condition III holds,} \\ \emptyset & \text{otherwise} \end{cases}$$

in this specific case. The definition of the distance measure $d_{\text{oim}}(P, P_\Delta)$ when Condition I or Condition III holds is trivial since there is only one element in the set Δ . When Condition II holds, and since Ψ is allpass, $\inf_{\Delta \in \Delta} \|\Delta\|_\infty = \|(P_\Delta - P)U^{-1}\|_\infty$. The following technical lemma states under what conditions can $Q(\infty) = 0$ be chosen.

Lemma 8

The choice $Q(\infty) = 0$ in the solution $\Delta = [(P_\Delta - P)U^{-1} \quad Q]\Psi$ for the $p > q$ case gives a $\Delta(\infty)$ that satisfies $\det(I - H_{11}\Delta)(\infty) \neq 0$ if and only if $\det(P_\Delta^* P)(\infty) \neq 0$.

Proof

Choose $Q(\infty) = 0$ in (23) for $p > q$. Then

$$\begin{aligned} \det(I - H_{11}\Delta)(\infty) \neq 0 &\iff \det[P \quad \check{P}_\Delta^*](\infty) \neq 0 \\ &\iff \det \left(\begin{bmatrix} P_\Delta^* \\ \check{P}_\Delta \end{bmatrix} [P \quad \check{P}_\Delta^*] \right) (\infty) \neq 0 \\ &\iff \det(P_\Delta^* P)(\infty) \neq 0 \end{aligned}$$

since $\check{P}_\Delta P_\Delta = 0$. □

Then, it follows from Definition 1 that—for a $U \in \mathcal{R}^{q \times q}$ that satisfies $U^*U = P_\Delta^*P_\Delta$ —the distance measure $d_{\text{oim}}(P, P_\Delta)$ for output inverse multiplicative uncertainty characterizations is given by:

$$d_{\text{oim}}(P, P_\Delta) := \begin{cases} \|(P_\Delta - P)U^{-1}\|_\infty & \text{when Condition I or II holds,} \\ \|(P_\Delta - P)P_\Delta^*(P_\Delta P_\Delta^*)^{-1}\|_\infty & \text{when Condition III holds,} \\ \infty & \text{otherwise.} \end{cases} \quad (27)$$

- (5) Write the winding number condition independent of Δ . The problem needs to be split again into three cases: square, tall and fat plants.

Square plants:

When Condition I is satisfied, using $\Delta \in \Delta^{\text{min}}$ given by Equation (24) in winding number condition Equation (2) gives

$$\eta(P_\Delta) - \eta(P) = \text{wno det}(P P_\Delta^{-1}) = \text{wno det}(P P_\Delta^*). \quad (28)$$

Note that Equation (28) can be simplified to $z(P_\Delta) = z(P)$, where $z(\cdot)$ are the number of open right half plane zeros of (\cdot) , but for consistency with the tall/fat plant cases we choose not to use this simpler formulation.

Tall plants:

Before tackling this case, note that an immediate corollary to Lemma 8 is as follows:

Corollary 9

Choosing $Q = 0$ in Equation (25) for the $p > q$ case gives a $\Delta \in \Delta^{\text{min}}$ if and only if $\text{det}(P_\Delta^*P)(\infty) \neq 0$.

Consequently, when Condition II and $\text{det}(P_\Delta^*P)(\infty) \neq 0$ are satisfied, using $\Delta \in \Delta^{\text{min}}$ given by Equation (25) with $Q = 0$ in winding number condition Equation (2) gives

$$\eta(P_\Delta) - \eta(P) = \text{wno det}(I - (P_\Delta - P)(P_\Delta^*P_\Delta)^{-1}P_\Delta^*) = \text{wno det}(P_\Delta^*P). \quad (29)$$

Fat plants:

When Condition III is satisfied, using $\Delta \in \Delta^{\text{min}}$ given by Equation (26) in winding number condition Equation (2) gives

$$\eta(P_\Delta) - \eta(P) = \text{wno det}(P P_\Delta^*(P_\Delta P_\Delta^*)^{-1}) = \text{wno det}(P P_\Delta^*). \quad (30)$$

- (6) State robust stability and robust performance theorems.

