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# Characterising discrete-time linear systems with the "mixed" positive real and bounded real property ☆

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ABSTRACT

In this paper, we present characterisations of linear, shift-invariant, discrete-time systems that exhibit mixtures of small gain-type properties and positive real-type behaviours in a certain manner. These "mixed" systems are already fairly well characterised in the continuous-time domain, but the widespread adoption of digital controllers makes it necessary to verify whether commonly used discretisation procedures preserve the characteristic of "mixedness". First, we analyse the effects of classical discretisation methods on the "mixed" property using Nyquist methods. A frequency domain feedback stability result is then presented. Finally, we develop a spectral-based characterisation of "mixed" discrete-time systems which provides a practical computational test that can also be applied to the MIMO case. Several examples validate the developed theory.



## 1. Introduction

Passivity-based analysis and control [10,25], a well-established theory for engineering systems that appears in a wide range of application areas such as circuit network theory [3], signal processing systems [46], mechanical networks [41] and robotics [30,12], has recently generated interest from the engineering community once again due to novel results on the stability of network flow control schemes [45]. Traditionally, passivity results guarantee the stability of a feedback interconnection of two stable systems if, for instance, both the systems are passive, and one of the systems is input strictly passive with finite gain [42]. Furthermore, parallel interconnections of passive systems are passive [25], and thus passivity is useful for designing stable control systems that are modular-based. Indeed, motivated by gradient update methods for the optimisation problem of maximising a sum of utility functions (dependent on packet rates from different information sources) subject to capacity constraints in the communication links, [45] developed a framework for stabilising source and link control laws by showing that they can be replaced by dynamic systems with prescribed passivity properties. It was also pointed out that the extra generality afforded in doing so issues one with an ability to design the controllers to improve robustness in stability and performance with respect to time delays, unmodelled flows and capacity variation.

It must always be noted that passivity is a sufficient condition for stability, and problems can arise from using purely passivity-based techniques for real-world applications. For example, unmodelled dynamics can destroy assumed or nominal passivity over certain frequency bandwidths [34,1], and meeting set passivity criteria can conflict with system performance requirements [16]. A range of work exists on investigating ways to relax passivity requirements while being able to maintain the stability properties

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of nominally passive control systems; for instance, [22,23] studied in depth the concepts of finite frequency positive realness (i.e., positive realness only over a certain frequency band) and "restricted passivity".

Building on the idea of finite frequency positive realness, "mixed" systems were recently examined in [17–20]. *"Mixed" system* was a term coined to refer to systems that combine notions of passivity and small gain type behaviour in a certain manner; for example, a "mixed" system has small gain behaviours over frequency bands where positivity is violated. "Mixed" systems were intended to aid in the formalisation and extension of the well-known engineering notion that keeping feedback-loop gain small at high frequencies where passivity might be violated avoids destabilisation of high frequency dynamics; for instance, see [33,5]. The stability of large-scale interconnections of "mixed" systems was considered in [20], and a spectral-based characterisation, leading to a test, for "mixed" systems in continuous-time was presented in [19].

While the study of systems with finite frequency positive realness (e.g. "mixed" systems [20,19]; see also [47,31]) has seen much progress over the past number of years, many basic questions remain. For instance, engineers rarely work with continuous-time systems exclusively. For simulation purposes, or for the purpose of control design, or in order to implement a controller, at some stage a discrete-time representation of the system must be considered. Thus, it is critical to establish whether discrete-time systems inherit fundamental properties of the continuous-time systems from which they are derived. System discretisation has currently become an issue of importance once again, and several papers [32,48,6,11,40,36] have recently appeared on this topic, particularly in the switched systems community.

The main purpose of this paper is to introduce the concept of "mixedness" to discrete-time systems. Initially, we emphasise that not all discretisation procedures preserve the "mixed" property. We provide some results that show that "mixedness" is preserved by Tustin's and Euler's backward methods. However, for some applications, these methods may not be suitable. Given the potential for "mixedness" to be lost upon discretisation, we then adapt previous results for the continuous-time case to the discrete-time domain. First, stability results for feedback interconnections of "mixed" discrete-time systems are derived. Then, a computationally viable spectral-based characterisation method, for determining whether a system in discrete-time is "mixed," is developed.

The remainder of the paper is organised as follows. In Section 2, "mixed" systems in discrete-time are defined and some preliminary work that is required later in the paper is provided. In Section 3, we motivate the need to characterise "mixed" systems in discrete-time, by observing how certain discretisation processes do not necessarily preserve the property of "mixedness". In Section 4, a feedback stability result for "mixed" discrete-time systems, based on the Nyquist stability theorem, is presented. A spectral-based characterisation of "mixed" discrete-time systems, based on their state-space information, is derived in Section 5. This spectral-based characterisation leads to a test for determining "mixedness" in discrete-time and constitutes one of the main results of the paper. Examples of the spectral-based test are provided in Section 6, and directions for our future research are presented in Section 7.

## 2. Mathematical preliminaries

First, some mathematical definitions and auxiliary results that are to be applied to obtain the main results of the paper are presented.

### 2.1. Notation

Let $\Re[\cdot]$ and $\Im[\cdot]$ denote the real and the imaginary parts of a complex number, respectively. The conjugate of a complex number $z=re^{j\theta}$, where $r$ is the magnitude of $z$, $\theta$ is the phase of $z$ and $j^2=-1$, will be denoted by $\overline{z}$. For any square matrix, $\rho(\cdot)$ denotes its spectral radius. For complex matrices $A\in\mathbb{C}^{n\times n}$, $A^*$ denotes its conjugate transpose and $A^T$ denotes its transpose. Moreover, for $A$ Hermitian, $A>0$ ($A\geq 0$) denotes that it is positive (semi)definite, whilst $\lambda_{min}(\cdot)$ and $\lambda_{max}(\cdot)$ denote its minimum and maximum eigenvalue, respectively.

### 2.2. Definitions

The following definitions will be required. We begin by defining the input–output stability of a discrete-time system.

**Definition 1** (*Santina and Stubberud [35, Section 10.1.3]*). A discrete-time system with proper, real-rational transfer function matrix $G(z)$ is said to be input–output stable if all of the poles of $G(z)$ lie inside the unit circle on the complex plane.

Suppose that $\theta:=\omega T$, where $T$ denotes a fixed sampling interval in seconds, and $\omega$ denotes any signal frequency in rad/s such that $\theta\in[-\pi,\pi]$. Suppose that $0\leq\overline{a}\leq\overline{b}\leq\pi$, where $\overline{a}$ and $\overline{b}$ are in radians.

**Definition 2.** An input–output stable discrete-time system with square, proper, real-rational transfer function matrix $M(z)$ is said to be input and output strictly positive over $[-\overline{b},-\overline{a}]\cup[\overline{a},\overline{b}]$ if there exist real numbers $k,\ell>0$ such that

$$-kM^*(e^{j\theta})M(e^{j\theta})+M^*(e^{j\theta})+M(e^{j\theta})-\ell I\geq 0$$

for all $\theta\in[-\overline{b},-\overline{a}]\cup[\overline{a},\overline{b}]$.

A system is said to be input strictly positive over $[-\overline{b},-\overline{a}]\cup[\overline{a},\overline{b}]$ if Definition 2 is satisfied with $k=0$; output strictly positive over $[-\overline{b},-\overline{a}]\cup[\overline{a},\overline{b}]$ if the definition is satisfied with $\ell=0$; and positive over $[-\overline{b},-\overline{a}]\cup[\overline{a},\overline{b}]$ if it is satisfied with $k=\ell=0$.

**Definition 3.** For an input–output stable discrete-time system with proper, real-rational transfer function matrix $M(z)$, define the system gain over $[-\overline{b},-\overline{a}]\cup[\overline{a},\overline{b}]$ as

$$\epsilon:=\min\{\overline{\epsilon}\in\mathbb{R}_+ : -M^*(e^{j\theta})M(e^{j\theta})+\overline{\epsilon}^2 I\geq 0 \text{ for all } \theta\in[-\overline{b},-\overline{a}]\cup[\overline{a},\overline{b}]\}.$$

The system is said to have a gain of less than one over $[-\overline{b},-\overline{a}]\cup[\overline{a},\overline{b}]$ if $\epsilon<1$.

A "mixed" discrete-time system, analogous to the description of a "mixed" continuous-time system provided in [20,19], is now defined.

