

# A combined iterative scheme for identification and control redesigns

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## SUMMARY

This work proposes a unified algorithm for identification and control. Frequency domain data of the plant is weighted to satisfy the given performance specifications. A model is then identified from this weighted frequency domain data and a controller is synthesised using the  $\mathcal{H}_\infty$  loopshaping design procedure. The cost function used in the identification stage essentially minimizes a tight upper bound on the difference between the achieved and the designed performance in the sense of the  $\mathcal{H}_\infty$  loopshaping design paradigm. Given a model, a method is also suggested to re-adjust model and weighting transfer functions to reduce further the worst case chordal distance between the weighted true plant and the model. Copyright © 2004 John Wiley & Sons, Ltd.

KEY WORDS: joint identification and control;  $v$ -gap metric; identification for control;  $\mathcal{H}_\infty$  loopshaping

## 1. INTRODUCTION

A number of iterative schemes have been proposed over the last few years for model-based iterative controller redesign. A common feature of these schemes is that iterations are performed of plant model updates (by identification with the most recent controller applied to the actual plant) and of model-based controller updates (the controller design being based on the most recent model). Representative examples of these schemes can be found in [1–4]. In general, there is no guarantee in these schemes that the succession of designed controllers stabilizes the actual plant. Some of these schemes are reviewed below.

Experience has shown that stability robustness is enhanced by applying cautious steps of plant modification and controller modification. Reference [5] uses the  $v$ -gap metric [6] and its

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related robust stability results as a tool to justify the need for cautious iterations (i.e. small controller modifications and, possibly, small plant model adjustments) in iterative identification and control design. In that paper, it is shown how to use these gap metric results to compute controller updates that are smaller than what would result from an optimal design based on the nominal design, and that stability of the actual closed-loop system is guaranteed. These cautious (or safe) adjustments were recently used in [7] in a reference model-based scheme.

Iterative identification and feedback design has been around in the form of dual Youla parameterization for a long time. The role of the dual Youla parameterization in identification for control is highlighted and exploited by several researchers, see e.g. [8–10] and references therein. The well-known Hansen scheme was modified in [8] in order to estimate the Youla parameters separately. This modification was motivated by the ultimate objective of using the identified estimates directly for control design. On the other hand, Królikowski [11] proposed an optimal combination of sequential identification and control for a linear bounded-input bounded-noise discrete-time particular plant. Various configurations of identifying/controlling sequences were investigated in order to find an optimal tradeoff. The optimal identifying input sequence was determined by maximization of a given identification accuracy measure while the control input sequence was derived to assure a good tracking of the control system.

An integrated identification and controller design approach, based on loopshaping principles, was presented in [12]. Starting with time-domain input–output data, a coprime factorization of the plant was identified using linear least-squares techniques. The residual error from the identification was then used to obtain estimates of the coprime factor uncertainty that were translated to sensitivity and complementary sensitivity bounds for the closed-loop system. These bounds were then used as a guide for the selection of a target loop and the controller was designed using  $\mathcal{H}_\infty$ -tools. An application on a paper machine simulator was also presented, illustrating the main ideas of the approach.

The self-tuning frequency-domain loop shaping control problem for stable minimum phase systems was considered in [13]. The resulting control scheme corresponded to an explicit self-tuning controller, where identification and control were implemented in the frequency domain, thereby taking advantage of the properties afforded by frequency domain identification schemes. The input and output signals to the system were transformed to the frequency domain via discrete Fourier transforms; subsequently the identification part recursively estimated the system's transfer function as a finite impulse response filter. The controller corresponded to a circular convolver, implemented in the frequency domain, and shapes the open-loop frequency response by providing the necessary supplement of magnitude and phase in order to match an a priori closed-loop frequency response.

Iterative solutions to the combined problem of identification and control design were also proposed in [14,15]. Different techniques were used for identification and control. Each of these papers demonstrated the applicability of their results through a case example and showed that the compensator corresponding to the eventual model controls the plant better than ever would be possible by the best design on an approximate open-loop model. Reference [16] presented an identification-based mechanism for introducing guaranteed stability when using a data-driven model-free iterative control design method known as iterative feedback tuning. Also, the use of unbiased estimates of the Hessian is shown to significantly improve the user control over the tuning procedure. A method for iterative feedback tuning for a robustness under frequency domain additive or coprime-factor uncertainty is described in [17].

Another approach to iterative design and control is through unfalsification of model. In [18], an asymptotically convergent iterative scheme based on model unfalsification is presented. The method presented in this paper yields a near-optimal worst-case servo performance under norm bounded parametric uncertainty. The technique presented in [18] was further developed to include a variety of robust performance objectives in [19]. An interesting variation of the unfalsified model based design method is presented in [20]. In this paper, the closed-loop data is used to identify an unfalsified set of controllers which *may* have produced that data and which can achieve a given level of performance. A new controller can then be chosen from this set of unfalsified controllers to improve upon the measured performance for the present controller.

A detailed survey paper on the progress of control-oriented identification throughout the 1990s with a number of applications in the chemical industry is given in [21].

In the work presented here, the following approach is taken: Given an unknown true plant  $P_0$ , the designer wishes to maximize some performance criterion as expressed by a function of plant and controller, say,  $J(P_0, C)$ . Since  $P_0$  is unknown, a model  $\hat{P}$  has to be found on the basis of which a controller can be designed. Since

$$J(P_0, C) \geq J(P, C) - |J(P, C) - J(P_0, C)|$$

for any controller  $C$  and model  $P$ , a promising approach to joint identification and control design will be to solve

$$\max_{P, C} \{J(P, C) - |J(P, C) - J(P_0, C)|\} \quad (1)$$

assuming that this term can be somehow captured in a cost function. An iterative approach is presented here to solve (1) which involves minimizing a tight bound on the second term and maximizing the first term. The identification stage in successive iterations does not involve new experiments. The performance criterion used in this paper is the  $\mathcal{H}_\infty$  loopshaping cost and the corresponding identification criterion is the  $\nu$ -gap metric (as approximated on a finite data). As will be seen in Section 2,  $\nu$ -gap does capture the difference in closed-loop performance from  $\mathcal{H}_\infty$  loopshaping point of view. The algorithm presented here is numerically tractable and may easily be implemented in MATLAB or similar commercial software. It can be used to design model-based controllers for multivariable and unstable plants, even for those with a poles on the imaginary axis. This will be demonstrated through a simulation example later.

