Published in IET Control Theory and Applications
Received on 21st August 2009
Revised on 30th December 2009
doi: 10.1049/iet-cta.2009.0429

ISSN 1751-8644

Brief Paper
Stability analysis of negative imaginary systems with real parametric uncertainty – the single-input single-output case

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Abstract: Real parametric uncertainty is common in many applications. In this study, the authors analyse the robust stability of feedback loops of single-input single-output systems with negative imaginary frequency response subject to real parametric uncertainty. A recent result on the stability of such feedback loops is specialised to this case, resulting in a very simple stability condition dependent only on the steady-state parameters of the systems. The structured singular value for this system type is also obtained, and it is shown that it yields related but more constrained, easily computable stability guarantees. In a numerical example, the robust stability of a control loop for a DC machine subject to real parametric uncertainty is analysed.

1 Introduction

Lanzon and Petersen [1] recently described a stability result for interconnections of systems with negative imaginary frequency response. Akin to positive real systems, in the single-input single-output (SISO) case, systems with negative imaginary frequency response are defined as having the positive-frequency branch of their Nyquist plot on or below the real axis. Such a frequency response arises for example when modelling the transfer function from a force actuator to a colocated position sensor, or from input voltage to shaft rotational velocity in a DC machine. The necessary and sufficient stability condition for an interconnection of two systems with negative imaginary frequency response is that the loop gain at zero frequency is less than unity. In this paper, we will analyse the robust stability of negative imaginary systems with real parametric uncertainty.

Uncertainty in the (often real-valued) parameters of physical systems is common and poses a challenge for control design. Variation in the parameters may lead to a loss of performance of the closed-loop system or may even cause instability. Several options are available to represent such uncertainty in the plant model. The most intuitive of these is to introduce uncertainty at the coefficient level into the transfer function or state-space model of the plant, and to restrict the uncertain coefficients to real numbers. This real parametric uncertainty accurately and intuitively captures the uncertainty in the physical parameters and its exact influence on the model. It is therefore easy to adopt when parametric variations of the plant can be quantified and the uncertainty is not caused by neglected (e.g. high-order) dynamics.

However, many robust stability results are based on the small-gain theorem [2] and hence are formulated for unstructured complex uncertainty [3, Table 9.1]. Unstructured means that a norm-bounded uncertainty term of compatible dimensions interacts with the plant model transfer function matrix. This interaction may be in an additive, multiplicative or other form–but generally the uncertainty affects the entire plant model with little flexibility in specifying whether some parameters are less uncertain than others. The set of possible plant models forms a region around some nominal plant model. It is usually not straightforward to find an unstructured uncertainty representation for a given set of measured or estimated plant models without also covering many other 'unwanted' plant models. Of course, robust stability results using an oversized uncertainty representation may be conservative.

To integrate specific knowledge about the structure of the uncertainty affecting the plant model into robust stability analysis, μ analysis was introduced by Doyle [4], (see also [5]). The central concept of this theory is the structured singular value μ, which quantifies the smallest destabilising uncertainty in a loop with a given transfer function matrix under some structural assumptions on the uncertainty matrix. The problem of computing μ becomes more complicated when the uncertainties are restricted to real numbers [6, 7].

This paper considers the application of both μ analysis and the stability theorem described in [1] to the robust stability analysis of interconnections of SISO negative imaginary systems with real parametric uncertainty. For both methods, specialised and easily computable results, depending only on the steady-state parameters of the systems, are obtained. An example using a DC machine model with parametric uncertainty highlights the easy applicability of the specific results obtained in this paper, and illustrates the resulting robust stability regions in parameter space.

1.1 Notation

The notation is standard. Let $\mathcal{RH}_\infty^{n\times m}$ denote the set of real-rational stable transfer function matrices of dimension $n \times m$. Also, let $\mathcal{F}_l(.,.)$ and $\mathcal{F}_u(.,.)$ denote lower and upper linear fractional transformations (LFT), respectively. Let $\overline{\lambda}(.)$ and $\overline{\sigma}(.)$ denote the largest eigenvalue and singular value of a matrix. Let A^T denotes the transpose of a matrix A, and denote by $\mathbf{0}_{n\times m}$ an $n \times m$ dimensional matrix with all 0 entries, and denote the n-dimensional identity matrix by I_n. Finally, let $\Im(.)$ denotes the imaginary part of a complex number, and let $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers, respectively.

