Feedback Control of Negative-Imaginary Systems

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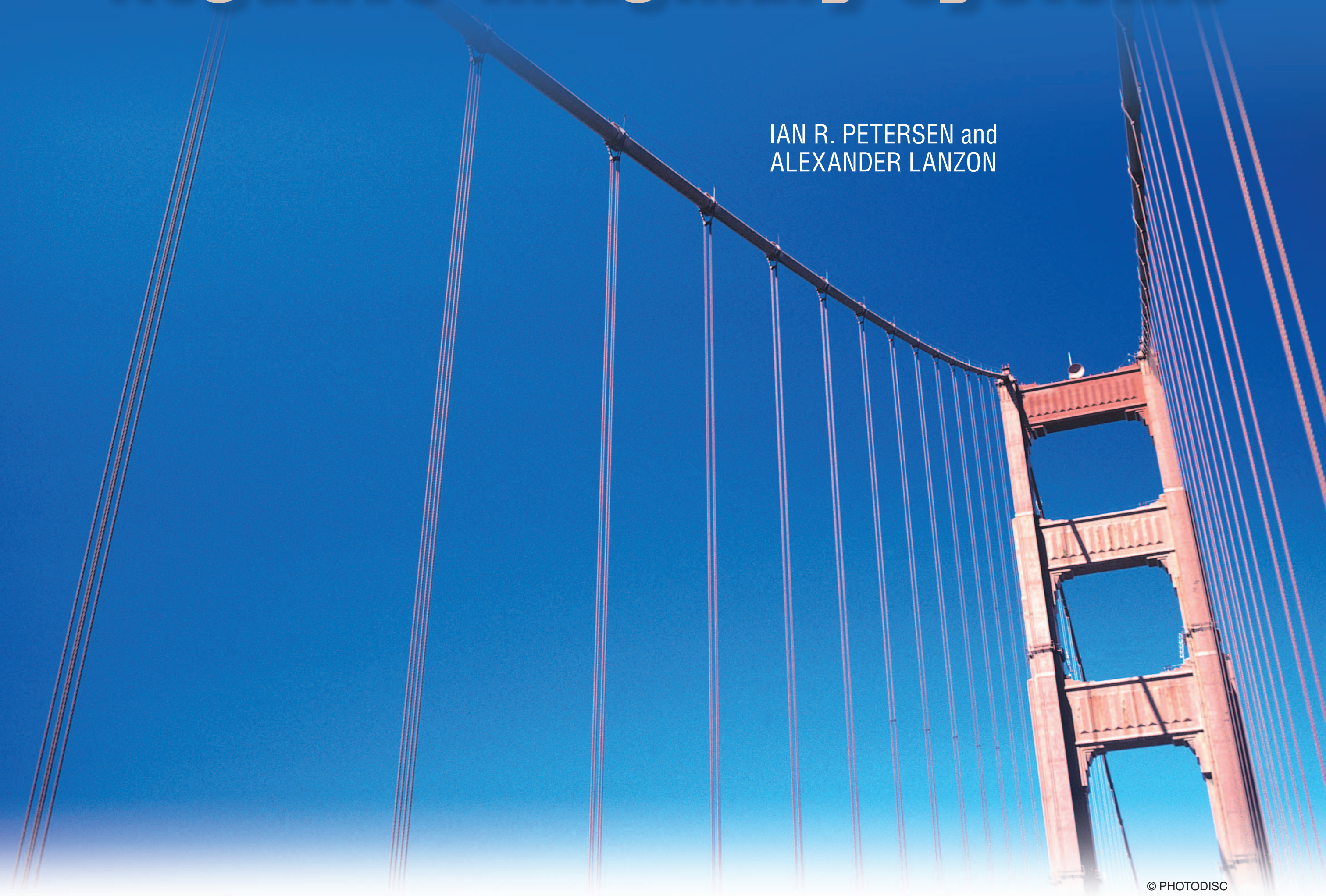

FLEXIBLE STRUCTURES WITH COLOCATED ACTUATORS AND SENSORS

Highly resonant dynamics can severely degrade the performance of technological systems. Structural modes in machines and robots, ground and aerospace vehicles, and precision instrumentation, such as atomic force microscopes and optical systems, can limit the ability of control systems to achieve the desired performance. Consequently, control systems must be designed to suppress the effects of these dynamics, or at least avoid exciting them beyond open-loop levels. Open-loop techniques for highly resonant systems, such as input shaping [1], as well as closed-loop techniques, such as damping augmentation [2], [3], can be used for this purpose.