Parts of this work have been presented in [22]. The identification algorithm used in this paper was first proposed in [23]. This algorithm minimizes an upper bound on the (pointwise) mismatch between the designed and achieved performance for *any*<sup>‡</sup> controller within a set of controllers. The weight optimization stage uses a convex optimization-based procedure, first proposed in [24], to synthesize weights and a controller that maximize the  $\mathcal{H}_\infty$  loopshaping performance criterion. Note that in  $\mathcal{H}_\infty$  loopshaping, the synthesized weights end up with a part of the implementable controller. Further, a new method of model and weight re-adjustment to minimize the relevant identification cost is presented here.

The rest of the paper is organized as follows. Section 1.1 introduces the notation used. Section 2 introduces the  $\mathcal{H}_\infty$  loopshaping design procedure and its relation with the  $\nu$ -gap metric. Section 3 outlines the new algorithm and Section 4 demonstrates it with a simulation example.

<sup>‡</sup> Although the controller does not explicitly appear in the identification problem, its general behaviour is however still captured through the associated weights.

### 1.1. Notation

$\mathbb{C}$  and  $\mathbb{R}$  represent real and complex numbers respectively.  $\mathbb{C}_+$  denote the open right half plane,  $\mathbb{C}_+ := \{s \in \mathbb{C} : \text{re}(s) > 0\}$ .  $\mathcal{R}^{m \times n}$  denotes the space of all real rational transfer functions with  $n$  inputs and  $m$  outputs. The superscript  $m \times n$  is dropped whenever the dimension of transfer matrix is irrelevant.  $\mathcal{L}_\infty$  denotes the normed space of all functions essentially bounded on  $j\mathbb{R}$  and having norm  $\|f\|_{\mathcal{L}_\infty} := \text{ess sup}_\omega \bar{\sigma}(f(j\omega))$ .  $\mathcal{H}_\infty$  denotes the subspace of functions in  $\mathcal{L}_\infty$  that are analytic and bounded in  $\mathbb{C}_+$ .  $\mathcal{RL}_\infty$  ( $\mathcal{RH}_\infty$ ) represents the subspace of real rational transfer functions in  $\mathcal{L}_\infty$  ( $\mathcal{H}_\infty$ ). For  $P \in \mathcal{R}$ ,  $P^\sim(s) := P^T(-s)$ .

$P = ND^{-1}$  ( $= \tilde{D}^{-1}\tilde{N}$ ) is called a normalized right (left) coprime factorization of plant  $P \in \mathcal{R}$  if  $N, D$  are right coprime ( $\tilde{D}, \tilde{N}$  are left coprime) and  $N^*N + D^*D = I$  ( $\tilde{N}\tilde{N}^* + \tilde{D}\tilde{D}^* = I$ ).  $G_i := [N_i^T D_i^T]^T$  and  $\tilde{G}_i := [-\tilde{D}_i \tilde{N}_i]$  are, respectively, called the normalized right and normalized left gap symbols of plant  $P_i$ .

## 2. $\mathcal{H}_\infty$ LOOPSHAPING AND THE $v$ -GAP METRIC

$\mathcal{H}_\infty$  loopshaping combines the traditional loopshaping wisdom with robust stabilization of normalized coprime factors. Given a model  $P$  of a plant, a pre-compensator  $W_1$  and a post-compensator  $W_2$  are selected so that  $P_s := W_2 P W_1$  has the desired loopshape. The loopshape is determined from the closed-loop performance specifications. Typically, this means choosing weights  $W_1$  and  $W_2$  such that  $\underline{\sigma}(P_s(j\omega)) \gg 1$  over some low frequency range,  $\bar{\sigma}(P_s) \ll 1$  over some high frequency range and  $\sigma_i(P_s)$  have a moderate roll off rate around crossover frequency. Unlike classical loopshaping, the designer need not shape phase explicitly. Following [24], the loopshaping constraints considered here are

$$\alpha(\omega) < \sigma_i(P_s(j\omega)) < \beta(\omega) \quad \forall \omega \in \Omega \text{ and } \forall i \quad (2)$$

where  $\alpha$  and  $\beta$  are non-negative, real scalar functions and  $\Omega = \{\omega_1, \omega_2, \dots, \omega_m\}$  specifies a set (or a grid) of frequencies of interest.  $\alpha$  and  $\beta$  may be chosen by translating time domain performance specifications (e.g. relating to servo performance) into frequency domain specifications (in terms of low frequency gain, high frequency roll-off and 0 dB bandwidth). The loopshaping weights  $W_1$  and  $W_2$  are often selected to be diagonal, stable and inversely stable transfer matrices. It is not always easy to see how the weights affect the singular values of  $P_s$  and hence often the designer has to rely on a process of trial and error to arrive at satisfactory weighting transfer functions. Following an algorithm proposed in [24], a method is suggested in Section 3.2 for selection of weights which alleviates these difficulties.

In  $\mathcal{H}_\infty$  loopshaping, the performance measure chosen for synthesizing a controller  $C$  stabilizing  $P_s$  is

$$b(P_s, C) = \|H(P_s, C)\|_\infty^{-1} \quad (3)$$

where the closed-loop transfer function  $H(P_s, C)$  is defined by

$$H(P_s, C) = \begin{bmatrix} P_s \\ I \end{bmatrix} (I - CP_s)^{-1} [-C \quad I] \quad (4)$$