2 Model set and exact stability criterion

The analysis in this paper is concerned with the positive feedback interconnection of two stable SISO systems with negative imaginary frequency response as displayed in Fig. 1. The set of negative imaginary SISO systems is defined as

$$\mathcal{C} := \{R(s) \in \mathcal{RH}_\infty: j[R(j\omega) - R(j\omega)^*] \geq 0\ \forall \omega \in (0, \infty)\} \tag{1}$$

and the set of strictly negative imaginary SISO systems is defined as

$$\mathcal{C}_s := \{R(s) \in \mathcal{RH}_\infty: j[R(j\omega) - R(j\omega)^*] > 0\,\forall \omega \in (0, \infty)\} \subset \mathcal{C} \tag{2}$$

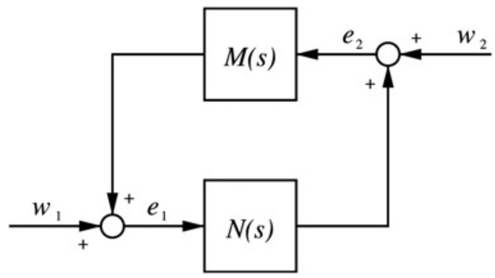

Figure 1 *Positive feedback interconnection*

The difference between the two sets is important, as we require one of the systems to be strictly negative imaginary, whereas the other one may be just negative imaginary. We can now define the transfer functions of the two systems M and N, which are connected in a positive feedback interconnection as shown in Fig. 1. We assume that N contains the uncertainties, and that all its nominal parameter values are known. M on the other hand is allowed to be any system with negative imaginary frequency response, and only its steady-state value $M(0)$ is required for the stability analysis. Let

$$\begin{aligned} M(s) &\in \mathcal{C} \\ N(s) &:= \frac{n(s) + \sum_{i=0}^{p} \delta_{n_i} s^i}{d(s) + \sum_{i=0}^{q} \delta_{d_i} s^i} \in \mathcal{C}_s \end{aligned}$$

In the definition of the uncertain transfer function $N(s)$, $n(s) = \sum_{i=0}^{p} n_i s^i$, $d(s) = s^q + \sum_{i=0}^{q-1} d_i s^i$ are real-coefficient polynomials of order p and q, respectively, which contain the nominal values of all parameters of $N(s)$. Some restrictions need to be placed on both the order of the polynomials as well as on the coefficients and uncertainties to ensure that $N(s) \in \mathcal{C}_s$. Firstly, the requirement that $N(s)$ be stable implies that $\overline{d}_i := d_i + \delta_{d_i} > 0\,\forall i = 0, \ldots, q-1$ and $1 + \delta_{d_q} > 0$. Secondly, it must be ensured that $\Im\{N(j\omega)\} < 0$ at all frequencies. At high frequencies ($\omega \to \infty$), this requires a relative degree $r = q - p$ between zero and two. At lower frequencies, the sign of the imaginary part depends on the relative location of poles and zeros. The parametric conditions can be derived explicitly for low-order functions by computing the roots of the polynomial $\Im\{N(j\omega)\}$.

For the following robust stability analysis of negative imaginary SISO systems, it is important to bear in mind that the restriction $N(s) \in \mathcal{C}_s$ imposes the above-mentioned limits on the uncertain parameters δ_{n_i} and δ_{d_i}. In some physical scenarios, these restrictions arise naturally because no possible uncertainty can cause the system to become non-negative imaginary, as will be illustrated in the example in Section 4.

doi: 10.1049/iet-cta.2009.0429

We will now state necessary and sufficient stability conditions for an interconnection of systems with negative imaginary frequency response when one of the systems is subject to parametric uncertainty.

Theorem 1: Given a positive feedback interconnection $[M(s), N(s)]$ as displayed in Fig. 1, with $M(s) \in \mathcal{C}$ and $N(s) = (n(s) + \sum_{i=0}^{p} \delta_{n_i} s^i)/(d(s) + \sum_{i=0}^{q} \delta_{d_i} s^i) \in \mathcal{C}_s$ satisfying $M(\infty)N(\infty) = 0$ and $N(\infty) \geq 0$. Then

$$[M(s), N(s)] \text{ is internally stable} \Leftrightarrow M(0)\delta_{n_0} - \delta_{d_0} < -(M(0)n_0 - d_0) \quad (3)$$

Proof: Reduction of [1, Theorem 5] for this system class. The result follows upon noting that

$$\bar{\lambda}(M(0)N(0)) < 1 \Leftrightarrow M(0)\frac{n_0 + \delta_{n_0}}{d_0 + \delta_{d_0}} < 1$$

□

As shown by the authors of [1], stability of an interconnection of systems with negative imaginary frequency response depends only on their steady-state gains. Uncertainties in the higher-order coefficients of $N(s)$ do not affect stability so long as they do not cause $N(s)$ itself to become unstable or otherwise violate the restrictions for a negative-imaginary frequency response. We add an immediate corollary to Theorem 2.1.