**Definition 4.** An input–output stable discrete-time system with square, proper, real-rational transfer function matrix $M(z)$ is said to be "mixed" if, for each $\theta\in[-\pi,\pi]$, either of the following hold:

(i) there exist $k,\ell>0$ such that $-kM^*(e^{j\theta})M(e^{j\theta})+M^*(e^{j\theta})+M(e^{j\theta})-\ell I\geq 0$;
(ii) there exists $\epsilon<1$ such that $-M^*(e^{j\theta})M(e^{j\theta})+\epsilon^2 I\geq 0$.

### 2.3. Preliminary results

A feedback stability result for "mixed" discrete-time systems is presented later in Section 4. The proof of the result is based on classical Nyquist techniques. Hence, we now recall a discrete-time version of the well-known Nyquist stability theorem, as follows.

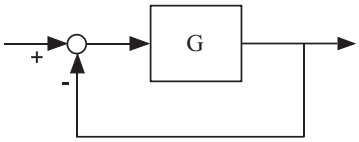



**Fig. 1.** A negative feedback-loop.

**Theorem 1** (*Glad and Ljung [14, p. 74], Åström and Wittenmark [4, Section 3.2]*). *Consider the feedback-loop depicted in Fig. 1. Suppose that $G(z)$ is a strictly proper, real-rational transfer function of a stable discrete-time system. Then the feedback-loop is stable if and only if the Nyquist plot of $1+G(e^{j\theta})$ for $-\pi\le\theta\le\pi$ does not make any encirclements of the origin.*

In the above theorem, stability is defined in the sense of [14, Section 3.7]. Note, also, the following observations concerning the Nyquist plot of $1+G(e^{j\theta})$ for $-\pi\le\theta\le\pi$.

**Observation 1.** *The Nyquist plot of $1+G(e^{j\theta})$ belongs to a family of Nyquist plots of $1+(1/\kappa)G(e^{j\theta})$, where $\kappa\in[1,\infty)$.*

**Observation 2.** *Each Nyquist plot of $1+(1/\kappa)G(e^{j\theta})$ is symmetrical about the real axis of the complex plane, where $\kappa\in[1,\infty)$.*[3]

**Observation 3.** *As $\kappa$ and $\theta$ vary continuously, the point in the complex plane on which the Nyquist plot of $1+(1/\kappa)G(e^{j\theta})$ lies varies continuously.*

**Observation 4.** *As $\kappa\to\infty$, $1+(1/\kappa)G(e^{j\theta})\to 1$.*

**Observation 5.** *Suppose that $\kappa$ is very large such that $1+(1/\kappa)G(e^{j\theta})$ is almost equal to 1 for all $\theta\in[-\pi,\pi]$. Then suppose that $\kappa$ is continuously decreased towards 1. Suppose that the Nyquist plot of $1+G(e^{j\theta})$ encircles the origin at least once. Then there must exist at least one $\kappa_0$ and one $\theta_0$ for which $1+(1/\kappa_0)G(e^{j\theta_0})=0$.*

From the previous observations and Theorem 1, we can establish the following corollary.

**Corollary 2.** *Adopt the hypotheses of Theorem 1. Then a sufficient condition for the Nyquist plot of $1+G(e^{j\theta})$ to make no encirclements of the origin is that, for all $\kappa\in[1,\infty)$ and all $\theta\in[-\pi,\pi]$, $1+(1/\kappa)G(e^{j\theta})\neq 0$.*

Finally, we present an additional preliminary result that will be required to obtain the feedback stability theorem of Section 4.

**Lemma 3.** *Let $G_1(z)$ and $G_2(z)$ be square, proper, real-rational transfer function matrices with no poles on or outside of the unit circle in the complex plane. Suppose that $G_1^*(e^{j\theta_0})+G_1(e^{j\theta_0})>0$ and $G_2^*(e^{j\theta_0})+G_2(e^{j\theta_0})\ge 0$ for some $\theta_0\in[-\pi,\pi]$. Then $\det[I+G_1(e^{j\theta_0})G_2(e^{j\theta_0})]\neq 0$.*

**Proof.** Since $G_1^*(e^{j\theta_0})+G_1(e^{j\theta_0})>0$, $\Re[\lambda_i[G_1(e^{j\theta_0})]]>0\ \forall i$ (where $\lambda_i[\cdot]$ denotes the $i$th eigenvalue) [24, Theorem 1 of Section 13.1] and so $G_1(e^{j\theta_0})$ is nonsingular. Then $G_1^{-*}(e^{j\theta_0})+G_1^{-1}(e^{j\theta_0})>0$ since $G_1^*(e^{j\theta_0})+G_1(e^{j\theta_0})$ and $G_1^{-1}(e^{j\theta_0})+G_1^{-*}(e^{j\theta_0})$ are Hermitian congruent [28, p. 415]. Then $G_1^{-*}(e^{j\theta_0})+G_2^*(e^{j\theta_0})+G_1^{-1}(e^{j\theta_0})+G_2(e^{j\theta_0})>0$. Hence $\Re[\lambda_i[G_1^{-1}(e^{j\theta_0})+G_2(e^{j\theta_0})]]>0\ \forall i$ and so $\det[G_1^{-1}(e^{j\theta_0})+G_2(e^{j\theta_0})]\neq 0$. Then $\det[I+G_1(e^{j\theta_0})G_2(e^{j\theta_0})]\neq 0$ since $\det[I+G_1(e^{j\theta_0})G_2(e^{j\theta_0})]=\det[G_1(e^{j\theta_0})]\det[G_1^{-1}(e^{j\theta_0})+G_2(e^{j\theta_0})]$ and $G_1(e^{j\theta_0})$ is nonsingular. □

## 3. On discretisation and the "mixed" property

Discrete-time systems are widely used, not only to model inherently discrete processes, but also to deal with applications in which continuous-time dynamics have to be approximated, such as in digital and networked control systems [9,21]. Unfortunately, it is known that some properties, such as controllability, may be lost when discretisation methods are applied to continuous-time dynamical systems [38]. Since qualitative properties of the controller are often exploited in control design, discretisation techniques are at present still being studied in the literature to ensure the preservation of such characteristics; see [43,44] in the context of LPV systems. Therefore, it is of interest to investigate whether the "mixed" properties, described in [20,19] for continuous-time systems, are carried through to exist in the form given by Definition 4 after a discretisation procedure is applied.

Next, we will investigate some discretisation methods that are well-established in the literature. To this end, we consider a continuous-time, linear, time-invariant (LTI) system given by its state-space realisation

$$\dot{x}(t)=Ax(t)+Bu(t)$$
$$y(t)=Cx(t)+Du(t)$$

with associated transfer matrix given by $M(s)=C(sI-A)^{-1}B+D$. For any given sampling period $T>0$, the discretisation procedures to be analysed yield a discrete-time, linear, shift-invariant system given by

$$x[k+1]=A_dx[k]+B_du[k]$$
$$y[k]=C_dx[k]+D_du[k]$$

where $x[k]:=x(kT)$, $u[k]:=u(kT)$ and $y[k]:=y(kT)$, for each $k\in\mathbb{N}$. This system has as its transfer matrix $M_d(z)=C_d(zI-A_d)^{-1}B_d+D_d$ and its realisation matrices $(A_d,B_d,C_d,D_d)$ depend on the adopted method. The methods to be analysed are the following:

(1) *Zero-order hold (ZOH)* [13,2]: it is also called a step-invariant transformation and, in this case, the discrete-time matrices are

$$A_d=e^{AT}\quad\text{and}\quad B_d=\int_0^T e^{A\xi}B\,d\xi,$$

together with $C_d=C$ and $D_d=D$. These matrices follow from the analytical solution of the original differential equation for a piece-wise constant input $u$. The method can be alternatively defined by its transfer function matrix

$$M_d(z)=(1-z^{-1})\mathcal{Z}\left\{\mathcal{L}^{-1}\left[\frac{M(s)}{s}\right]\Bigg|_{t=kT}\right\}$$

(2) *Euler's approximations* [2,37]: there are two *rectangular* methods that approximate the integrator, which are Euler's forward and backward methods. For the *forward approximation*, the discrete-time transfer function matrix is given by

$$M_d(z)=M\left(\frac{z-1}{T}\right),$$

which implies the following approximation in the state space matrices: $A_d=I+TA$ and $B_d=TB$. For the *backward approximation*, we have the discrete-time transfer function matrix

$$M_d(z)=M\left(\frac{z-1}{Tz}\right)$$

and this case imposes $A_d=(I-TA)^{-1}$ and $B_d=T(I-TA)^{-1}B$.