$b(P_s, C)$  represents the robustness of the closed-loop against bounded perturbations of normalized coprime factors of the shaped plant  $P_s$  and is also called the generalized robust

stability margin. The best achievable generalized robust stability margin is defined by

$$b_{\text{opt}}(P_s) := \max_{C \text{ stabilising}} b(P_s, C)$$

and can be explicitly computed [25]. The same reference also provides a characterization of all controllers  $C$  achieving  $b(P_s, C) < b_{\text{opt}}(P_s)$ .  $b_{\text{opt}}(P_s)$  is an indicator of success of the loopshaping design stage; a large (resp. small)  $b_{\text{opt}}(P_s)$  indicates compatibility (resp. incompatibility) between the designed loopshape and robust stability. As a rule of thumb,  $b(P_s, C) > 0.3$  would be considered adequate in most cases. The final controller is given by re-aligning weights with the controller;  $C_s = W_1 C W_2$ . Besides robust stability, another motivation for the use of  $b(P_s, C)$  as a performance measure comes from the fact that the size of each of the four closed-loop transfer matrices in  $H(P, C_s)$  can be bounded from above at each frequency  $\omega$  in terms of  $b(P_s, C)$ ,  $\bar{\sigma}(P_s)(j\omega)$ ,  $\underline{\sigma}(P_s)(j\omega)$  and the condition numbers of  $W_1(j\omega)$ ,  $W_2(j\omega)$ . More details on  $\mathcal{H}_\infty$  loopshaping may be found in [26]. A generalization of  $\mathcal{H}_\infty$  loopshaping procedure through introduction of scalar parameter is discussed in [27].  $\mathcal{H}_\infty$  loopshaping has been used successfully in a variety of applications; see [28] and references therein.

A metric called  $v$ -gap metric was suggested in [6] as a natural dual to  $b(P_s, C)$ . The  $v$ -gap between two plants  $P_1, P_2 \in \mathcal{R}$  can be defined as

$$\begin{aligned} \delta_v(P_1, P_2) &= \inf_{Q, Q^{-1} \in \mathcal{L}_\infty} \|G_1 - G_2 Q\|_\infty \quad \text{if } I(P_1, P_2) = 0 \\ &= 1 \quad \text{otherwise} \end{aligned} \quad (5)$$

where  $I(P_1, P_2) := \text{wno det}(G_2^* G_1) = \text{wno det}(\tilde{G}_1 \tilde{G}_2^*)$  and  $\text{wno}(g)$  denotes the winding number of  $g(s)$  evaluated on the standard Nyquist contour indented to the right around any poles on  $j\mathbb{R}$ . For a real rational transfer matrix  $X$  such that  $X, X^{-1} \in \mathcal{R}\mathcal{L}_\infty$ , winding number  $\text{wno det}(X)$  is the excess of number of zeros of  $X$  in  $\mathbb{C}_+$  over the number of poles of  $X$  in  $\mathbb{C}_+$ . When  $I(P_1, P_2) = 0$ ,  $\delta_v(P_1, P_2)$  equals  $\mathcal{L}_2$ -gap, defined by

$$\delta_{\mathcal{L}_2}(P_1, P_2) := \|\tilde{G}_2 G_1\|_\infty = \sup_{\omega} \kappa(P_1, P_2) \quad (6)$$

where  $\kappa(P_1, P_2)(j\omega)$  is the pointwise chordal distance,

$$\kappa(P_1, P_2)(j\omega) := \bar{\sigma}((I + P_2 P_2^*)^{-\frac{1}{2}}(P_2 - P_1)(I + P_1^* P_1)^{-\frac{1}{2}})(j\omega)$$

It is known that [6] any controller stabilizing  $P_1$  and achieving  $b(P_1, C) > \alpha$  stabilizes the plant set

$$\{P_2 : \delta_v(P_1, P_2) \leq \alpha\}$$

More importantly,  $\delta_v(P_1, P_2)$  is a measure of the ‘closeness’ of the closed-loop performance of  $P_1$  and  $P_2$  for any stabilizing controller. The following result can be easily derived from the proof of Theorem 3.8 in [29]:

#### Lemma 1

Suppose a controller  $C$  stabilizes a given pair of plants  $P_1, P_2$ . Then

$$\frac{1}{\bar{\sigma}(H(P_1, C))(j\omega)} \geq \frac{1}{\bar{\sigma}(H(P_2, C))(j\omega)} - \kappa(P_1, P_2)(j\omega) \quad (7)$$

From (3)–(7), it follows that any controller  $C$  that stabilizes  $P_2$  with a good  $b(P_2, C)$  also stabilizes  $P_1$ , *without any significant deterioration in performance* (in terms of  $b(P_1, C)$ ), provided  $\delta_v(P_1, P_2)$  is small.

To pose a control-oriented identification problem with finite data, some relevant quantities need to be defined. As *a posteriori* information in the identification process, suppose that a block matrix of (not necessarily uniformly spaced) frequency response samples of the true plant  $P_0(s) \in \mathcal{R}^{p \times n}$  at measurement frequencies  $\omega_i$ ,  $i = 1, 2, \dots, m$  is given:

$$P_\Omega := [P_0(j\omega_1) \quad P_0(j\omega_2) \quad \dots \quad P_0(j\omega_m)] \quad (8)$$

It is implicitly assumed here that some qualitative *a priori* information is available about the plant to decide a sensible grid of frequencies. Care has to be taken not to miss significant dynamics e.g. pairs of resonant poles not too far from each other; flexible structures often display such dynamics.

Next, define

$$\begin{aligned} \delta_\Omega(P_1, P_2) &:= \max_{i \in [1, m]} \kappa(P_1, P_2)(j\omega_i) \quad \text{if } I(P_1, P_2) = 0 \\ &= 1 \quad \text{otherwise.} \end{aligned} \quad (9)$$

Finally, for any model  $P_1$  and a controller  $C$  stabilizing both the plant  $P_0$  and model  $P_1$ , define a performance measure over finite frequency set,

$$b_\Omega(P_k, C) := \left\{ \max_{i \in [1, m]} \bar{\sigma}(H(P_k, C))(j\omega_i) \right\}^{-1}, \quad k = 0, 1 \quad (10)$$

Then from (7), it is easy to show that

$$b_\Omega(P_0, C) \geq b_\Omega(P_1, C) - \delta_\Omega(P_0, P_1) \quad (11)$$

holds, where  $\delta_\Omega(P_0, P_1)$  is as defined in (9).