Corollary 1: Given a positive feedback interconnection $[M(s), N(s)]$ as shown in Fig. 1, with $M(s) \in \mathcal{C}$ and $N(s) = (n(s) + \sum_{i=0}^{p} \delta_{n_i} s^i)/(d(s) + \sum_{i=0}^{q} \delta_{d_i} s^i) \in \mathcal{RH}_\infty$ satisfying $M(\infty)N(\infty) = 0$ and $N(\infty) \geq 0$. If $M(0)\delta_{n_0} - \delta_{d_0} < -(M(0)n_0 - d_0)$ holds,

$$[M(s), N(s)] \text{ not internally stable} \Rightarrow N(s) \notin \mathcal{C}_s$$

Proof: We carry out a proof by contradiction. Assume that $N(s) \in \mathcal{C}_s$. Since all other assumptions of Theorem 1 are satisfied, we then have

$$[M(s), N(s)] \text{ is not internally stable} \Leftrightarrow M(0)\delta_{n_0} - \delta_{d_0} \geq -(M(0)n_0 - d_0)$$

which contradicts our assumption that $M(0)\delta_{n_0} - \delta_{d_0} < -(M(0)n_0 - d_0)$. Hence $N(s) \notin \mathcal{C}_s$. □

3 Analysis with the structured singular value

Having established the exact stability criterion for this system structure, we will now compute the structured singular value μ for comparison. In order to carry out μ analysis, the system structure needs to be transformed into a loop of two different system matrices, one being the uncertainty matrix Δ and the other being the generalised plant $G(s)$. The uncertainty matrix Δ is a member of the set

$$\boldsymbol{\Delta} = \left\{ \Delta = \text{diag}(\delta_{d_q}, \ldots \delta_{d_0}, \delta_{n_p}, \ldots \delta_{n_0}) : \Delta \in \mathbb{R}^{(q+p+2)\times(q+p+2)} \right\} \quad (4)$$

Note that for some $\Delta \in \boldsymbol{\Delta}$, $N(s) \notin \mathcal{C}_s$. The structured singular value $\mu_{\boldsymbol{\Delta}}(G)$ for a $G \in \mathbb{C}^{(q+p+2)\times(q+p+2)}$ is defined as

$$\mu_{\boldsymbol{\Delta}}(G) := [\min_{\Delta \in \boldsymbol{\Delta}} \{\bar{\sigma}(\Delta) : \det(I - G\Delta) = 0\}]^{-1}$$

unless no $\Delta \in \boldsymbol{\Delta}$ solves $\det(I - G\Delta) = 0$, in which case $\mu_{\boldsymbol{\Delta}}(G) := 0$.

For a transfer function $G(j\omega)$, μ is computed pointwise at each frequency. By finding the supremum of μ over all frequencies, the inverse of the size of the smallest destabilising uncertainty at any single frequency is obtained. Zhou *et al.* [3, Theorem 11.8] reformulate the small gain theorem for structured uncertainty in this fashion. A robust stability margin $b(G)$ quantifying the size of the smallest destabilising uncertainty for the feedback interconnection $[G(s), \Delta]$ (which is equivalent to the smallest uncertainty that destabilises $[M(s), N(s)]$) is given by the following expression

$$\frac{1}{b(G)} = \sup_{\omega} \mu_{\boldsymbol{\Delta}}(G(j\omega)) \quad (5)$$

We transform the system into the standard structure for μ analysis in two steps, illustrated in Fig. 2. Firstly, find a $\Sigma(s) \in \mathcal{RH}_\infty^{(q+p+3)\times(q+p+3)}$ which satisfies

$$N(s) = \mathcal{F}_l(\Sigma(s), \Delta) \quad (6)$$

That is extract the uncertainty block Δ from $N(s)$ in an LFT fashion. The resulting interconnection structure is illustrated in Fig. 2. The second step is the calculation of the generalised plant $G(s)$, again using the LFT mechanism, as follows

$$G(s) = \mathcal{F}_u(\Sigma(s), M(s)) \quad (7)$$

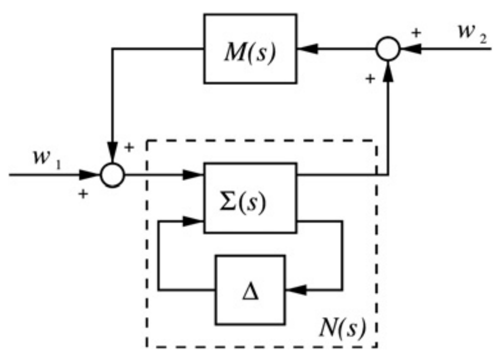

Figure 2 *Feedback structure with extracted uncertainty*

Let $N_{\text{nom}}(s) := n(s)/d(s)$. Then a suitable $\Sigma(s)$ is given by (see (8))