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Structural dynamics are often difficult to model with high precision due to sensitivity to boundary conditions as well as aging and environmental effects. Therefore, active damping augmentation to counteract the effects of external commands and disturbances must account for parametric uncertainty and unmodeled dynamics. This problem is simplified to some extent by using force actuators combined with colocated measurements of velocity, position, or acceleration, where colocated refers to the fact that the sensors and actuators have the same location and the same direction. Colocated control with velocity measurements, called *negative-velocity feedback*, can be used to directly increase the effective damping, thereby facilitating the design of controllers that guarantee closed-loop stability in the presence of plant parameter variations and unmodeled dynamics [1], [4]. This guaranteed stability property can be established by using results on passive systems [5], [6]. However, the theoretical properties of negative-velocity feedback are based on the idealized assumption of colocation and require the availability of velocity sensors, which may be expensive. Also, the choice of measured variable may depend on whether the desired objective is shape control or damping augmentation.

An alternative approach to negative-velocity feedback is *positive-position feedback*, where position sensors are used in place of velocity sensors. Although position sensors can facilitate the objective of shape control, it is less obvious how they can be used for damping augmentation. Nevertheless, it is shown in [7] and [8] that a positive-position feedback controller can be designed to increase the damping of the modes of a flexible structure. Furthermore, this controller is robust against uncertainty in the modal frequencies as well as unmodeled plant dynamics. As shown in [7]–[10], the robustness properties of positive-position feedback are similar to those of negative-velocity feedback.

This article investigates the robustness of positive-position feedback control of flexible structures with colocated force actuators and position sensors. In particular, the theory of negative-imaginary systems [9], [10] is used to reveal the robustness properties of multi-input, multi-output (MIMO) positive-position feedback controllers and related types of controllers for flexible structures [1], [11]–[14]. The negative-imaginary property of linear systems can be extended to nonlinear systems through the notion of counterclockwise input-output dynamics [15], [16]. It is shown in [17] for the single-input, single-output (SISO) linear case that the results of [15] and [16] guarantee the stability of a positive-position feedback control system in the presence of unmodeled dynamics and parameter uncertainties that maintain the negative-imaginary property of the plant.

Positive-position feedback can be regarded as one of the last areas of classical control theory to be encompassed by modern control theory. In this article, positive-position feedback, negative-imaginary systems, and related control methodologies are brought together with the underlying systems theory.

Table 1 summarizes notation used in this article, while Table 2 lists acronyms.

FLEXIBLE STRUCTURE MODELING

In modeling an undamped flexible structure with a single actuator and a single sensor, modal analysis can be applied to the relevant partial differential equation [18], leading to the transfer function

$$P(s) = \sum_{i=1}^{\infty} \frac{\phi_i(s)}{s^2 + \omega_i^2}, \tag{1}$$

where each $\omega_i > 0$ is a modal frequency, the functions $\phi_i(s)$ are first-order polynomials, and $\omega_i \neq \omega_j$ for $i \neq j$. In the case of a structure with a force actuator and colocated velocity sensor, the form of the numerator of (1) is determined by the passive nature of the flexible structure. Since the product $u(t)y(t)$ of the force actuator input $u(t)$ and the velocity sensor output $y(t)$ represents the power provided by the actuator to the structure at time t, conservation of energy implies

$$E(t) \leq E(0) + \int_0^t u(\tau)y(\tau)d\tau \tag{2}$$

TABLE 1 Notation.

A^*	Complex conjugate transpose of the complex matrix A.
A^T	Transpose of the matrix A.
$A > 0$	The matrix A is positive definite.
$A \geq 0$	The matrix A is positive semidefinite.
$\Re[s]$	Real part of the complex number s.
$\Im[s]$	Imaginary part of the complex number s.
$\Im_H[A]$	Hermitian-imaginary part $-\frac{1}{2}\jmath[A - A^*]$.
$\lambda_{max}(A)$	Maximum eigenvalue of the matrix A whose eigenvalues are all real.
$\sigma_{max}(A)$	Maximum singular value of the matrix A.
CRHP	Closed right half of the complex plane.
ORHP	Open right half of the complex plane.
CLHP	Closed left half of the complex plane.
OLHP	Open left half of the complex plane.