### 3. A UNIFIED ALGORITHM FOR IDENTIFICATION AND $\mathcal{H}_\infty$ LOOPSHAPING CONTROL

#### 3.1. Outline of the algorithm

From (11), a sensible—although intractable—joint identification and control problem from  $\mathcal{H}_\infty$  loopshaping point of view would be

$$\max_{C, W_1, W_2, \hat{P}} b_\Omega(\hat{P}, C) - \delta_\Omega(W_2 P_0 W_1, \hat{P}) \quad (12)$$

where the weighting transfer function matrices  $W_1, W_2$  and the model  $\hat{P}$  are constrained to appropriate sets,  $C$  belongs to the set of controllers stabilizing  $\hat{P}$  and  $W_2 P_0 W_1(j\omega)$  satisfies the loopshaping constraints (2). Comparing with (1), note that the second term in (12) is independent of controller (although the controller  $C$  does not explicitly appear in (12), the general behaviour of the implementable controller  $C_s = W_1 C W_2$  is however still

captured through the weights.). Expression (12) gives a lower bound on pointwise version of generalized robust stability margin for the true plant;  $b_{\Omega}(W_2 P_0 W_1, C)$ . This lower bound is tight in a certain sense; see [6] for details. Note that the choice of  $v$ -gap-like cost for identification is a logical result of  $\mathcal{H}_{\infty}$  loopshaping cost as the choice of control design criterion. Outside the loopshaping design context, the identification algorithm presented still makes sense as a numerical tractable method to develop black-box models of unstable multivariable plants from frequency response data. However, in the light of (1), one expects that different identification schemes would be appropriate for different control design objectives.

Here, an algorithm which minimizes a cost similar to (12) is outlined. Details of its numerical implementation are discussed in the subsequent sections.

**Given:**  $P_{\Omega}$  as in (8) and the loopshaping specifications (2).

1. **Initial Weight Synthesis:** Find some stable, minimum phase diagonal weighting transfer matrices  $W_{1,0}, W_{2,0}$  such that the ‘shaped’ frequency response samples  $W_{2,0} P_0 W_{1,0}(j\omega_i)$  satisfy the loopshaping specifications (2) for all  $\omega_i, i \in [1, m]$ .
2. **Identification:** Solve

$$\min_{P \in \mathcal{S}} \delta_{\Omega}(W_{2,0} P_0 W_{1,0}, P) \quad (13)$$

where  $\mathcal{S}$  is an appropriate model set. Let  $\lambda_1$  be the achieved minimum cost and let  $\hat{P}_1 \in \mathcal{S}$  be any model which achieves it.

3. **Weight and Model Re-adjustment:** Given  $\hat{P}_1$ , solve

$$\min_{\substack{W_1 \in \mathcal{W}_1 \\ W_2 \in \mathcal{W}_2}} \max_i \kappa(W_2 P_0 W_1, -(W_1^{-1} \hat{P}_1 \sim W_2^{-1}) \sim)(j\omega_i) \quad (14)$$

subject to (2) being satisfied. Here  $\bar{P} = -(W_{1,0} \hat{P}_1 \sim W_{2,0}) \sim$  and  $\mathcal{W}_1, \mathcal{W}_2$  are appropriate model sets such that  $W_{1,0} \in \mathcal{W}_1, W_{2,0} \in \mathcal{W}_2$ . Let  $W_{2,1}, W_{1,1}$  be the weights obtained on solving (14) and let  $\lambda_2$  be the achieved minimum cost.

4. **Robust Stabilization:** Find the controller  $C$  which achieves  $b(\hat{P}_2, C) = b_{\text{opt}}(\hat{P}_2)$ . Here,  $\hat{P}_2 = -(W_{1,1}^{-1} \hat{P}_1 \sim W_{2,1}^{-1}) \sim$ . The final controller is given by  $C_s = W_{1,1} C W_{2,1}$ .
5. If  $\lambda_2 \ll b(\hat{P}_2, C)$ , then the plant model and the controller are considered adequate. Otherwise, go to Step 2 for further iterations.

Note that Steps 3 and 4 together form the controller design stage, as the weights form an integral part of the implementable controller  $C_s$ . The rationale behind (14) will be clear later from Lemma 2 and discussion in Section 3.4. Compared with (12), it may be seen that the step 4 above maximizes the first term (over the set of stabilizing controllers for the model) and steps 1–3 approximately minimize the second term (over the set of models and the sets of weighting transfer functions).

Assuming that global minimum exists (and is found) for each of the optimization problems (13)–(14), a nice property of the above algorithm is non-increasing cost:

#### Lemma 2

Let  $\lambda_1$  and  $\lambda_2$  be the achieved cost in the optimizations (13) and (14), respectively. Then

$$\lambda_2 \leq \lambda_1$$

*Proof*

Since,  $W_{1,0} \in \mathcal{W}_1, W_{2,0} \in \mathcal{W}_2$ ,

$$\begin{aligned} \lambda_2 &= \min_{\substack{W_1 \in \mathcal{W}_1 \\ W_2 \in \mathcal{W}_2}} \max_i \kappa(W_2 P_0 W_1, -(W_1^{-1} \bar{P} \sim W_2^{-1}) \sim)(j\omega_i) \\ &\leq \max_i \kappa(W_{2,0} P_0 W_{1,0}, -(W_{1,0}^{-1} \bar{P} \sim W_{2,0}^{-1}) \sim)(j\omega_i) \\ &= \max_i \kappa(W_{2,0} P_0 W_{1,0}, \hat{P}_1)(j\omega_i) = \lambda_1 \end{aligned}$$

where the last step uses  $\bar{P} = -(W_{1,0} \hat{P}_1 \sim W_{2,0}) \sim$ .  $\square$

If  $\lambda_1$  is deemed sufficiently small, step 3 is not required. On the other hand, if  $\lambda_2$  is deemed to be too large at the end of step 3, it is possible to iterate through steps 2 and 3 till the cost becomes sufficiently small. Similarly, if  $b_{\text{opt}}(\hat{P}_2)$  is too small for the final model  $\hat{P}_2$ , it may be necessary to relax the loopshaping specifications and return to step 1 again. Simulation experience indicates that more than two iterations are rarely required.

Note that the controller design and model identification stages in the above procedure are interleaved, since the weights form part of the final controller  $C_s$ . Step 4 of this procedure is standard and is described in many robust control textbooks, e.g. [30]. The choice of model sets and numerical implementation in the first three steps is described in the subsequent sections.