Theorem 10 (Robust Stability—Output Inverse Multiplicative)

Given a plant $P \in \mathcal{R}^{p \times q}$, a perturbed plant $P_\Delta \in \mathcal{R}^{p \times q}$ and a controller $C \in \mathcal{R}^{q \times p}$. Define a stability margin $b_{\text{oim}}(P, C)$ as in (21), a distance measure $d_{\text{oim}}(P, P_\Delta)$ as in (27), and an object $\Xi = \begin{cases} P P_\Delta^* & \text{when } p \leq q \\ P_\Delta^* P & \text{otherwise} \end{cases}$. Furthermore, suppose $d_{\text{oim}}(P, P_\Delta) < b_{\text{oim}}(P, C)$ and when $p > q$, suppose also $\text{det}(P_\Delta^*P)(\infty) \neq 0$. Then

$$[P_\Delta, C] \text{ is internally stable} \iff \text{wno det}(\Xi) = \eta(P_\Delta) - \eta(P),$$

where the winding number is evaluated on a contour indented to the right around any imaginary axis poles of P and P_Δ .

Proof

This theorem specializes Theorem 1 using formulae derived in the above subsection. The supposition $d_{\text{oim}}(P, P_\Delta) < b_{\text{oim}}(P, C)$ implies that either Condition I or II or III must hold since $d_{\text{oim}}(P, P_\Delta) < b_{\text{oim}}(P, C) \leq \infty$. Note also that the supposition that ‘ H is stabilizable’ is automatically fulfilled in this specific design case. \square

Theorem 11 (Robust Performance—Output Inverse Multiplicative)

Given the suppositions of Theorem 10 and furthermore assuming $\text{wno det}(\Xi) = \eta(P_\Delta) - \eta(P)$, where the winding number is evaluated on a contour indented to the right around any imaginary axis poles of P and P_Δ . Then

$$\left| 1 - \frac{b_{\text{oim}}(P, C)}{b_{\text{oim}}(P_\Delta, C)} \right| \leq \|P_\Delta(I - CP_\Delta)^{-1}C\|_\infty d_{\text{oim}}(P, P_\Delta) \tag{31}$$

and

$$\frac{\|\mathcal{F}_I(H_\Delta, C) - \mathcal{F}_I(H, C)\|_\infty}{\|\mathcal{F}_I(H, C)\|_\infty} \leq \|P_\Delta(I - CP_\Delta)^{-1}C\|_\infty d_{\text{oim}}(P, P_\Delta), \tag{32}$$

where

$$H = \begin{bmatrix} I & P \\ I & P \end{bmatrix}$$

and

$$H_\Delta = \begin{bmatrix} I & P_\Delta \\ I & P_\Delta \end{bmatrix}.$$

Proof

This theorem specializes Theorem 2 using formulae derived in the above subsection. The result follows on choosing $k=0$ and noting that $S = S_z S_w = I \in \mathcal{RL}_\infty$ thereby giving $\|\mathcal{F}_I(H_\Delta, C) - S\|_\infty = \|P_\Delta(I - CP_\Delta)^{-1}C\|_\infty$. \square

Note that the object $\|P_\Delta(I - CP_\Delta)^{-1}C\|_\infty$ corrupts the right side of inequalities (31) and (32). This quantity is typically very close to unity in the pass-band, very small in the stop-band and not too big around crossover. Hence, it is a factor that assists in tightening the inequalities in the stop-band. Again, the discrepancy between nominal and perturbed stability margin and closed-loop transfer function given in inequalities (31) and (32) appear naturally in multiplicative form.