(3) *Tustin's approximation* [2,37]: this method is based on the numerical integration trapezoidal rule, which yields the transformation $s=(2/T)(z-1)/(z+1)$ and implies the discrete-time transfer function matrix given by

$$M_d(z)=M\left(\frac{2}{T}\frac{z-1}{z+1}\right).$$

[3] Since $1+(1/\kappa)G(e^{-j\omega T})=\overline{1+(1/\kappa)G(e^{j\omega T})}$.

The state-space realisation expressions of Tustin's method are rather complicated and can be found in [2], but it is important to emphasise that $A_d = (I - TA/2)^{-1}(I + TA/2)$ for this method.

(4) *Simpson's approximation* [37]: this method is based on the approximation of the integrator with Simpson's rule for numerical integration, yielding the discrete-time transfer function

$$M_d(z) = M\left(\frac{12}{T}\frac{z(z-1)}{5z^2+8z-1}\right)$$

**Observation 6.** *Not all discretisation methods presented above may preserve essential properties such as stability. Indeed, three methods that guarantee stability are the step-invariant transformation, Tustin's approximation and Euler's backward method. Tustin's and Euler's backward approximations can be seen as bilinear transformations that map $\Re[s] < 0$ onto $|z| < 1$, in the first case, and onto a subset of the unit circle, in the latter, and, therefore, stability is preserved by both methods. Since ZOH is based on the solution of the state-space differential equations, its transformation also preserves stability.*

**Observation 7.** *As stated before, some of the discretisation methods presented above are based on the application of classical quadrature formulas to approximate the unit integrator transfer function $1/s$ [37]. Euler's, Tustin's and Simpson's methods are obtained from the rectangular, trapezoidal and quadratic interpolations to numerically compute integrals.*

**Observation 8.** *From the state-space point of view, the continuous-time dynamic matrix $A$ is mapped onto the discrete-time dynamic matrix $A_{zoh} = e^{AT}$ for the ZOH method. The discrete-time dynamic matrices $A_d = I + TA$, for Euler's forward method, $A_d = (I - TA)^{-1}$, for Euler's backward method, and $A_d = (I - TA/2)^{-1}(I + TA/2)$, for Tustin's approximation, can all be seen as Padé approximants for the matrix exponential* [15].

We apply these discretisation methods in the following example, which gives some insights with respect to the behaviour of the "mixed" property.

**Example 1.** Consider the following transfer function:

$$m(s) = -0.2\frac{s^2+3s-13}{(s+1)(s+2)},$$

whose Nyquist Diagram is illustrated in Fig. 2. From [20, Definition 3], we can conclude that this continuous-time system is "mixed." First, we use the `c2d` routine of MATLAB R2011b with the `'zoh'` option in order to obtain the corresponding discrete-time transfer function with a given sampling period $T > 0$. Taking the sampling period $T = 1.0$ s, we obtain

$$m_{zoh}(z) = \frac{-0.2z^2+0.7z+0.2105}{z^2-0.5032z+0.04979},$$

whose Nyquist Diagram is shown by the outer blue line in Fig. 3, which also shows the continuous-time transfer function $m(s)$ plot with the inner blue line. The grey region approximately contains all the Nyquist Diagrams for $T \in (0, 1)$ and the red curve is obtained for $T = 0.53$ s, marginally violating the "mixed" property, which is not satisfied for greater values of the sampling period in the considered interval. Indeed, for the particular case $T = 1.0$ s, we observe that there is a frequency range from 0.757 rad/s to 0.952 rad/s (and another range from $-0.952$ rad/s to $-0.757$ rad/s) over which neither Property (i) nor Property (ii) of Definition 4 holds, since the Nyquist Diagram of this transfer function is neither inside the unit circle nor contained by the right half-plane whenever $\omega$ is taken belonging to these intervals.

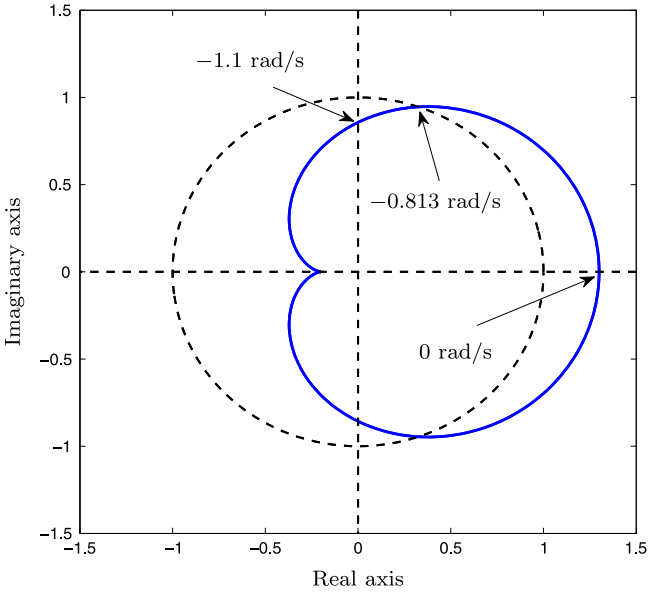



**Fig. 2.** Nyquist diagram of $m(s)$.

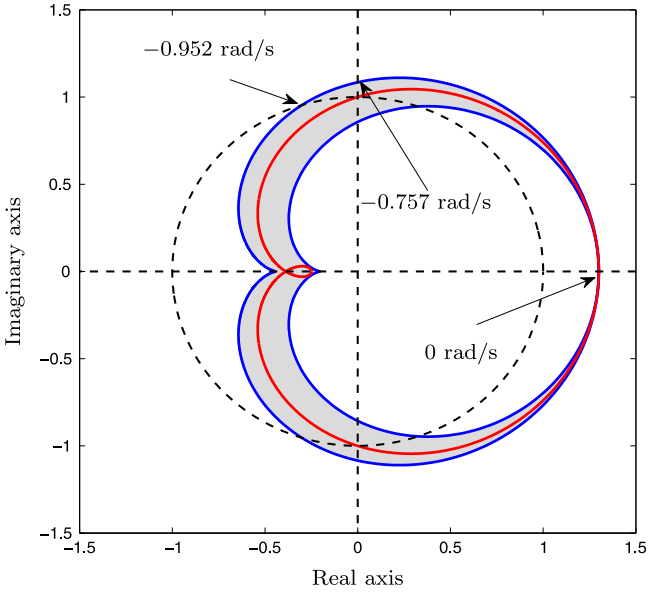



**Fig. 3.** Nyquist diagrams of $m_{zoh}(z)$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this paper.)

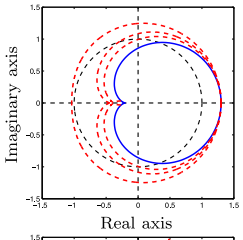
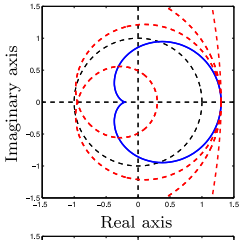
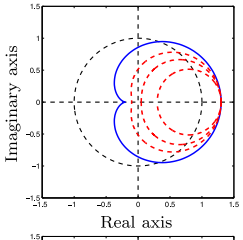
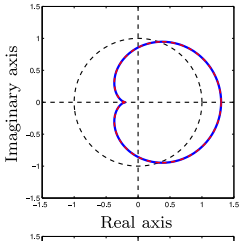
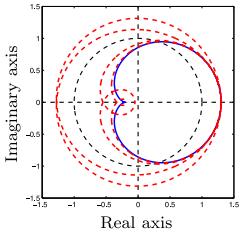
Proceeding in the same way, we conclude that Euler's forward and Simpson's approximations also violate the "mixed" property for a large enough sampling period. Indeed, "mixedness" is marginally violated with $T = 0.27$ s for Euler's forward method and $T = 1.59$ s for Simpson's and it is preserved for any sampling period smaller than these values.

Now, we discretise $m(s)$ with Tustin's approximation procedure using `c2d` with the option `'tustin'` and, for $T = 1.0$ s, we obtain

$$m_{tus}(z) = \frac{0.05z^2+0.5667z+0.25}{z^2-0.3333z}.$$

The Nyquist Diagram of this transfer function overlaps the one of $m(s)$ and, therefore, the "mixed" property is not lost when the system was discretised, contrarily to the ZOH case. Adopting the same procedure with Euler's backward formula, we observe that "mixedness" is preserved for this example since the Nyquist plot of

**Table 1**
Results of Example 1.

| Method | Preserves "mixedness"? | Nyquist plots |
|---|---|---|
| ZOH | No, it fails with $T \geq 0.53$s |  |
| Euler's forward | No, it fails with $T \geq 0.27$s |  |
| Euler's backward | Yes |  |
| Tustin's approximation | Yes |  |
| Simpson's approximation | No, it fails for $T \geq 1.59$s |  |



the discrete-time transfer function is always inside the region delimited by the continuous-time curve.