### 3.2. Initial weight synthesis

Given a true plant  $P_0$ , a controller  $C_{\text{init}}$  stabilizing  $P_0$  and *any* stable minimum phase transfer functions  $W_1$  and  $W_2$ , recall that

$$H(P_0, C_{\text{init}}) \in \mathcal{H}_\infty \Leftrightarrow H(W_2 P_0 W_1, W_1^{-1} C_{\text{init}} W_2^{-1}) \in \mathcal{H}_\infty$$

In the weight optimization procedure of [24], the aim is to find stable, minimum phase weights  $W_1$  and  $W_2$  such that

1.  $b_\Omega(W_2 P_0 W_1, W_1^{-1} C_{\text{init}} W_2^{-1})$  is maximized and
2. the ‘shaped’ plant  $P_s(s) := (W_2 P_0 W_1)(s)$  satisfies the loopshaping specifications (2).

An outline of the procedure for weight synthesis from Reference [24] is given here.

1. Let  $\Gamma_q$  denote the set of real diagonal  $q \times q$  matrices. For ease of notation, let  $P_{\omega_i} = P_0(j\omega_i) \in \mathbb{C}^{p \times q}$  and let  $C_{\omega_i} = C_{\text{init}}(j\omega_i)$ . Without loss of generality, it is assumed that  $p \geq q$ . The case when  $p < q$  can be handled using a dual problem; see [24]. Given  $P_\Omega$  as in (8) and the loopshaping constraints (2), solve the following quasi-convex optimization problem at each frequency  $\omega_i$ ,  $i = 1, 2, \dots, m$ :

$$\inf_{X_i \in \Gamma_q, Y_i \in \Gamma_p} \gamma_i \tag{15}$$



subject to

$$\begin{bmatrix} 0 & P_{\omega_i} \\ 0 & I \end{bmatrix}^* \begin{bmatrix} X_i & 0 \\ 0 & Y_i \end{bmatrix} \begin{bmatrix} 0 & P_{\omega_i} \\ 0 & I \end{bmatrix} < \gamma_i \begin{bmatrix} I & P_{\omega_i} \\ C_{\omega_i} & I \end{bmatrix}^* \begin{bmatrix} X_i & 0 \\ 0 & Y_i \end{bmatrix} \begin{bmatrix} I & P_{\omega_i} \\ C_{\omega_i} & I \end{bmatrix} \quad (16)$$

$$\alpha^2(\omega_i) Y_i < P_{\omega_i}^* X_i P_{\omega_i} < \beta^2(\omega_i) Y_i \quad (17)$$

$$X_i > 0, Y_i > 0 \quad (18)$$

Let  $\hat{X}_i, \hat{Y}_i, i = 1, 2, \dots, m$  be the solutions of the pointwise optimization problems (15)–(18) and let  $\hat{\gamma}_i$  be the optimum cost at each  $i$ . Let  $\hat{\gamma} = \max_i \hat{\gamma}_i$ .

2. Construct diagonal transfer function matrices  $W_{1,0}(s), W_{2,0}(s)$  that are units in  $\mathcal{RH}_\infty$  by fitting minimum phase stable transfer function to each magnitude function on the diagonal of  $\hat{Y}_i^{-1/2}$  and  $\hat{X}_i^{-1/2}$ , respectively.

If  $W_{1,0}$  and  $W_{2,0}$  interpolate  $\hat{Y}_i^{-1/2}$  and  $\hat{X}_i^{-1/2}$  exactly, it can be easily shown that (17) is equivalent to

$$\alpha(\omega_i) < \sigma_k((W_{2,0} P_0 W_{1,0})(j\omega_i)) < \beta(\omega_i) \quad \forall i \in [1, m] \quad \forall k \in [1, q]$$

and (16) is equivalent to

$$\bar{\sigma}(H(W_{2,0} P_0 W_{1,0}, W_{1,0}^{-1} C_{\text{init}} W_{2,0}^{-1}))(j\omega_i) < \sqrt{\hat{\gamma}_i} \quad \forall i \in [1, m] \quad (19)$$

Proof of this fact may be found in [24]. In practice, an approximate, low order fit to  $\hat{Y}_i^{-1/2}$  and  $\hat{X}_i^{-1/2}$  should still ensure that  $P_s = W_{2,0} P_0 W_{1,0}$  adheres to the loopshaping specifications. While approximating  $\hat{Y}_i^{-1/2}$  and  $\hat{X}_i^{-1/2}$ , it must be kept in mind that  $P_s(j\omega)$  should have a moderate roll-off rate around crossover frequency.

It is possible to include additional constraints on optimization (15) for better numeric conditioning of  $X_i, Y_i$ ; see [24] for details.

### 3.3. Identification in the $v$ -gap metric

A method for identification in  $v$ -gap metric was presented in [23]. An outline of the same is given here. SISO case is discussed here for simplicity; extension to MIMO case is straightforward. Let  $S_n$  denote the set of finite impulse response (FIR) models of degree less than  $n$ . Define a candidate model set for approximation of coprime factors

$$S_{1,2} = \{f : f = [f_1 f_2]^T, f_1 \in S_{n_1}, f_2 \in S_{n_2}\}$$

Lastly, let  $\mathcal{R}_n$  denote the set of real rational transfer functions of order less than  $n$ .

1. Let  $\omega_{\max} = \max_i \omega_i$ . Take  $\omega_s = 2\pi/T_s > 2\omega_{\max}$ <sup>§</sup> and let

$$e^{j\theta_i} = \frac{1 + j\omega_i T_s/2}{1 - j\omega_i T_s/2}$$

<sup>§</sup>  $\omega_s$  should not be too large as compared to  $\omega_{\max}$  as this may place the poles and zeros of model too close to each other and may cause numerical difficulties in optimization.

Let  $P_{s,\omega_i} = W_{2,0}(j\omega_i)P_{\omega_i}W_{1,0}(j\omega_i)$  and define  $F_i = \left[ \frac{P_{s,\omega_i}(1+P_{s,\omega_i}^*P_{s,\omega_i})^{-1/2}}{(1+P_{s,\omega_i}^*P_{s,\omega_i})^{-1/2}} \right]$ . Here,  $W_{1,0}(s)$  and  $W_{2,0}(s)$  are stable, minimum phase weights obtained using the algorithm described in Section 3.2.