$$\Sigma(s) = \left[\begin{array}{c:c} \Sigma_{11}(s) & \Sigma_{12}(s) \\ \hdashline \Sigma_{21}(s) & \Sigma_{22}(s) \end{array}\right] = \left[\begin{array}{c:cccccccc} N_{\text{nom}}(s) & -\frac{s^q}{d(s)} & -\frac{s^{q-1}}{d(s)} & \cdots & -\frac{1}{d(s)} & \frac{s^p}{d(s)} & \frac{s^{p-1}}{d(s)} & \cdots & \frac{1}{d(s)} \\ \hdashline N_{\text{nom}}(s) & -\frac{s^q}{d(s)} & -\frac{s^{q-1}}{d(s)} & \cdots & -\frac{1}{d(s)} & \frac{s^p}{d(s)} & \frac{s^{p-1}}{d(s)} & \cdots & \frac{1}{d(s)} \\ \vdots & \vdots & & & & & & & \vdots \\ N_{\text{nom}}(s) & -\frac{s^q}{d(s)} & -\frac{s^{q-1}}{d(s)} & \cdots & -\frac{1}{d(s)} & \frac{s^p}{d(s)} & \frac{s^{p-1}}{d(s)} & \cdots & \frac{1}{d(s)} \\ 1 & 0 & & & \cdots & & & & 0 \\ \vdots & \vdots & \ddots & & & & & & \\ 1 & 0 & & & \cdots & & & & 0 \end{array}\right] \quad (8)$$

where the submatrices $\Sigma_{11}(s)$, $\Sigma_{12}(s)$, $\Sigma_{21}(s)$ and $\Sigma_{22}(s)$ have dimensions 1×1, $1 \times (p+q+2)$, $(p+q+2) \times 1$ and $(p+q+2) \times (p+q+2)$, respectively, and $\Sigma_{22}(s)$ has $q+1$ non-zero rows and $p+1$ zero rows ($\Sigma_{21}(s)$ is partitioned accordingly). A simple but space-consuming calculation confirms that (6) is satisfied for $\Sigma(s)$ as given in (8). It should be noted that the $\Sigma(s)$ provided in (8) is not unique. Other structures are possible, but the stability results below are unaffected by the choice of $\Sigma(s)$, as (in terms of structure) they depend only on the uncertainty matrix Δ.

Having established the form of $\Sigma(s)$, we can now compute $G(s)$ using (7)

$$\begin{aligned} G(s) &= \Sigma_{22}(s) + \frac{M(s)}{1 - N_{\text{nom}}(s)M(s)}\Sigma_{21}(s)\Sigma_{12}(s) \\ &= \frac{1}{1 - N_{\text{nom}}(s)M(s)}\begin{bmatrix} 1 & \cdots & 1 & M(s) & \cdots & M(s) \end{bmatrix}^{\mathrm{T}} \\ &\quad \times \begin{bmatrix} -\frac{s^q}{d(s)} - \frac{s^{q-1}}{d(s)} \cdots - \frac{1}{d(s)} \frac{s^p}{d(s)} \frac{s^{p-1}}{d(s)} \cdots \frac{1}{d(s)} \end{bmatrix} \end{aligned} \quad (9)$$

To be able to apply μ analysis, we require $G(s) \in \mathcal{RH}_\infty^{(q+p+2)\times(q+p+2)}$. It is obvious from (9) that internal stability of the nominal feedback loop $[M(s), N_{\text{nom}}(s)]$ is sufficient to fulfill this condition.

With the form of $G(s)$ as described by (9) and the set of allowable uncertainties given in (4), we can now compute the structured singular value for this feedback interconnection $[G(s), \Delta]$, which is an equivalent reformulation of our initital problem setting $[M(s), N(s)]$. The following theorem summarises the stability results obtained with this approach.

Theorem 2: Given a positive feedback interconnection $[M(s), N(s)]$ as shown in Fig. 1 with $M(s) \in \mathcal{C}$ and $N(s) = (n(s) + \sum_{i=0}^{p} \delta_{n_i} s^i)/(d(s) + \sum_{i=0}^{q} \delta_{d_i} s^i) \in \mathcal{C}_s$ satisfying $M(\infty)N(\infty) = 0$, $N(\infty) \geq 0$ and $[M(s), N_{\text{nom}}(s)]$ internally stable, where $N_{\text{nom}}(s) = n(s)/d(s)$. Then the feedback interconnection $[M(s), N(s)]$ is robustly stable for all

$$\Delta \in \mathbf{\Delta} = \left\{ \text{diag}(\delta_{d_q}, \ldots \delta_{d_0}, \delta_{n_p}, \ldots \delta_{n_0}) : \Delta \in \mathbb{R}^{(q+p+1)\times(q+p+1)} \right\}$$

for which

$$\overline{\sigma}(\Delta) < \begin{cases} \left| \dfrac{d_0 - M(0)n_0}{M(0) - 1} \right| & \text{if } M(0) \leq 0 \\ \left| \dfrac{d_0 - M(0)n_0}{M(0) + 1} \right| & \text{if } M(0) \geq 0 \end{cases}$$