5.4. *Input inverse multiplicative*

The problem of input inverse multiplicative uncertainty characterization, i.e. $P_\Delta = P(I - \Delta)^{-1}$, with an input inverse multiplicative performance measure is simply a *dual problem* to that discussed above for output inverse multiplicative uncertainty characterization, i.e. $P_\Delta = (I - \Delta)^{-1}P$, with an output inverse multiplicative performance measure. Consequently, the results in this case follow trivially by performing the simple substitutions $P \rightarrow P^T$, $P_\Delta \rightarrow P_\Delta^T$ and $\Delta \rightarrow \Delta^T$ in the definitions, theorems and results of Section 5.3.

6. NUMERICAL EXAMPLE

A physically motivated numerical example illustrates the effectiveness of the robust stability and robust performance results derived in the preceding sections. We consider a spring-mass-damper system which is a small variation of a benchmark problem in robust control theory ([30] and the references therein; see also [31]). The schematic diagram of the system is depicted in Figure 5. The plant is a single-input single-output (SISO) system, where the input u is a force applied to one mass, and the output y is the measured acceleration of the same mass amplified with a sensor gain $k = 10$. This resembles a practical situation in the control of mechanical structures, where an accelerometer is used to capture information about the state of the system. Elementary mechanical modelling yields the following nominal SISO plant transfer function from u to y :

$$P(s) = 10 \frac{s^2(s^2 + 2s + 2)}{s^4 + 4s^3 + 7s^2 + 6s + 3}.$$

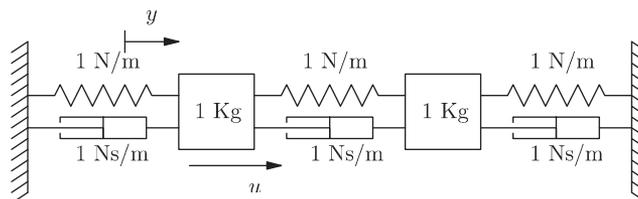


Figure 5. The spring-mass-damper system studied in the example. u is a force input and y an acceleration output.

Based on the nominal plant, an output feedback controller C is synthesized using the Matlab routine 'ncfsyn'. It achieves an optimal robust stability margin in the v -gap metric, i.e. with respect to normalized coprime factor uncertainty [32]. The synthesized controller is of third order, and its transfer function is given by

$$C(s) = -\frac{0.4587s^3 + 1.165s^2 + 1.456s + 0.5978}{s^3 + 2.173s^2 + 2.349s + 0.35}.$$

This robust controller achieves a normalized coprime factor robust stability margin of $b(P, C) = 0.5053$ [8]. Using the definitions of this paper, robust stability margins for other uncertainty structures can also be computed. In this example, we focus on multiplicative uncertainty, for which the robust stability margin is given in Equation (8), and is calculated as $b_{\text{im}}(P, C) = 1.0142$. We could similarly use additive or inverse multiplicative robust stability margins, but these uncertainty structures are left out for brevity.

Given this combination of nominal plant and controller, we will now study the impact of neglected actuator dynamics on the stability and performance of the feedback system. Let

$$U(s) = \frac{as + 10}{s + 10} U'(s),$$

where u' is the input to the actuator and $a \in \mathbb{R}$ is an uncertain parameter. This represents a low-pass filter with uncertain high frequency gain. As a consequence, we will consider the uncertain plant

$$P_{\Delta}(s, a) = P(s) \frac{as + 10}{s + 10}$$

and study the effect that varying values of a have on stability and performance. Figure 6 shows the Bode plots of several members of the uncertain plant family $P_{\Delta}(s, a)$ for values $a \in [-0.21, 1]$. The nominal plant $P(s)$ is recovered for $a = 1$.