All the results obtained in this example are summarised in Table 1. The Nyquist plots column contains the diagrams for the continuous-time system (continuous blue lines) and for the discretised systems (dashed red lines) with sampling periods of 0.5 s, 1.0 s and 2.0 s.

The previous example shows that the "mixed" property may be lost when we adopt one of the following methods: ZOH, Euler's forward approximation and Simpson's rule. Nevertheless, "mixedness" is preserved by either Euler's backward or Tustin's approximations, due to the effects these transformations have on the Nyquist plot of the continuous-time system. This result is stated formally, as follows.

**Theorem 4.** *Consider a continuous-time LTI system given by its transfer function m(s). The following statements hold.*

(i) *If $m(s)$ is input–output stable and its Nyquist plot is a Jordan curve (simple closed curve), then Euler's backward transformation is "contractive" on the Nyquist plot of $m(s)$; that is, the Nyquist plot of $m_d(z)$ is always contained by the region delimited by the Nyquist plot of $m(s)$.*
(ii) *The Nyquist plot of $m(s)$ is "invariant" under Tustin's bilinear transformation; that is, both Nyquist plots delimit the same region in the complex plane.*

**Proof.** For the first statement, the proof is based on the following observation. From the definition of the discrete-time transfer function for Euler's backward method, the frequency response of the discretised system is given by

$$m_d(e^{j\theta}) = m\left(\frac{1-e^{-j\theta}}{T}\right) = m\left(\frac{1-\cos\,\theta}{T} + j\frac{\sin\,\theta}{T}\right),$$

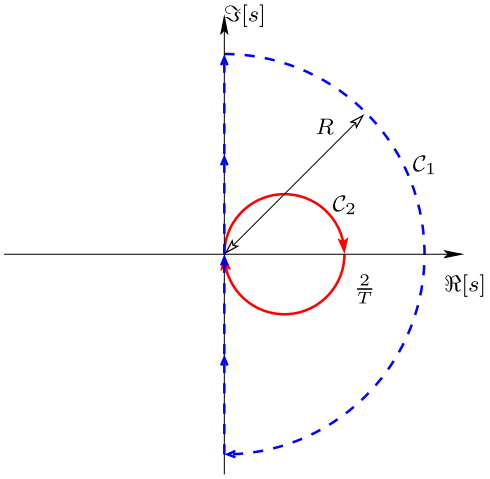



**Fig. 4.** Nyquist curves. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this paper.)

which allows us to conclude that the Nyquist plot of $m_d(z)$ can be obtained by mapping the positively oriented circle described by the equations above using $m(s)$. The red continuous line in Fig. 4 shows the circle $\mathcal{C}_2$ defined above, for a given sampling period $T>0$, and the dashed blue line represents the classical Nyquist stability criterion curve $\mathcal{C}_1$, where $R\to\infty$. Define $\mathcal{C}_1^*$ and $\mathcal{C}_2^*$ as the curves obtained by mapping $\mathcal{C}_1$ and $\mathcal{C}_2$ under $m(s)$. Since $m(s)$ is analytic in the closed right half-plane and its Nyquist plot is a Jordan curve, then it follows from [27, p. 154] that the interior of $\mathcal{C}_1$ is mapped onto the interior of $\mathcal{C}_1^*$. Since $\mathcal{C}_2$ is contained by the interior of $\mathcal{C}_1$, it is true that $\mathcal{C}_2^*$ is contained by the region delimited by $\mathcal{C}_1^*$. Then, the statement follows.

Now we prove item (ii). The frequency response of the discrete-time transfer function is given by

$$m_d(e^{j\theta}) = m\left(\frac{2}{T}\frac{e^{j\theta}-1}{e^{j\theta}+1}\right) = m\left(j\frac{2}{T}\tan\left(\frac{\theta}{2}\right)\right) = m(j\omega)$$

where $\theta\in[-\pi,\pi]$ and $\omega=(2/T)\tan(\theta/2)\in\mathbb{R}\cup\{\pm\infty\}$. Hence, since there is a bijective function that relates $\theta$ and $\omega$ in both frequency responses, it follows that both curves in the complex plane are overlapped. Therefore, the Nyquist plot of $m(s)$ is "invariant" under Tustin's transformation. The proof is complete. □

**Corollary 5.** *Consider a "mixed" continuous-time system, as defined in [20]. Then, for any given sampling period, its discretised system obtained either by Tustin's approximation or by Euler's backward method is "mixed," provided the assumptions of Theorem 4 are valid.*

The previous corollary is a direct consequence of Theorem 4 and adequately states the behaviour observed in the example.

**Observation 9.** *The additional hypotheses on $m(s)$ in the first statement of Theorem 4 are necessary to ensure the regularity of the mapping defined by $m(s)$ and to precisely define the bounded region of the closed curves used in its proof. It is important to assert that the stability condition is always satisfied by "mixed" systems. Moreover, one must note that none of these conditions are needed to prove the second statement.*

This section shows that most discretisation methods do not maintain the "mixed" property of a continuous-time system for an arbitrary sampling period, especially the widely adopted ZOH method. This method has the singular property of being exact for piece-wise constant inputs, which are frequent in practical engineering applications. Thus, it is really necessary to have a computational test to detect when the "mixedness" of a given continuous-time system is lost under discretisation. This is one of the main contributions of this paper and is presented in Section 5.

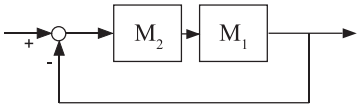



**Fig. 5.** A negative feedback interconnection of "mixed" systems.

## 4. Feedback stability

The assumption that, upon discretisation, systems retain certain properties, such as "mixedness" or passivity, is not always a valid one, as demonstrated in Section 3. In fact, this issue with system discretisation is well documented in the case of passivity [29]. The following result shows that, if "mixedness" has been established in discrete-time, then a feedback stability result holds. (A test for determining whether a system is "mixed" in discrete-time is the subject of Section 5.)

Theorem 6 draws an analogy to the feedback stability result presented in [20] for "mixed" continuous-time systems. A simpler version of Theorem 6 was presented in [7, Proposition 4]. (In Theorem 6, we have substituted the bandwidth condition in [7, Proposition 4] and with a more general condition.)

**Theorem 6.** *Suppose that $M_1(z)$ and $M_2(z)$ denote the transfer functions of "mixed" discrete-time systems, interconnected as depicted in Fig. 5, where one of these transfer functions is strictly proper. Suppose that there exist two closed sets of $\theta$: (a) a set denoted by $\Theta_p$ that consists of $\theta\in[-\pi,\pi]$ over which both $M_1(e^{j\theta})$ and $M_2(e^{j\theta})$ have associated with them Property (i) as given in Definition 4; and (b) a set denoted by $\Theta_s$ that consists of $\theta\in[-\pi,\pi]$ over which both $M_1(e^{j\theta})$ and $M_2(e^{j\theta})$ have associated with them Property (ii) as given in Definition 4. Finally, suppose that $\Theta_p\cup\Theta_s=\{\theta\in\mathbb{R}:-\pi\le\theta\le\pi\}$. Under these assumptions, the feedback-loop in Fig. 5 is stable.*

**Proof.** The goal is to show that, for all $\kappa\in[1,\infty)$ and all $\theta\in[-\pi,\pi]$, $1+(1/\kappa)M_1(e^{j\theta})M_2(e^{j\theta})\neq 0$. From Corollary 2, this is a sufficient condition for stability of the feedback-loop. Subsequently, the proof is split into two parts: first, it is shown that $1+(1/\kappa)M_1(e^{j\theta})M_2(e^{j\theta})\neq 0$ for all $\kappa\in[1,\infty)$ and all $\theta\in\Theta_s$; and second, it is shown that $1+(1/\kappa)M_1(e^{j\theta})M_2(e^{j\theta})\neq 0$ for all $\kappa\in[1,\infty)$ and all $\theta\in\Theta_p$.