Proof: We construct the proof by first obtaining an upper bound to $b(G)$, where $G(s)$ is the generalised plant given in (9), and then showing that a smaller destabilising uncertainty cannot be found by making use of Theorem 1 and Corollary 1. From definition (5), it is obvious that

$$\frac{1}{b(G)} \geq \mu_{\mathbf{\Delta}}(G(j0)) = \frac{1}{\min\{\overline{\sigma}(\Delta)\, n\Delta \in \mathbf{\Delta}, \det(I - G(j0)\Delta) = 0\}} \quad (10)$$

Using the notation

$$d_{11} = \left[\frac{\delta_{d_0}}{d_0 - n_0 M(0)} \cdots \frac{\delta_{d_0}}{d_0 - n_0 M(0)} \right]^{\mathrm{T}},$$

$$d_{12} = \left[-\frac{\delta_{n_0}}{d_0 - n_0 M(0)} \cdots -\frac{\delta_{n_0}}{d_0 - n_0 M(0)} \right]^{\mathrm{T}},$$

$$d_{21} = \left[\frac{M(0)\delta_{d_0}}{d_0 - n_0 M(0)} \cdots \frac{M(0)\delta_{d_0}}{d_0 - n_0 M(0)} \right]^{\mathrm{T}} \text{ and}$$

$$d_{22} = \left[-\frac{M(0)\delta_{n_0}}{d_0 - n_0 M(0)} \cdots -\frac{M(0)\delta_{n_0}}{d_0 - n_0 M(0)} \right]^{\mathrm{T}}$$

(where $d_{11}, d_{12} \in \mathbb{R}^{q\times 1}$ and $d_{21}, d_{22} \in \mathbb{R}^{p\times 1}$), the instability

IET Control Theory Appl., 2010, Vol. 4, Iss. 11, pp. 2631–2638
doi: 10.1049/iet-cta.2009.0429

condition for this lower bound can be reformulated as follows

$$\det(I - G(j0)\Delta) = 0$$

$$\Leftrightarrow \quad \det\begin{pmatrix} I_q & d_{11} & 0 & d_{12} \\ \mathbf{0}_{1\times q} & 1+\frac{\delta_{d_0}}{d_0 - n_0 M(0)} & \mathbf{0}_{1\times p} & -\frac{\delta_{n_0}}{d_0 - n_0 M(0)} \\ \mathbf{0}_{p\times q} & d_{21} & I_p & d_{22} \\ \mathbf{0}_{1\times q} & \frac{M(0)\delta_{d_0}}{d_0 - n_0 M(0)} & \mathbf{0}_{1\times p} & 1 - \frac{M(0)\delta_{n_0}}{d_0 - n_0 M(0)} \end{pmatrix} = 0$$

$$\Leftrightarrow \quad M(0)\delta_{n_0} - \delta_{d_0} = d_0 - M(0)n_0 \tag{11}$$

We will proceed to calculate the size of the smallest uncertainty $\Delta_0 \in \mathbf{\Delta}$ for which (11) holds. Since only δ_{d_0} and δ_{n_0} are relevant for stability, without loss of generality we can set $\delta_{d_i} = \delta_{n_i} = 0 \, \forall i > 0$. Hence

$$\overline{\sigma}(\Delta_0) = \max\{|\delta_{d_0}|, |\delta_{n_0}|\}$$

Since we are seeking the smallest $\overline{\sigma}(\Delta_0)$ which fulfills (11), we are in effect computing the intersection between the square $\overline{\sigma}(\Delta_0)$ and the line described by (11) in the parameter space spanned by $(\delta_{d_0}, \delta_{n_0})$. At the intersection, we have

$$|\delta_{d_0}| = |\delta_{n_0}|$$

which allows two possible solutions to (11)

$$\delta_{n_0} = \delta_{d_0} = \frac{d_0 - M(0)n_0}{M(0) - 1}$$

$$\delta_{n_0} = -\delta_{d_0} = \frac{d_0 - M(0)n_0}{M(0) + 1}$$

The size of both solutions depends on the sign of $M(0)$, and hence

$$\overline{\sigma}(\Delta_0) = \begin{cases} \left|\dfrac{d_0 - M(0)n_0}{M(0) - 1}\right| & \text{if } M(0) \le 0 \\ \left|\dfrac{d_0 - M(0)n_0}{M(0) + 1}\right| & \text{if } M(0) \ge 0 \end{cases}$$