For a set of values of $a \in [-0.21, 1]$, the distance between the nominal and the perturbed plant is calculated using both the multiplicative distance measured given in Equation (18), and the v -gap distance for normalized coprime factor uncertainty (see e.g. [8]). From Equation (18), and since $P(\infty)$ has full rank and $P^{-1}P_{\Delta} \in \mathcal{RL}_{\infty}$, we have

$$d_{\text{im}}(P, P_{\Delta}) = \|P^{-1}(P - P_{\Delta})\|_{\infty}.$$

Using left- and right-normalized coprime factorizations $\{\tilde{M}, \tilde{N}\}$ and $\{N_{\Delta}, M_{\Delta}\}$ of P and P_{Δ} , respectively, the v -gap is defined as (cf. [8]; the winding number and determinant conditions hold in this case)

$$\delta_v(P, P_{\Delta}) = \left\| \begin{bmatrix} -\tilde{M} & \tilde{N} \end{bmatrix} \begin{bmatrix} N_{\Delta} \\ M_{\Delta} \end{bmatrix} \right\|_{\infty}.$$

Both these distance measures are plotted in Figure 7 for the interval $a \in [-0.21, 1]$ together with the corresponding robust stability margins. Using Theorem 5, robust stability is guaranteed for all plants for which $d_{\text{im}}(P, P_{\Delta}) < b_{\text{im}}(P, C)$ and $\eta(P) = \eta(P_{\Delta})$. Hence, all $P_{\Delta}(a, s)$ with $a \in [-0.014, 1]$

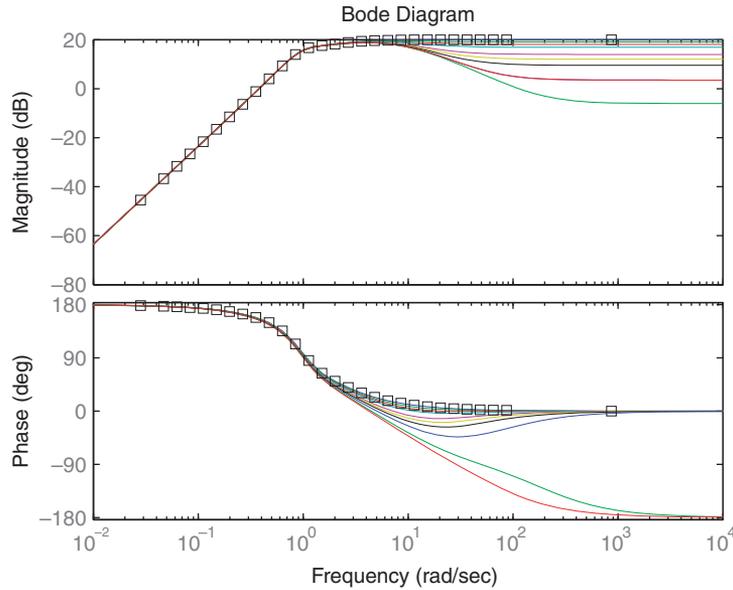


Figure 6. Members of the uncertain plant family for several values of $a \in [-0.21, 1]$. The nominal plant is denoted by the squares.

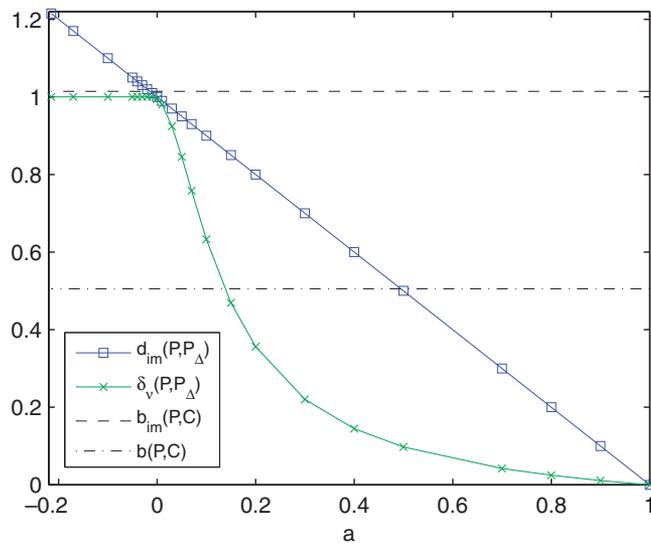


Figure 7. Distance measures for uncertain plants in the interval $a \in [-0.21, 1]$, together with the corresponding robust stability margins.

are guaranteed to be stable using the methods proposed in this paper for input multiplicative uncertainty. Similarly, using the v -gap theory, robust stability can be guaranteed for all plants for which $\delta_v(P, P_\Delta) < b(P, C)$ (assuming that certain winding number conditions hold, which are satisfied in this case; cf. [8]). Therefore, using normalized coprime factor uncertainty, robust stability is guaranteed for the set of uncertain plants with $a \in [0.138, 1]$.