Part I: for any $\theta\in\Theta_s$. From Property (ii) of Definition 4, $|M_i(e^{j\theta})|<1$ for $i=1,2$ and hence $|M_1(e^{j\theta})M_2(e^{j\theta})|<1$. Then $(1/\kappa)|M_1(e^{j\theta})M_2(e^{j\theta})|<(1/\kappa)\le 1$ for any $\kappa\ge 1$; i.e. $|(1/\kappa)M_1(e^{j\theta})M_2(e^{j\theta})|<1$ since $(1/\kappa)|M_1(e^{j\theta})M_2(e^{j\theta})|=|(1/\kappa)M_1(e^{j\theta})M_2(e^{j\theta})|$. So $(1/\kappa)M_1(e^{j\theta})M_2(e^{j\theta})\neq -1$ for any $\kappa\in[1,\infty)$.

Part II: for any $\theta\in\Theta_p$. From Property (i) of Definition 4, $M_i^*(e^{j\theta})+M_i(e^{j\theta})>0$ for $i=1,2$. Observe that $M_i^*(e^{j\theta})+M_i(e^{j\theta})>0$ if and only if $(1/\sqrt{\kappa})M_i^*(e^{j\theta})+(1/\sqrt{\kappa})M_i(e^{j\theta})>0$, where $\kappa>0$. Then, from Lemma 3, $1+(1/\kappa)M_1(e^{j\theta})M_2(e^{j\theta})\neq 0$ for any $\kappa>0$, and hence for any $\kappa\ge 1$. □

It is supposed in Theorem 6 that $M_1(z)$ and $M_2(z)$ denote the transfer functions of "mixed" discrete-time systems. While the product $M_1(z)M_2(z)$ will not necessarily denote a "mixed" discrete-time system (since the sum of the phases of two complex numbers, each with phase less than 90°, may be greater than 90°), Theorem 6 guarantees that the feedback-loop will nonetheless be stable.

Another interesting result is presented in the following theorem. Conversely to Theorem 6, Theorem 7 suggests that as long as the product of $n$ transfer functions (the product of the transfer functions being labelled as $G(z)$ in Theorem 7) is "mixed," then a feedback-loop is stable, even if some or all of the $n$ transfer functions do not each individually represent "mixed" discrete-time systems.

**Theorem 7.** *Suppose that $G(z)$ is the strictly proper transfer function of a discrete-time "mixed" system. The feedback-loop in Fig. 1 is stable.*

**Proof.** The goal is to show that, for all $\kappa \in [1,\infty)$ and all $\theta \in [-\pi,\pi]$, $1+(1/\kappa)G(e^{j\theta}) \neq 0$. From Corollary 2 and Theorem 1, this is a sufficient condition for stability of the feedback-loop. Indeed, for any given $\theta \in [-\pi,\pi]$, we have that either Property (i) or Property (ii) from Definition 4 hold. Therefore, we can split the proof into two parts.

(i) If $\theta \in [-\pi,\pi]$ is such that Property (i) holds, then $G(e^{j\theta}) + G^*(e^{j\theta}) > 0$ and, thus, $\Re[G(e^{j\theta})] > 0$. Therefore, $(1/\kappa)G(e^{j\theta}) \neq -1$ for all $\kappa > 0$ and, in particular, for all $\kappa \in [1,\infty)$.
(ii) If $\theta \in [-\pi,\pi]$ is such that Property (ii) holds, then $|G(e^{j\theta})| < 1$. Hence, as a clear consequence, it follows that $(1/\kappa)G(e^{j\theta}) \neq -1$ for all $\kappa \in [1,\infty)$.

Based on this reasoning, the goal is achieved and the proof is complete. □

**Observation 10.** *Note that both stability results depend only on the closed-loop characteristic equations for the systems, which are $1+G(z)=0$ and $1+M_1(z)M_2(z)=0$. Thus, they remain valid for any well-posed feedback configuration that presents the same closed-loop characteristic equation (as one of the previous ones). Moreover, in Theorem 7, no assumption is necessary concerning each isolated system, since the result focuses on the product $G(z)$. This is similar to the closely related classical results in input–output feedback stability: the Passivity Theorem and the Small-gain Theorem [10,25].*

To conclude this section, it is also worth noting that, while some classical results for passive systems also hold for "mixed" systems (e.g. the feedback stability result of the style described above), other classical results do not. For instance, parallel interconnections of passive systems are passive [25, Theorem 8.2], whereas parallel interconnections of "mixed" systems are not necessarily "mixed." For a simple example of this, consider the transfer functions of two "mixed" continuous-time systems $m_1(s) = 3/(s^2+3s+2)$ and $m_2(s) = 4.314/(s^2+4.5s+2)$, compared to the parallel interconnection of these two systems given by $m(s) = m_1(s)+m_2(s)$. (The system $m(s)$ does not exhibit the "mixed" property.)

## 5. Spectral-based characterisation

Now we provide a more general procedure for testing whether a discrete-time system is "mixed," which is particularly interesting since it allows us to adequately analyse the MIMO case. Consider an arbitrary, causal, linear, shift-invariant system, described by its state-space realisation

$$x(k+1) = Ax(k)+Bu(k), \quad x(0)=x_0,$$
$$y(k) = Cx(k)+Du(k),$$

where, for all $k \in \mathbb{N}$, $x(k) \in \mathbb{R}^n$ is the state vector, $u(k) \in \mathbb{R}^m$ is the input and $y(k) \in \mathbb{R}^m$ is the output. The real matrices $A$, $B$, $C$ and $D$ have compatible dimensions and $A$ is Schur stable[4] and nonsingular. Denoting $M(z) := C(zI-A)^{-1}B+D$ and $M^*(z) := [M(z^{-1})]^T$ gives

$$M^*(z) = \left[\begin{array}{c|c} A^{-T} & -A^{-T}C^T \\ \hline B^TA^{-T} & D^T - B^TA^{-T}C^T \end{array}\right] \tag{1}$$

from [49, Section 21.4].[5] Let $G_1(e^{j\theta}) := -kM^*(e^{j\theta})M(e^{j\theta}) + M^*(e^{j\theta}) + M(e^{j\theta}) - \ell I$ and $G_2(e^{j\theta}) := -M^*(e^{j\theta})M(e^{j\theta}) + \epsilon^2 I$. Consider the following two results.

[4] I.e. $\rho(A) < 1$ [49, Section 21.1].
[5] The notation on the right-hand side of (1) denotes a state-space realisation.

**Lemma 8.** *Suppose that $k, \ell \in \mathbb{R}$ and consider $G_1(e^{j\theta})$ as defined above. Let $Y := I-kD$ and suppose that $X_1 := -kD^TD+D^T+D-\ell I$ and $\tilde{X}_1 := X_1 - B^TA^{-T}C^TY$ are invertible. For some $\theta_0 \in [-\pi,\pi]$, the matrix $G_1(e^{j\theta_0})$ has a zero eigenvalue if and only if the simplectic matrix $S_1$ has an eigenvalue on the unit circle at the point $e^{j\theta_0}$, where*

$$S_1 := \begin{pmatrix} E_1+U_1E_1^{-T}V_1 & -U_1E_1^{-T} \\ -E_1^{-T}V_1 & E_1^{-T} \end{pmatrix}$$