2. Solve  $\mathcal{L}_2$ -gap approximation problem:

$$\begin{aligned} & \min_{P \in \mathcal{B}_n} \max_i \kappa(W_{2,0}P_0W_{1,0}, P)(e^{j\theta_i}) \\ & = \min_{f \in S_{1,2}} \max_i \inf_{Q_i} \bar{\sigma}(F_i - f(e^{j\theta_i})Q_i) \end{aligned} \quad (20)$$

Note that, for a fixed  $\hat{Q}_i \in \mathbb{C}$ ,  $i = 1, \dots, m$ ,

$$\min_{f \in S_{1,2}} \max_i \bar{\sigma}(F_i - f(e^{j\theta_i})\hat{Q}_i) \quad (21)$$

is an LMI optimization in parameters of  $f$ . On the other hand, for a fixed  $\hat{f} \in S_{1,2}$ , at each  $\theta_i$ ,

$$\inf_{Q_i} \bar{\sigma}(F_i - \hat{f}(e^{j\theta_i})Q_i) \quad (22)$$

is a linear least squares problem in  $Q_i \in \mathbb{C}$  and has a (pointwise) closed-form solution. Using these facts, (20) may be solved iteratively in  $f$  and  $Q_i$  and the cost is non-increasing through iterations; see [23] for details.

3. Let  $\hat{f} = [f_1 \ f_2]^T$  be the result of  $\mathcal{L}_2$ -gap approximation. Then the discrete time model is given by  $\hat{P}_d = f_1 f_2^{-1}$ . Note that  $Q$  does not appear in  $\hat{P}_d$ . Hence it is not parameterized and is evaluated only pointwise in (22). The continuous time model is obtained by bilinear transformation:

$$\hat{P}_x = \hat{P}_d \left( \frac{1 + sT_s/2}{1 - sT_s/2} \right)$$

4. The procedure for approximation does not guarantee that the true ‘shaped’ plant and the model will satisfy the winding number condition, i.e. a controller stabilizing the shaped plant with an adequate stability margin may still fail to stabilize  $\hat{P}_x$ . A model  $\hat{P}_1$  such that  $I(W_{2,0}P_0W_{1,0}, \hat{P}_1) = 0$  can be obtained from  $\hat{P}_x$  and any controller  $C_x$  stabilizing the shaped plant by a procedure described in [23]. Note that this procedure is not specific to the identification algorithm described so far; it may be used even if a model  $\hat{P}_x$  obtained by *any other* identification method is de-stabilized by a controller that stabilizes the true plant. If the true plant and  $\hat{P}_x$  are both stable or are both stabilized by the same controller, this procedure is not required and  $\hat{P}_1 = \hat{P}_x$ . See [23] for details of this procedure.

At the end of this identification procedure described above, a model  $\hat{P}_1$  is obtained which is a suboptimal solution to

$$\inf_{P \in \mathcal{B}} \delta_\Omega(W_{2,0}P_0W_{1,0}, P) \quad (23)$$

It is instructive to compare this with (13).

### 3.4. Weight and model re-adjustment

In Step 3 of our proposed algorithm (i.e. optimization problem (14)), the weights and the plant model are adjusted simultaneously to reduce cost  $\delta_{\Omega}(W_2 P_0 W_1, P)$  further, with the adjusted weights still satisfying loopshaping constraints (2). Let  $\hat{P}_1$  be the model and  $W_{1,0}, W_{2,0}$  be the weights obtained at the end of the 2nd step of the algorithm in Section 3.1. The solution to (14) rests on the following lemma, which tells us exactly how to use the weight optimization procedure discussed in the previous subsection to solve the chordal distance optimization given in (14).

#### Lemma 3

Given  $\bar{P} = -(W_{1,0} \hat{P}_1 \sim W_{2,0}) \sim$  and frequency response samples  $P_0(j\omega_i)$ , suppose  $W_{1,1} \in \mathcal{W}_1$  and  $W_{2,1} \in \mathcal{W}_2$  is any pair of diagonal transfer function matrices which is solution to

$$\min_{\substack{W_1 \in \mathcal{W}_1 \\ W_2 \in \mathcal{W}_2}} \max_i \kappa(W_2 P_0 W_1, -(W_1^{-1} \bar{P} \sim W_2^{-1}) \sim)(j\omega_i) \quad (24)$$

subject to (2) being satisfied. Then  $W_{1,1}$  and  $W_{2,1}$  also solve

$$\min_{\substack{W_1 \in \mathcal{W}_1 \\ W_2 \in \mathcal{W}_2}} \max_i \bar{\sigma}(H(W_2 P_0 W_1, W_1^{-1} \bar{P} \sim W_2^{-1}))(j\omega_i) \quad (25)$$

subject to (2) being satisfied. Here  $H(P, C)$  is as defined in (4).

#### Proof

The proof follows from the following relation from Reference [29]: Given  $P, C$  at any frequency  $\omega$ ,

$$\frac{1}{\bar{\sigma}(H(P, C))(j\omega)} = \sqrt{1 - (\kappa(P, -C \sim)(j\omega))^2} \quad \square$$

Let  $C_{\omega_i} = \bar{P} \sim(j\omega_i) = -(W_{1,0} \hat{P}_1 \sim W_{2,0})(j\omega_i)$ . Consider the following problem at each frequency  $\omega_i, i = 1, 2, \dots, m$ :

$$\inf_{X_i \in \Gamma_q, Y_i \in \Gamma_p} \gamma_i \quad (26)$$

subject to

$$\begin{aligned} & \begin{bmatrix} 0 & P_{\omega_i} \\ 0 & I \end{bmatrix}^* \begin{bmatrix} X_i & 0 \\ 0 & Y_i \end{bmatrix} \begin{bmatrix} 0 & P_{\omega_i} \\ 0 & I \end{bmatrix} \\ & < \gamma_i \begin{bmatrix} I & P_{\omega_i} \\ C_{\omega_i} & I \end{bmatrix}^* \begin{bmatrix} X_i & 0 \\ 0 & Y_i \end{bmatrix} \begin{bmatrix} I & P_{\omega_i} \\ C_{\omega_i} & I \end{bmatrix} \end{aligned} \quad (27)$$