Consequently, for this example, the multiplicative distance measure and robust stability margin of this paper enable less conservative robust stability guarantees for the plant with uncertain actuator dynamics. One of the strengths of the distance measure concept is that it also allows the design engineer to bound robust performance degradation. For illustration, we compute the lower bound on the remaining multiplicative robust stability margin, using Theorem 6 and—when all

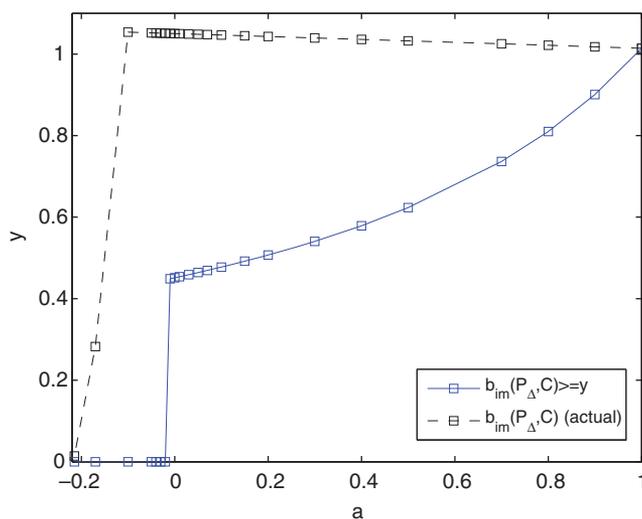


Figure 8. Lower bounds on the robust stability margins provided by Theorem 6 and [8, Theorem 3.8] together with the actual robust stability margins. Note that the performance theorems only apply to those plants where the distance is less than the corresponding robust stability margin.

assumptions hold—reformulate inequality (19) as follows:

$$b_{\text{im}}(P_{\Delta}, C) \geq \frac{b_{\text{im}}(P, C)}{1 + \|(I - CP_{\Delta})^{-1}\|_{\infty} d_{\text{im}}(P, P_{\Delta})}$$

The result is plotted in Figure 8, together with the actual values of the multiplicative robust stability margin as computed for each value of a . It can be seen that the guarantee becomes conservative as the distance to the nominal plant grows.

As a final remark, it must be stressed that the insight that multiplicative uncertainty enables less conservative robust stability and performance guarantees should lead the designer to reconsider the choice of the controller, which was chosen to maximize the normalized coprime factor robust stability margin in this example. The input multiplicative robust stability margin would be an obvious choice for an objective function in this specific case, but synthesis methods are beyond the scope of this paper.

7. CONCLUSIONS

This paper has derived specific robust stability tests as well as bounds on robust performance for several uncertainty structures (additive, input/output multiplicative, output/input inverse multiplicative) commonly used by the practising community. The robust stability theorems of this paper go far beyond previously known results by extending the space of allowable perturbations from \mathcal{RH}_{∞} to \mathcal{RL}_{∞} . Furthermore, the degradation of performance of a given feedback loop upon replacing the nominal plant by the perturbed plant is quantified in various robust performance theorems relying only on the computation of two infinity norms for each uncertain plant being considered. A physically motivated example shows that a suitably chosen uncertainty characterization is essential in ensuring least conservative stability and performance guarantees for a given perturbation class. Four-block uncertainty—though able to capture a large set of possible uncertainties—cannot always guarantee less conservative robust stability results for specific classes of perturbed plants for which other less general uncertainty structures are better suited (as shown in the example).

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