*and $E_1 := A-BX_1^{-1}Y^TC$, $U_1 := -BX_1^{-1}B^T$, $V_1 := kC^TC+C^TYX_1^{-1}Y^TC$.*

**Proof.** Given that

$$\begin{bmatrix} (e^{j\theta_0}I-A)^{-1} & 0 \\ -k(e^{j\theta_0}I-A^{-T})^{-1}A^{-T}C^TC(e^{j\theta_0}I-A)^{-1} & (e^{j\theta_0}I-A^{-T})^{-1} \end{bmatrix}$$
$$= \left[e^{j\theta_0}I - \begin{pmatrix} A & 0 \\ -kA^{-T}C^TC & A^{-T} \end{pmatrix}\right]^{-1},$$

note that $G_1(e^{j\theta_0}) = -k[-B^TA^{-T}(e^{j\theta_0}I-A^{-T})^{-1}A^{-T}C^T+D^T-B^TA^{-T}C^T][C(e^{j\theta_0}I-A)^{-1}B+D]-B^TA^{-T}(e^{j\theta_0}I-A^{-T})^{-1}A^{-T}C^T+D^T-B^TA^{-T}C^T+C(e^{j\theta_0}I-A)^{-1}B+D-\ell I = \overline{C}(e^{j\theta_0}I-\overline{A})^{-1}\overline{B}+\tilde{X}_1$, where

$$\overline{A} := \begin{pmatrix} A & 0 \\ -kA^{-T}C^TC & A^{-T} \end{pmatrix}, \quad \overline{B} := \begin{pmatrix} B \\ A^{-T}C^TY \end{pmatrix}$$

and

$$\overline{C} := (Y^TC+kB^TA^{-T}C^TC \;\; -B^TA^{-T}),$$

using [8, Lemma 3]. Then, in the manner of [39, Lemma 1],

$$\begin{aligned} \det(G_1(e^{j\theta_0})) &= \det(\overline{C}(e^{j\theta_0}I-\overline{A})^{-1}\overline{B}+\tilde{X}_1) \\ &= \det(\tilde{X}_1)\det(I+\tilde{X}_1^{-1}\overline{C}(e^{j\theta_0}I-\overline{A})^{-1}\overline{B}) \\ &= \det(\tilde{X}_1)\det(I+(e^{j\theta_0}I-\overline{A})^{-1}\overline{B}\tilde{X}_1^{-1}\overline{C}) \\ &\quad (\because \text{Sylvester's Determinant Theorem}) \\ &= \det(\tilde{X}_1)\det((e^{j\theta_0}I-\overline{A})^{-1})\det(e^{j\theta_0}I-\overline{A}+\overline{B}\tilde{X}_1^{-1}\overline{C}) \\ &= \det(\tilde{X}_1)\det((e^{j\theta_0}I-A)^{-1})\det((e^{j\theta_0}I-A^{-T})^{-1}) \\ &\quad \times \det(e^{j\theta_0}I-\tilde{H}_1), \end{aligned}$$

where $\tilde{H}_1 := \overline{A}-\overline{B}\tilde{X}_1^{-1}\overline{C}$. Since $A$ is Schur stable, then $\det(e^{j\theta_0}I-A) \neq 0$ for any $\theta_0 \in \mathbb{R}$; and $e^{j\theta_0}I-A$ is invertible and so $\det((e^{j\theta_0}I-A)^{-1}) \neq 0$. Similarly, $\det((e^{j\theta_0}I-A^{-T})^{-1}) \neq 0$ noting that $(-1)^n \det(e^{j\theta_0}I)\det(e^{-j\theta_0}I-A)\det(A^{-1}) = \det(e^{j\theta_0}I-A^{-1}) = \det(e^{j\theta_0}I-A^{-T})$ from [26, Eq. 6.1.4]. Thus, $G_1(e^{j\theta_0})$ has a zero eigenvalue if and only if $\det(e^{j\theta_0}I-\tilde{H}_1)=0$, i.e. $\tilde{H}_1$ has an eigenvalue on the unit circle at the point $e^{j\theta_0}$. Finally, $\tilde{H}_1 = S_1$ via matrix inversion identities [49, Section 2.3]. □

**Lemma 9.** *Suppose that $\epsilon \in \mathbb{R}\backslash\{0\}$ and consider $G_2(e^{j\theta})$ as defined at the beginning of the section. Suppose that $-DD^T+\epsilon^2 I$, $X_2 := -D^TD+\epsilon^2 I$ and $\tilde{X}_2 := X_2+B^TA^{-T}C^TD$ are invertible. For some $\theta_0 \in [-\pi,\pi]$, the matrix $G_2(e^{j\theta_0})$ has a zero eigenvalue if and only if the simplectic matrix $S_2$ has an eigenvalue on the unit circle at the point $e^{j\theta_0}$, where*

$$S_2 := \begin{pmatrix} E_2+U_2E_2^{-T}V_2 & -U_2E_2^{-T} \\ -E_2^{-T}V_2 & E_2^{-T} \end{pmatrix}$$

*and $E_2 := A+BX_2^{-1}D^TC$, $U_2 := -BX_2^{-1}B^T$, $V_2 := \epsilon^2C^T(-DD^T+\epsilon^2 I)^{-1}C$.*

**Proof.** Given (5) with $k=1$, note that $G_2(e^{j\theta_0}) = -[-B^TA^{-T}(e^{j\theta_0}I-A^{-T})^{-1}A^{-T}C^T+D^T-B^TA^{-T}C^T][C(e^{j\theta_0}I-A)^{-1}B+D]+\epsilon^2 I = \overline{C}(e^{j\theta_0}I-\overline{A})^{-1}\overline{B}+\tilde{X}_2$, where

$$\overline{A} := \begin{pmatrix} A & 0 \\ -A^{-T}C^TC & A^{-T} \end{pmatrix}, \quad \overline{B} := \begin{pmatrix} B \\ -A^{-T}C^TD \end{pmatrix}$$

and

$$\overline{C} := (-D^TC+B^TA^{-T}C^TC \;\; -B^TA^{-T}),$$

from [8, Lemma 3]. Then, in the manner of [39, Lemma 1] and similar to the proof of Lemma 8, $\det(G_2(e^{j\theta_0})) = \det(\tilde{X}_2)\det((e^{j\theta_0}I - A)^{-1})\det((e^{j\theta_0}I - A^{-T})^{-1})\det(e^{j\theta_0}I - \tilde{H}_2)$, where $\tilde{H}_2 := \overline{A} - \overline{B}\tilde{X}_2^{-1}\overline{C}$. The remainder of this proof follows in the manner of the proof to Lemma 8. □

**Observation 11.** *Lemmas 8 and 9 are presented in a general mathematical sense, in that $k$, $\ell$ and $\epsilon$ can take a wider range of values than are applicable for the "mixed" systems framework.*

Lemmas 8 and 9 can be utilised for testing whether a system of the form given at the beginning of the section is "mixed" in the following manner.

Let $\tilde{G}_1(e^{j\theta})$ denote $G_1(e^{j\theta})$, where $k = \ell = 0$. Similarly, let $\tilde{G}_2(e^{j\theta})$ denote $G_2(e^{j\theta})$, where $\epsilon = 1$. Upon applying Lemmas 8 and 9 to $\tilde{G}_1(e^{j\theta})$ and $\tilde{G}_2(e^{j\theta})$, respectively, set

$\Psi_p := \{\theta \in [-\pi, \pi] : S_1$ has an eigenvalue on the unit circle at $e^{j\theta}\}$

and

$\Psi_s := \{\theta \in [-\pi, \pi] : S_2$ has an eigenvalue on the unit circle at $e^{j\theta}\}$.

**Observation 12.** *It has been assumed that the system does not have a strictly proper transfer function in order to facilitate the application of Lemma 8 to $\tilde{G}_1(e^{j\theta})$.*

Next, divide two intervals of $-\pi$ to $\pi$ up into smaller intervals, where any elements of $\Psi_p$ and $\Psi_s$ are set as open interval endpoints, as follows:

Division Group 1 $:= [-\pi, \theta_{p_1}), (\theta_{p_1}, \theta_{p_2}), \ldots, (\theta_{p_{\overline{n}-1}}, \theta_{p_{\overline{n}}}), (\theta_{p_{\overline{n}}}, \pi]$

Division Group 2 $:= [-\pi, \theta_{s_1}), (\theta_{s_1}, \theta_{s_2}), \ldots, (\theta_{s_{\overline{m}-1}}, \theta_{s_{\overline{m}}}), (\theta_{s_{\overline{m}}}, \pi]$

where $\overline{n}$ is the number of elements in $\Psi_p$; $\overline{m}$ is the number of elements in $\Psi_s$; $\theta_{p_1}, \theta_{p_2}, \ldots, \theta_{p_{\overline{n}}}$ denote the elements of $\Psi_p$ listed in increasing order; and $\theta_{s_1}, \theta_{s_2}, \ldots, \theta_{s_{\overline{m}}}$ denote the elements of $\Psi_s$ listed in increasing order. If $\Psi_p$ is empty, then $\overline{n} = 0$ and Division Group 1 consists of the single interval $[-\pi, \pi]$; similarly, if $\Psi_s$ is empty, then $\overline{m} = 0$ and Division Group 2 consists of the single interval $[-\pi, \pi]$. If $\theta_{p_1} = -\pi$ and $\theta_{p_{\overline{n}}} = \pi$, then Division Group 1 becomes $(-\pi, \theta_{p_2}), (\theta_{p_2}, \theta_{p_3}), \ldots, (\theta_{p_{\overline{n}-1}}, \pi)$. Similarly, if $\theta_{s_1} = -\pi$ and $\theta_{s_{\overline{m}}} = \pi$, then Division Group 2 becomes $(-\pi, \theta_{s_2}), (\theta_{s_2}, \theta_{s_3}), \ldots, (\theta_{s_{\overline{m}-1}}, \pi)$.