Having established an upper bound on the stability margin $b(G)$, it remains to be shown that no smaller destabilising uncertainty can be found for which $N(s) \in \mathcal{C}_s$. If no smaller destabilising uncertainty can be found, then the stability margin can be computed from $\mu_{\mathbf{\Delta}}(G(0))$ and hence inequality (10) is indeed an equality. To this end, note that Δ_0 lies on the line described by (11) in the space spanned by $(\delta_{d_0}, \delta_{n_0})$, and that this is exactly the stability boundary given by (3) in Theorem 1. Also note that by assumption, the origin $(\delta_{d_0}, \delta_{n_0}) = (0, 0)$ lies in the stable region of the parameter space. Hence, for all $\Delta \in \mathbf{\Delta}$ with $\overline{\sigma}(\Delta) < \overline{\sigma}(\Delta_0)$ we have

$$M(0)\delta_{n_0} - \delta_{d_0} < -(M(0)n_0 - d_0)$$

Since all other assumptions of Corollary 1 are fulfilled, for all $\overline{\sigma}(\Delta) < \overline{\sigma}(\Delta_0)$

$$[M(s), N(s)] \text{ not internally stable} \Rightarrow N(s) \notin \mathcal{C}_s$$

□

With this theorem, robust stability guarantees for negative imaginary systems can be easily expressed in a manner similar to μ analysis. While Theorem 1 gives an exact parametric stability region in the form of a subset of an open half-space in the multidimensional parameter space, Theorem 2 yields a robust stability region in the form of a subset of an open hyper-cube centred around the origin in the parameter space, which is contained in the former open half-space given via Theorem 1. This generic situation is illustrated in Fig. 3, and will be further detailed in the example in the following section. Although stability depends only on the uncertainties in the zero-order coefficients, the robust stability condition $\overline{\sigma}(\Delta) < \overline{\sigma}(\Delta_0)$ in Theorem 2 takes into account all parameters δ_{d_i} and δ_{n_i}. This is a consequence of characterising this theorem as closely aligned with standard μ analysis results which also depend on uncertainties in the higher-order coefficients of $N(s)$.

It should be pointed out that this method of computing the structured singular value for negative imaginary systems is computationally less expensive than generic methods which do not take advantage of the specific system structure. μ analysis has been shown to belong to the class of NP-hard problems [8, 9], and is therefore computationally intractable for systems with a large number of uncertainty blocks. For the SISO system class considered in this paper and using the method presented herein, μ can be computed in linear time, that is proportional to the

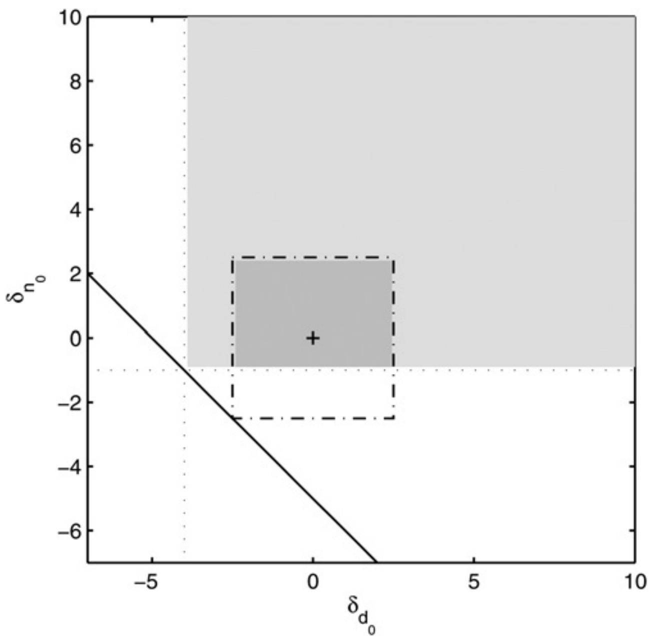

Figure 3 *Generic shapes of the stability regions obtained through Theorem 1 (light grey) and Theorem 2 (dark grey)*

number of uncertain parameters. Only the singular value of the uncertainty matrix Δ needs to be computed, which reduces to finding the maximum absolute value among the uncertain parameters since the matrix is diagonal.

The method applied in this paper to the analysis of SISO systems can be extended to multiple-input, multiple-output (MIMO) systems. The uncertain parameters for a plant of any dimension can be extracted using a method described in [3, pp. 261–264] to obtain a generalised plant in an LFT framework. Using this generalised plant, singular value-type stability conditions can be obtained. However, the derivations are not straightforward, and are beyond the scope of this paper.