TABLE 2 List of Acronyms.

SISO	Single-input, single-output
MIMO	Multi-input, multi-output
NI	Negative-imaginary
SNI	Strictly negative-imaginary
LMI	Linear matrix inequality
RLC	Resistor, inductor, capacitor

This article investigates the robustness of positive-position feedback control of flexible structures with colocated force actuators and position sensors.

for all $t \geq 0$, where $E(t) \geq 0$ represents the energy stored in the system at time t, and $E(0)$ represents the initial energy stored in the system. In this case, the variables $u(t)$ and $y(t)$ are *dual*. The passivity condition (2) implies that the transfer function $P(s)$ is positive real according to the following definition [5].

Definition 1 [19], [20]

The square transfer function matrix $P(s)$ is *positive real* if the following conditions are satisfied:

1) All of the poles of $P(s)$ lie in the closed left half of the complex plane (CLHP).
2) For all s in the open right half of the complex plane (ORHP),

$$P(s) + P^*(s) \geq 0. \tag{3}$$

If $P(s)$ is positive real, then it follows that [19], [20]

$$P(\jmath\omega) + P^*(\jmath\omega) \geq 0 \tag{4}$$

for all $\omega \in \mathbb{R}$ such that $s = \jmath\omega$ is not a pole of $P(s)$. If $P(s)$ is a SISO transfer function, then, for all $\omega \in \mathbb{R}$ such that $s = \jmath\omega$ is neither a pole nor a zero of $P(s)$, (4) is equivalent to the phase condition $\angle P(\jmath\omega) \in [-\pi/2, \pi/2]$.

Definition 2 [19]

The nonzero square transfer function matrix $P(s)$ is *strictly positive real* if there exists $\varepsilon > 0$ such that the transfer function matrix $P(s-\varepsilon)$ is positive real.

If $P(s)$ is strictly positive real, then it follows [19] that all of the poles of $P(s)$ lie in the open left half of the complex plane (OLHP) and

$$P(\jmath\omega) + P^*(\jmath\omega) > 0 \tag{5}$$

for all $\omega \in \mathbb{R}$. If P(s) is a SISO transfer function, then (5) holds for all $\omega \in \mathbb{R}$ such that $s = \jmath\omega$ is neither a pole nor a zero of $P(s)$ if and only if the phase condition $\angle P(\jmath\omega) \in (-\pi/2, \pi/2)$ holds for all $\omega \in \mathbb{R}$ such that $s = \jmath\omega$ is neither a pole nor a zero of $P(s)$.

Now consider the positive-real transfer function from force actuation to velocity measurement given by

$$P(s) = \sum_{i=1}^{\infty} \frac{\psi_i^2 s}{s^2 + \kappa_i s + \omega_i^2}, \tag{6}$$

where, for all i, $\kappa_i > 0$ is the viscous damping constant associated with the ith mode and $\omega_i > 0$. The transfer function (6) satisfies the phase condition $\angle P(\jmath\omega) \in (-\pi/2, \pi/2)$ for all $\omega > 0$. However, (6) has a zero at the origin, and thus (5) is not satisfied for $\omega = 0$. Hence, (6) is not strictly positive real.

Now consider a lightly damped flexible structure with m colocated sensor and actuator pairs. Let $u_1(t), \ldots, u_m(t)$ denote the force actuator input signals, and let $y_1(t), \ldots, y_m(t)$ denote the corresponding velocity sensor output signals. The actuator and sensor in the ith colocated actuator and sensor pair are *dual* when the product $u_i(t)y_i(t)$ is equal to the power provided to the structure by the ith actuator at time t. Now, we let

$$Y(s) = P(s)U(s),$$

where

$$U(s) = \begin{bmatrix} U_1(s) \\ \vdots \\ U_m(s) \end{bmatrix}, \quad Y(s) = \begin{bmatrix} Y_1(s) \\ \vdots \\ Y_m(s) \end{bmatrix}.$$