Finally, identify the sign definiteness of $\tilde{G}_1(e^{j\theta})$ over each of the individual intervals in Division Group 1, and the sign definiteness of $\tilde{G}_2(e^{j\theta})$ over each of the individual intervals in Division Group 2. Determining the sign definiteness over any of these intervals can be achieved by checking the sign definiteness at a single $\theta$ from within the interval, e.g. at the interval midpoint. Let $I_{\tilde{G}_1}$ denote the set of $\theta$ belonging to those intervals over which $\tilde{G}_1(e^{j\theta}) > 0$, and $I_{\tilde{G}_2}$ denote the set of $\theta$ belonging to those intervals over which $\tilde{G}_2(e^{j\theta}) > 0$. Then, implement the following result.

**Theorem 10.** *The following two statements are equivalent.*

(a) *A discrete-time system, as described at the beginning of the section, is "mixed."*
(b) $I_{\tilde{G}_1} \cup I_{\tilde{G}_2} = \{\theta \in \mathbb{R} : -\pi \le \theta \le \pi\}$

**Proof.** Let us suppose that (a) holds. Then, from Definition 4, an input–output stable, discrete-time system with square, proper, real-rational transfer function matrix $M(z)$ is "mixed" if, for each $\theta \in [-\pi, \pi]$, Property (i) and/or Property (ii) hold. Whenever valid for $\theta \in [-\pi, \pi]$, Properties (i) and (ii) clearly imply that $\tilde{G}_1(e^{j\theta}) > 0$ and $\tilde{G}_2(e^{j\theta}) > 0$, respectively. Therefore, the "mixedness" of the system ensures $I_{\tilde{G}_1} \cup I_{\tilde{G}_2} = \{\theta \in \mathbb{R} : -\pi \le \theta \le \pi\}$.

Conversely, let us suppose that (b) is valid. Define the set $\mathcal{K} = I_{\tilde{G}_1} \cap I_{\tilde{G}_2}$, which corresponds to the overlapping intervals where $\tilde{G}_1(e^{j\theta}) > 0$ and $\tilde{G}_2(e^{j\theta}) > 0$. This set is then written as

$$\mathcal{K} = \bigcup_{k=1}^{K} L_k$$

where the intervals $L_k$, $k \in \{1, \ldots, K\}$, satisfy the following property: for all $\theta \in L_k$ and all $\phi \in L_{k+1}$, $\theta < \phi$. Note that this *topological sorting* of the intervals $L_k$ can always be done. Thus, for each $k \in \{1, \ldots, K\}$, take any $\theta_k$ which is an interior point of $L_k$. Without any loss of generality, let us take $\theta_k$ as the midpoint of the $L_k$. Define the closed intervals $I_k = [\theta_k, \theta_{k+1}]$, $k \in \{0, \ldots, K\}$, where $\theta_0 = -\pi$ and $\theta_{K+1} = \pi$. By construction, it is important to note that $I_k \subset I_{\tilde{G}_1}$ or $I_k \subset I_{\tilde{G}_2}$, for each $k \in \{0, \ldots, K\}$. Indeed, since the endpoints of $I_k$ are elements of $I_{\tilde{G}_1} \cap I_{\tilde{G}_2}$, if $I_k \not\subset I_{\tilde{G}_1}$ or $I_k \not\subset I_{\tilde{G}_2}$, the hypothesis $I_{\tilde{G}_1} \cup I_{\tilde{G}_2} = \{\theta \in \mathbb{R} : -\pi \le \theta \le \pi\}$ would not be satisfied, from the definition of $L_k$. Now, the compactness of $I_k$ together with the validity of either $\tilde{G}_1(e^{j\theta}) > 0$ or $\tilde{G}_2(e^{j\theta}) > 0$, for all $\theta \in I_k$, implies that either Property (i) or Property (ii) are valid for all $\theta \in I_k$, respectively. Finally, noticing that, from the definition of $L_k$ and $I_k$, $\bigcup_{k=0}^{K} I_k = [-\pi, \pi]$, we conclude that Properties (i) or (ii) are valid for each $\theta \in [-\pi, \pi]$, which implies that the system is "mixed." □

Some remarks can be made at this point on the proposed algorithm. It is essential to note that the results developed in this section must be adapted in order to deal with some special cases that were not considered. Indeed, for $k = \ell = 0$, a necessary condition for the nonsingularity of the matrix $X_1$ in Lemma 8 is $D \neq 0$. Therefore, as stated in Remark 12, strictly proper transfer functions cannot be analysed with the straightforward adoption of the procedure described above. Moreover, in Lemma 9, it is demanded that $X_2$ is nonsingular, which requires that $D^{TD}$ or $DD^T$ does not have 1 as an eigenvalue. These cases may be treated computationally, but the test becomes only sufficient for the determination of the "mixed" property of a system (as follows).

In order to deal with these cases, we briefly describe a procedure that is based on its continuous-time version, presented in [19].

- *$X_1$ is singular for $k = \ell = 0$.* In this case, we must define the tolerances $\text{tol}_k$ and $\text{tol}_\ell$, which are related to the parameters $k$ and $\ell$, respectively. It is recommended that these constants are taken as very small positive numbers, which allows us to get arbitrarily close to the limit case $k = \ell = 0$, if it is necessary. Then, we perform the same analysis as we did in the previous setting, but we consider $G_1(e^{j\theta})$ instead of $\tilde{G}_1(e^{j\theta})$ to define Division Group 1 and the set $I_{G_1}$ for given $k \ge \text{tol}_k$ and $\ell \ge \text{tol}_\ell$. If $I_{G_1} \cup I_{\tilde{G}_2} = [-\pi, \pi]$, then the system is "mixed." Else, we must update the parameters $k, \ell$, reducing their value and redoing the whole process until the affirmative case is obtained or the tolerances on $k, \ell$ are violated, case in which the test is inconclusive.
- $X_2$ is singular for $\epsilon = 1$. Analogous to the previous case, define the tolerance $\text{tol}_\epsilon$, which has to be taken arbitrarily close to 1, but smaller, in order to include the pathological case as its limit. Then, the same procedure adopted for the first case is considered here; that is, $G_2(e^{j\theta})$ is used to define Division Group 2 and the set $I_{G_2}$ for a given $\epsilon \le \text{tol}_\epsilon$. Thus, if $I_{\tilde{G}_1} \cup I_{G_2} = [-\pi, \pi]$, the system is "mixed." Else, the same iterative procedure has to be adopted but $\epsilon$ shall have its value increased.
- $X_1$ is singular for $k = \ell = 0$ and $X_2$ is singular for $\epsilon = 1$. This case is treated combining both techniques described above.

It is important to note that the tolerances adopted in the procedure described above cannot be taken arbitrarily small in practice. Therefore, from a computational point of view, the test may be deemed inconclusive as to whether the system is "mixed" due to numerical limitations. In this situation, another procedure has to be adopted in order to verify "mixedness". Finally, we

recall that the test proposed in this section can also be used to verify the "mixed" property of continuous-time systems. Indeed, a continuous-time system can be discretised by Tustin's Approximation (which preserves "mixedness") and then be tested by the proposed algorithm.

## 6. Examples

The following examples demonstrate the spectral-based characterisation developed in Section 5.