$$\alpha^2(\omega_i) Y_i < P_{\omega_i}^* X_i P_{\omega_i} < \beta^2(\omega_i) Y_i \quad (28)$$

$$X_i > 0, Y_i > 0 \quad (29)$$

Similar to (19), (27) can be shown to be equivalent to the constraint

$$\bar{\sigma}(H(W_2 P_0 W_1, W_1^{-1} \tilde{P} W_2^{-1}))(j\omega_i) < \sqrt{\gamma_i} \forall i \in [1, m]$$

with  $X_i = W_2^* W_2(j\omega_i)$ ,  $Y_i = W_1^{-1} W_1^{-*}(j\omega_i)$ . Thus optimization (26) subject to constraints (27)–(29) is simultaneously affine in  $W_1^{-1} W_1^{-*}(j\omega_i)$  and  $W_2^* W_2(j\omega_i)$ . Hence (26) may be solved as an LMI optimization problem in  $X_i, Y_i$  at each  $\omega_i$  and then minimum phase stable diagonal weights could be fitted to  $X_i^{1/2}, Y_i^{-1/2}$ , as in Section 3.2. Alternatively, the matrix functions  $W_1^{-1} W_1^{-*}$  and  $W_2^* W_2$  may be affinely parameterized such that the parameterization includes  $W_{1,0}^{-1} W_{1,0}^{-*}$  and  $W_{2,0}^* W_{2,0}$ . This would ensure a non-increasing cost from (13) to (14), as stated in Lemma 2.

Let  $W_{1,1}$  and  $W_{2,1}$  be the weights obtained by this procedure. Then  $W_{1,1}$  and  $W_{2,1}$  also solve (24), as mentioned in Lemma 3. The final model is given by  $\hat{P}_2 = -(W_{1,1}^{-1} \tilde{P} W_{2,1}^{-1})^\sim$ . Note that, in this step, *both* the weights and the model are adjusted to reduce the worst case chordal distance between the weighted plant and the model, while the changed weights are such that the weighted plant still satisfies (2). The fact that the worst case chordal distance is further reduced (or at least not increased) in this step follows from Lemma 2.

If  $\delta_\Omega(W_{2,1} P_0 W_{1,1}, \hat{P}_2)$  is still deemed too large, the next logical step would be to minimize

$$\min_{P \in \mathcal{S}} \delta_\Omega(W_{2,1} P_0 W_{1,1}, P) \quad (30)$$

using the procedure outlined in Section 3.3. Provided  $\hat{P}_2 \in \mathcal{S}$ , solution of (30) will not increase worst case chordal distance. Further iterations of steps 2 and 3 of the procedure in Section 3.1 are possible, though simulation experience indicates that any iterations beyond (30) will be rarely required.

### 3.5. Effect of noise

So far in this discussion, noise-free frequency response samples of the true plant  $P_0$  are assumed to be available. In practice, it is far more likely that noisy frequency response samples will be available as a result of an identification experiment. In this case, the procedure outlined above may be carried out using the noisy samples. Suppose,  $P_{n,\omega_i}$  represents a noisy frequency response sample at frequency  $\omega_i$ . Let  $\hat{P}$  be the model and  $W_{1,1}, W_{2,1}$  be the weights obtained using the procedure detailed above (with  $P_0(\omega_i)$  replaced with  $P_{n,\omega_i}$ ) and let  $C$  be the designed controller. Then it is easy to show that

$$b_\Omega(W_{2,1} P_0 W_{1,1}, C) \geq b_\Omega(\hat{P}, C) - \kappa(W_{2,1}(j\omega_i) P_{n,\omega_i} W_{1,1}(j\omega_i), \hat{P}(j\omega_i)) - \varepsilon$$

where  $\varepsilon = \kappa((W_{2,1} P_0 W_{1,1})(j\omega_i), W_{2,1}(j\omega_i) P_{n,\omega_i} W_{1,1}(j\omega_i))$

The term  $\kappa(W_{2,1}(j\omega_i) P_{n,\omega_i} W_{1,1}(j\omega_i), \hat{P}(j\omega_i))$  may be minimized using the algorithm in Section 3.1. The size of  $\varepsilon$  needs to be controlled at identification experiment stage. If  $\varepsilon$  is small (or equivalently, the effect of noise around the crossover frequencies of the shaped plant is small), the possible deterioration of robustness margin due to noise is small.

## 4. SIMULATION EXAMPLE

Consider an unstable continuous time plant

$$P_0(s) = \begin{bmatrix} \frac{1}{s+2} & \frac{1}{s} \\ \frac{1}{s-1} & \frac{3}{s^2+s+1} \end{bmatrix}$$

Frequency response samples (matrices) of this plant at 50 frequencies, logarithmically spaced between 0.1 and 100 rad/s are used for estimation and weight selection. For choosing weights, the loopshaping specifications (2) were

$$\sigma_k(P_s(j\omega_i)) \geq 10 \quad \forall \omega_i \leq 0.3 \text{ rad/s}, \quad k = 1, 2$$

$$\sigma_k(P_s(j\omega_i)) \leq 0.1 \quad \forall \omega_i \geq 30 \text{ rad/s}, \quad k = 1, 2$$

Given an initial stabilizing controller, weights  $W_{1,1}$  of degree 4,  $W_{2,1}$  of degree 6 and a model  $\hat{P}_2$  of order 9 is obtained using the first 3 steps of the algorithm in Section 3.1. MATLAB's LMI toolbox is used for identification step and its *fitmag* command is used for fitting weighting transfer functions. This yields  $\max_i \kappa(W_{2,1}P_0W_{1,1}, \hat{P})(j\omega_i) = 0.0550$ . Then (30) is solved to obtain a model  $\hat{P}_3$  of order 9 which yields a further reduction in worst case chordal distance. The final  $\max_i \kappa(W_{2,1}P_0W_{1,1}, \hat{P}_3)(j\omega_i) = 0.0464$  and  $\delta_v(W_{2,1}P_0W_{1,1}, \hat{P}_3) = 0.0476$ . Also,