For $i = 1, 2, \ldots, m$, $U_i(s)$ and $Y_i(s)$ are the Laplace transforms of $u_i(t)$ and $y_i(t)$, respectively, and $P(s)$ is the transfer function matrix of the system. Then $P(s)$ is positive real and has the form

$$P(s) = \sum_{i=1}^{\infty} \frac{s}{s^2 + \kappa_i s + \omega_i^2} \psi_i \psi_i^{\mathrm{T}}, \tag{7}$$

where, for all i, $\kappa_i > 0$, $\omega_i > 0$, and ψ_i is an $m \times 1$ vector. A review of positive-real and passivity theory is given in "What Is Positive-Real and Passivity Theory?"

NEGATIVE-IMAGINARY SYSTEMS

Mechanical structures with colocated force actuators and position sensors do not yield positive-real systems because the product of force and position is not equal to the power provided by the actuator [9], [10]. In this case, the transfer function matrix from the force actuator inputs $u_1(t), \ldots, u_m(t)$ to the position sensor outputs $y_1(t), \ldots, y_m(t)$ is of the form

$$P(s) = \sum_{i=1}^{\infty} \frac{1}{s^2 + \kappa_i s + \omega_i^2} \psi_i \psi_i^{\mathrm{T}}, \tag{8}$$

where, for all i, $\kappa_i > 0$, $\omega_i > 0$, and ψ_i is an $m \times 1$ vector. Therefore, the *Hermitian-imaginary part*

What Is Positive-Real and Passivity Theory?

A SISO positive-real transfer function has a positive real part at all frequencies; a typical frequency response is depicted in Figure S1. The passivity theorem, which underpins much of the robust and adaptive control literature [S1], concerns the internal stability of the negative-feedback interconnection, as shown in Figure S2, of two positive-real transfer function matrices.

DEFINITION S1 [S2]

The feedback interconnection of two systems with transfer function matrices $M(s)$ and $N(s)$ as shown in Figure S2 is *internally stable* if the interconnection does not contain an algebraic loop and the transfer function matrix from exogenous signals to internal signals has no poles in CRHP.

The following result is the *passivity theorem* [5], [6, Sec. 6.5].

THEOREM S1

The negative-feedback interconnection of the positive-real transfer function matrix $M(s)$ and the strictly positive-real transfer function matrix $N(s)$ is internally stable.

The SISO positive-real transfer function $M(s)$ satisfies $\angle M(\jmath\omega) \in [-\pi/2, \pi/2]$ for all $\omega \geq 0$. Also, the SISO strictly positive-real transfer function $N(s)$ satisfies $\angle N(\jmath\omega) \in (-\pi/2, \pi/2)$ for all $\omega \geq 0$. From $\angle M(\jmath\omega) \in [-\pi/2, \pi/2]$ and $\angle N(\jmath\omega) \in (-\pi/2, \pi/2)$ for all $\omega \geq 0$, it follows that $\angle M(\jmath\omega)N(\jmath\omega) = \angle M(\jmath\omega) + \angle N(\jmath\omega) \in (-\pi, \pi)$ for all $\omega \geq 0$, and hence the Nyquist plot of $M(\jmath\omega)N(\jmath\omega)$ cannot intersect the negative real axis. Consequently, the Nyquist plot of $M(s)N(s)$ cannot encircle the Nyquist point $s = -1 + \jmath 0$, and internal stability of the negative-feedback interconnection of $M(s)$ and $N(s)$ follows from the Nyquist stability criterion as depicted in Figure S3.

The above concepts relating to positive-real systems and the passivity theorem generalize to MIMO linear time-invariant systems and also to a nonlinear and time-varying setting [5].

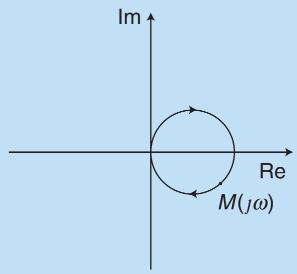

FIGURE S1 The Nyquist plot of the positive-real transfer function $M(s) = 1/(s+1)$. This plot illustrates the fact that, for a single-input, single-output positive-real transfer function, the real part of its frequency response is positive for all frequencies. Consequently, the Nyquist plot is contained in the closed right half of the complex plane.