**Example 2** (*SISO system*). Consider again the continuous-time transfer function defined in Example 1. Using several Nyquist Diagrams, we concluded that the system was "mixed" for $T<0.53$ s, and that this property was lost for $0.53\text{ s}\le T\le 1.00$ s. This characteristic is also perceived by the result presented in Theorem 10. For example, for $T=0.30$ s, we have the associated simplectic matrices

$$S_1=\begin{pmatrix}0.3568 & 2.2778 & -1.6761 & 0.2172\\ 0.1920 & 1.3985 & -0.2932 & 0.0380\\ 0 & -4.9103 & 5.3863 & -0.8729\\ 0 & 9.1248 & -4.8012 & 1.6222\end{pmatrix},$$

$$S_2=\begin{pmatrix}-0.0972 & 1.1332 & 2.3539 & -0.7440\\ 0.2325 & 1.9736 & 2.0223 & -0.6392\\ 0 & 5.6938 & 4.9680 & -2.4293\\ 0 & 2.3801 & 6.3770 & -1.0155\end{pmatrix}.$$

Both matrices have eigenvalues on the unit circle. The matrix $S_1$ has two eigenvalues on the unit circle, $\exp(\pm j0.2898)$, which indicates for us to break the phase interval $[-\pi,\pi]$ into $[-\pi,-0.2898)$, $(-0.2898, 0.2898)$ and $(0.2989,\pi]$. Examining the sign of $\tilde{G}_1(e^{j\theta})$ at some phase point from each of these intervals, we may conclude that $I_{\tilde{G}_1}=(-0.2898, 0.2898)$. The matrix $S_2$ also presents two eigenvalues with unitary moduli, $\exp(\pm j0.2556)$, and breaking the phase interval up in a similar way, and evaluating $\tilde{G}_2(e^{j\theta})$ at one point from each of these intervals, gives us $I_{\tilde{G}_2}=[-\pi,-0.2556)\cup(0.2556,\pi]$. Therefore, since $I_{\tilde{G}_1}\cup I_{\tilde{G}_2}=[-\pi,\pi]$, we conclude that, as expected, the system is "mixed."

Similarly, now let us consider $T=1.00$ s. It is possible to verify that $S_1$ has two eigenvalues on the unit circle, at $\exp(\pm j0.7569)$, and so does $S_2$, at $\exp(\pm j0.9523)$. Performing the same steps that we did previously, it follows that $I_{\tilde{G}_1}=(-0.7569, 0.7569)$ and $I_{\tilde{G}_2}=[-\pi,-0.9523)\cup(0.9523,\pi]$, and therefore, the system is not "mixed." Fig. 6 illustrates both cases, showing the values of $\tilde{G}_1(e^{j\theta})$ and $\tilde{G}_2(e^{j\theta})$, where $\theta\in[-\pi,\pi]$, for $T=0.30$s (shown by the dashed blue lines) and $T=1.00$ s (shown by the continuous red lines).

**Example 3** (*MIMO system*). Consider the MIMO discrete-time system given by its state-space realisation $(A,B,C,D)$ as described at the beginning of Section 5 with

$$A=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0.102 & -0.16 & -0.3\end{bmatrix},\quad B=\begin{bmatrix}0.3 & 0\\ 0.3 & 0.3\\ 0 & 0.3\end{bmatrix},$$

$$C=\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix},\quad D=\begin{bmatrix}0.5 & 0\\ 0 & 0.1\end{bmatrix}$$

This linear system is clearly asymptotically stable because the eigenvalues of the dynamic matrix $A$ are 0.30, $-0.30\pm j0.50$. Adopting the same procedure that we did in the previous example, we first analyse the eigenvalues of the simplectic matrices $S_1$ and $S_2$ that are on the unit circle. Since $S_1$ has two eigenvalues on the unit circle, $\exp(\pm j1.4803)$, we conclude that $I_{\tilde{G}_1}=(-1.4803, 1.4803)$. Similarly, $S_2$ has unitary eigenvalues $\exp(\pm j0.7816)$ and this implies that $I_{\tilde{G}_2}=[-\pi,-0.7816)\cup(0.7816,\pi]$. Since $I_{\tilde{G}_1}\cup I_{\tilde{G}_2}=[-\pi,\pi]$, we conclude that this MIMO system is "mixed." This result is validated in Fig. 7, where we show the minimum eigenvalues of $G_1(e^{j\theta})$ and $G_2(e^{j\theta})$, for $k=\ell=0.1$, $\epsilon^2=0.9$ and $\theta\in[-\pi,\pi]$, and this allows us to verify that Properties (i) and (ii) are valid for each point in this interval.

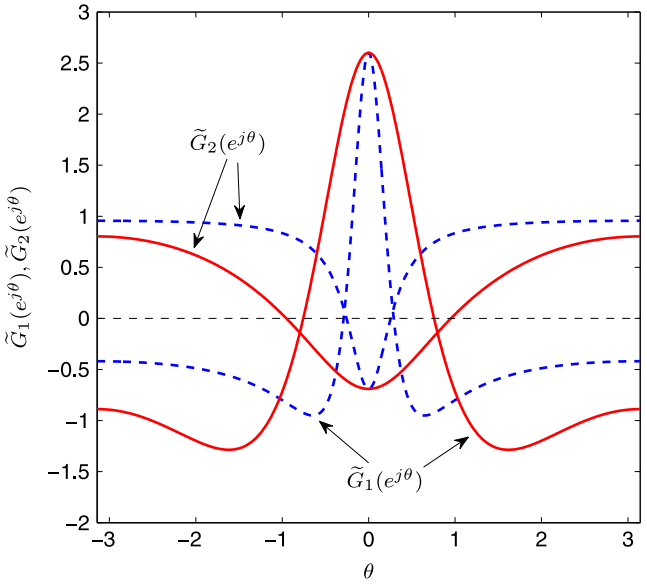



**Fig. 6.** Values of $\tilde{G}_1(e^{j\theta})$ and $\tilde{G}_2(e^{j\theta})$ for $T=0.30$, 1.00 s. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this paper.)

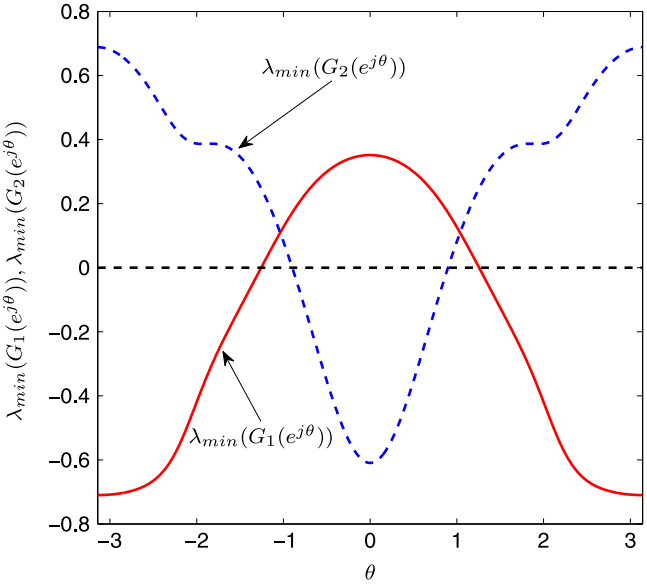



**Fig. 7.** Minimum eigenvalues of $G_1(e^{j\theta})$ and $G_2(e^{j\theta})$ for $k=\ell=0.1$, $\epsilon^2=0.9$.

**Example 4** (*Strictly proper SISO system*). Consider the strictly proper SISO discrete-time system given by its state-space realisation $(A,B,C,D)$ as described at the beginning of Section 5 with

$$A=\begin{bmatrix}0 & 1\\ 0.21 & 0.4\end{bmatrix},\quad B=\begin{bmatrix}0.4\\ 0.4\end{bmatrix},\quad C=\begin{bmatrix}1 & 0\end{bmatrix},\quad D=0$$

It is clear that $A$ is Schur stable because its eigenvalues are $-0.30$ and 0.70. Since $D=0$, $X_1$ is singular for $k=\ell=0$, which means that the modified version of our analysis has to be adopted. Therefore, using $k=\ell=0.01$, the eigenvalues of the simplectic matrix $S_1$ that are on

the unit circle are $\exp(\pm j0.6896)$ and, inspecting the sign of $G_1(e^{j\theta})$ in each interval of the modified Division Group 1, we obtain $I_{G_1} = (-0.6896, 0.6896)$. Since $X_2$ is nonsingular, we perform the exact same procedure we did in the previous examples. Thus, we verify that $S_2$ has two eigenvalues on the unit circle ($\exp(\pm j0.4661)$), which implies $I_{\tilde{G}_2} = [-\pi, -0.4661) \cup (0.4661, \pi]$. Since $I_{G_1} \cup I_{\tilde{G}_2} = [-\pi, \pi]$, we conclude that the system is "mixed." This conclusion is easily validated by examining the Nyquist plot of the system, as we did in Section 2.

## 7. Conclusions

In this paper, "mixed" systems were characterised in a discrete-time setting. Classical discretisation methods were investigated to verify if "mixedness" is preserved when these procedures are adopted. Since this remarkable property is not always preserved, a computational method to test whether a system is "mixed" or not was developed. A feedback stability result based on Nyquist's criterion that resembles the classical passivity and small-gain theorems was also presented. All these results provide the foundations for future studies concerning discretisation methods that preserve "mixedness."

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