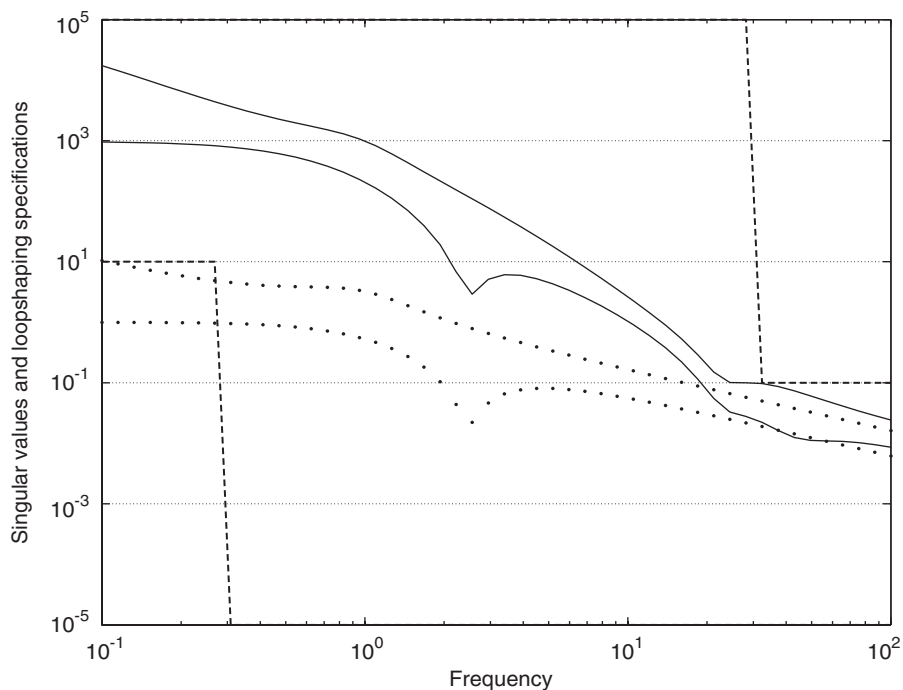


Figure 1. Singular value plots—true plant and loopshaping specifications.

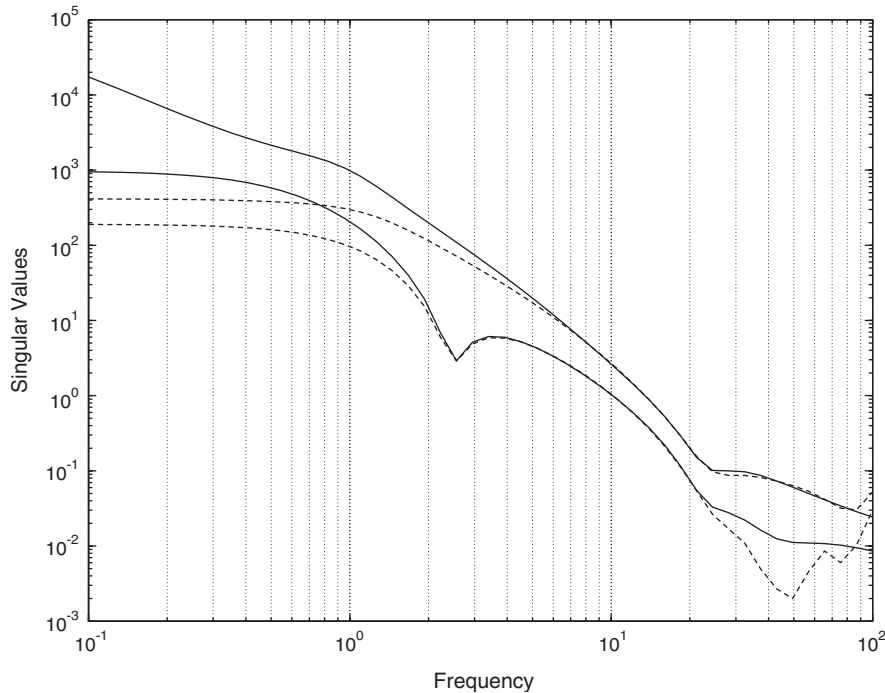


Figure 2. Singular value plots—weighted plant and model.

$b_{\text{opt}}(\hat{P}_3) = 0.292$  and the controller  $C_{\text{final}}$  which achieves  $b_{\text{opt}}(\hat{P}_3)$  yields  $b(W_{2,1}P_0W_{1,1}, C_{\text{final}}) = 0.248$ .

In Figure 1, the singular values of unshaped (original plant) are shown by dotted line, the loopshaping specifications are shown by dashed line while the achieved loopshape is shown by solid line. In Figure 2, the singular values of shaped plant are shown by solid line and the singular values of the model are shown by dashed line. It is seen that the minimization of worst case chordal distance in the identification step *automatically* yields a good fit around cross-over frequencies of singular values while (possibly) sacrificing accuracy of fit around very high and very low gains. This is a direct result of the definition of chordal distance stated in Section 3.3.

The  $v$ -gap metric identification step in the proposed procedure is somewhat time-consuming and converges slowly. However, this step is reasonably structured, automated and requires little manual input. Furthermore, since this identification is assumed to be offline, time required for computation should not be a huge problem. The important contribution of this algorithm is that it automates black-box modelling and control design for multivariable and possibly unstable plants, including those with poles on the imaginary axis. It is also worth mentioning here that a good worst case chordal distance fit is *not* necessarily reflected in a convergence of the parameter values in parameterized multivariable systems. Hence, simply checking whether the parameter values converged does not make much sense. Since these numbers throw no extra light on the procedure, they are omitted here for clarity.

## 5. CONCLUSION

A structured algorithm for identification and control using  $\mathcal{H}_\infty$  loopshaping is presented. The implementation of the final controller design stage in the algorithm is available in commercial software. The earlier steps of weight synthesis, identification and weight/model adjustment are based on LMI optimization and can as such be implemented easily using any LMI solver. A simulation example demonstrates the use of this algorithm. It is believed that this algorithm has a potential to reduce substantially the time normally required to identify a model from data and then to synthesise a controller which yields satisfactory closed-loop performance with the true system.

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