4 Robust stability of a DC machine and controller

DC machines are very common in many applications and are well understood. Nevertheless, parametric uncertainty in the mechanical and electrical model parameters is not unusual. This is a consequence of the parameter identification methods, which suffer from changes in environmental parameters, for example the machine temperature [10]. In this example, we will work with the transfer function from input voltage to shaft rotational velocity for a separately excited DC machine in armature mode (i.e. keeping the field voltage fixed), which has the following nominal form:

$$N_{\mathrm{nom}}(s) = \frac{K_{\mathrm{t}}}{JL_{\mathrm{a}}s^2 + (JR_{\mathrm{a}} + BL_{\mathrm{a}})s + BR_{\mathrm{a}} + K_{\mathrm{t}}K_{\mathrm{b}}}$$

where K_{t} and K_{b} are, respectively, the torque constant and back-EMF constant, J is the moment of inertia of the shaft and the load attached to it, B is the coefficient of viscous friction for shaft and load, L_{a} is the armature inductance and R_{a} is the armature resistance. The following nominal coefficient values are chosen for this example: $K_{\mathrm{t}} = 0.3\,\mathrm{N\,m}/A$, $K_{\mathrm{b}} = 0.3\,\mathrm{V\,s/rad}$, $R_{\mathrm{a}} = 1.2\,\Omega$, $L_{\mathrm{a}} = 0.01\,\mathrm{H}$, $B = 0.2\,\mathrm{N/ms}$ and $J = 0.02\,\mathrm{kg\,m^2}$ (see [11] for a more detailed description of the motor model). Hence, we obtain the following nominal transfer function (which we immediately normalise to comply with earlier notation)

$$N_{\mathrm{nom}}(s) = \frac{0.3}{2\times 10^{-4}s^2 + 0.026s + 0.33} = \frac{1500}{s^2 + 130s + 1650}$$

Let $\delta_{n_0}, \delta_{d_0} \in \mathbb{R}$ be uncertainties in the zero-order numerator and denominator coefficients, respectively. The uncertain transfer function is then given by

$$N(s) = \frac{1500 + \delta_{n_0}}{s^2 + 130s + 1650 + \delta_{d_0}}$$

To ensure that $N(s) \in \mathcal{C}_{\mathrm{s}}$, the uncertain parameters δ_{d_0} and δ_{n_0} must fullfill the following inequalities

$$1650 + \delta_{d_0} > 0 \qquad (12)$$

$$1500 + \delta_{n_0} > 0 \qquad (13)$$

Physically, these are quite natural assumptions, as neither $\overline{d}_0 = d_0 + \delta_{d_0}$ nor $\overline{n}_0 = n_0 + \delta_{n_0}$ is negative in any DC machine and hence any uncertainty affects only their magnitude but not their sign.

Even though usually higher-order coefficients are also subject to uncertainty, we limit the analysis in this example to the zero-order coefficients. As shown in Theorem 1, stability depends only on these. Higher-order parametric uncertainty would introduce additional constraints to ensure $N(s) \in \mathcal{C}_s$, which would unnecessarily complicate the notation without adding additional insight for the stability analysis in this example. It is clear from Theorem 2 that the μ stability condition would also restrict any uncertainties in higher-order coefficients to a certain interval around the origin, even though these uncertainties are in fact not relevant for stability. As mentioned above, this is due to formulating Theorem 2 as closely aligned as possible with standard results on the structured singular value.

A proportional controller is chosen to improve the settling time of the closed-loop system. Since the preceding developments assume a positive-feedback interconnection, a negative gain denotes negative feedback. Hence, let

$$M(s) = -5$$

Clearly, $N_{\mathrm{nom}}(s) \in \mathcal{C}_{\mathrm{s}}$ and $M(s) \in \mathcal{C}$. Furthermore, $N(\infty) = 0$. Therefore we can apply Theorem 1 to obtain the exact stability condition for the interconnection $[M(s), N(s)]$

$$[M(s), N(s)] \text{ is internally stable} \Leftrightarrow \delta_{d_0} + 5\delta_{n_0} > -9150$$

The resulting parametric stability region is illustrated in Fig. 4, along with the restrictions imposed by inequalities (12) and (13). It can be seen that in this example, no perturbation for which $N(s) \in \mathcal{C}_{\mathrm{s}}$ causes instability. Furthermore, the stability region is unbounded for $\delta_{d_0}, \delta_{d_0} \to \infty$.

To apply Theorem 2, note that the nominal feedback system is internally stable, and that in this example

$$\boldsymbol{\Delta} = \left\{ \begin{bmatrix} \delta_{d_0} & 0 \\ 0 & \delta_{n_0} \end{bmatrix} : \delta_{d_0}, \delta_{d_0} \in \mathbb{R} \right\}$$

Since $M(0) < 0$, robust stability is guaranteed for all $\Delta \in \boldsymbol{\Delta}$ for which $N(s) \in \mathcal{C}_{\mathrm{s}}$ and

$$\overline{\sigma}(\Delta) < \left| \frac{d_0 - M(0)}{M(0) - 1} \right| = 1525$$

The resulting area of robust stability is indicated in Fig. 4 by

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doi: 10.1049/iet-cta.2009.0429

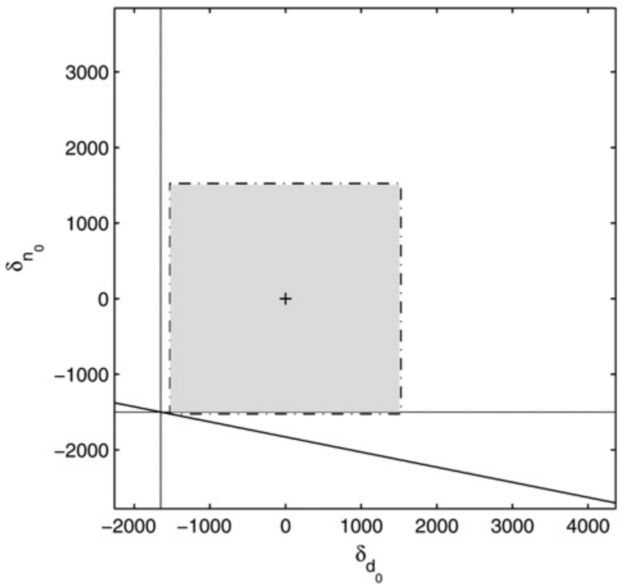

Figure 4 *Parametric stability regions for [M(s), N(s)] in the example*

Region $N(s) \in \mathcal{C}_s$ is bounded by the two dotted lines. The stability boundary given by Theorem 1 is shown by the solid line. The area for which Theorem 2 guarantees stability is shaded grey

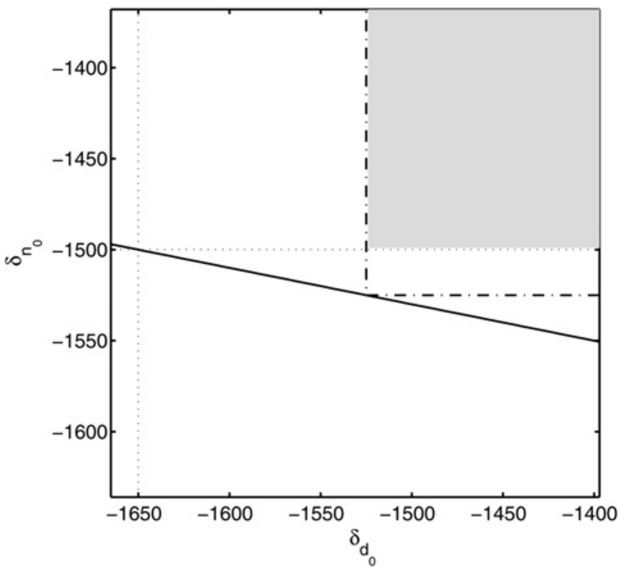

Figure 5 *Detailed view of the robust stability region obtained through Theorem 2 (grey area) and its boundaries defined by $\overline{\sigma}(\Delta) < 1525$ (dash-dot line) and $\delta_{n_0} > -1500$ (dotted line)*

the grey area (while the dashed square shows the region for which $\overline{\sigma}(\Delta) < 1525$). A detailed view in Fig. 5 shows that some part of the region $\overline{\sigma}(\Delta) < 1525$ is not guaranteed to be robustly stable because $N(s) \notin \mathcal{C}_s$.

5 Conclusions

In this paper, it was shown that robust stability regions for SISO systems with negative imaginary frequency response with real parametric uncertainty can easily be obtained using just the steady-state parameters. The result of [1] was specialised to a simple theorem, in which the stability condition takes the form of a scalar inequality in the steady-state parameters of the systems. The structured singular value μ was also computed in an exact form for this system class. The resulting robust stability guarantees are equally simple to compute, but potentially conservative because of the definition of μ. Both methods were applied to obtain robust stability guarantees for a feedback loop involving a DC machine with parametric uncertainty and a proportional controller.

6 Acknowledgments

The financial support of the Engineering and Physical Sciences Research Council (EPSRC grant number EP/F06022X/1) and the Royal Society (via a research grant) is gratefully acknowledged.

A preliminary version of this work has been presented at the European Control Conference 2009, see [12].

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doi: 10.1049/iet-cta.2009.0429