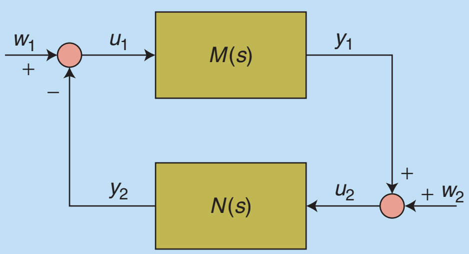

FIGURE S2 A negative-feedback interconnection. This figure shows the negative-feedback interconnection of the transfer functions $M(s)$ and $N(s)$. The stability of this feedback interconnection can be guaranteed using the passivity theorem if $M(s)$ and $N(s)$ are positive real and either $M(s)$ or $N(s)$ is strictly positive real.

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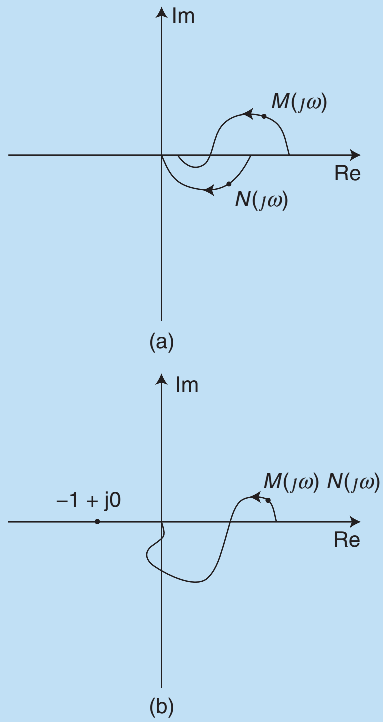

FIGURE S3 The passivity theorem. This plot shows two single-input, single-output positive-real transfer functions $M(s)$ and $N(s)$, both of whose Nyquist plots are contained in the closed right half of the complex plane and one of which is contained in the open right half of the complex plane. Therefore, the Nyquist plot of the loop transfer function $M(s)\ N(s)$ cannot intersect the negative real axis. Since the critical point $s = -1 + \jmath 0$ cannot be encircled, it follows from the Nyquist stability criterion that the negative-feedback interconnection of $M(s)$ and $N(s)$ must be internally stable.

Applications to Electrical Circuits

The properties of a flexible structure with colocated actuators and sensors have counterparts in passive electrical circuits driven by voltage or current sources. Consider an RLC electrical circuit with m voltage or current sources. Assume that, for each voltage source input, the current flowing through the source is the corresponding output of the system. Also, assume that, for each current source input to the system, the voltage across the source is the corresponding output of the system. Let $v_1(t), \ldots, v_m(t)$ denote the voltage signals, and let $i_1(t), \ldots, i_m(t)$ denote the current signals. These signals are dual in the sense that the product $v_k(t)\, i_k(t)$ is equal to the power provided to the circuit by the kth source at time t. Then, let $u(t)$ be the vector of voltage- or current-source inputs at time t, and let $y(t)$ be the vector of voltage or current outputs at time t. Writing

$$Y(s) = P(s)\,U(s),$$

where $P(s)$ is the transfer function matrix of the circuit, it follows that the total power provided to the circuit by the sources at time t is given by $u^{\mathrm{T}}(t)\,y(t)$. As in the case of a flexible structure with colocated sensors and actuators, the transfer function matrix $P(s)$ is positive real.

Now suppose that each voltage source is connected in series with a capacitor and that the corresponding system output is the voltage across this capacitor divided by the capacitance. Also, suppose that each current source is connected in parallel with an inductor and that the corresponding system output is the inductor current divided by the inductance. This situation, which is illustrated in Figure S4, is analogous to the case of a flexible structure with colocated force actuation and position measurements since the current through a capacitor is equal to the capacitance multiplied by the derivative of the voltage across it. Also, the voltage across an inductor is equal to the inductance multiplied by the derivative of the current flowing through it. Hence each output variable is such that its derivative is a variable that is dual to the corresponding source variable. Therefore, the transfer function matrix of the circuit $P(s)$ satisfies the negative-imaginary condition

$$\jmath(P(\jmath\omega) - P^{\mathrm{T}}(-\jmath\omega)) \geq 0.$$

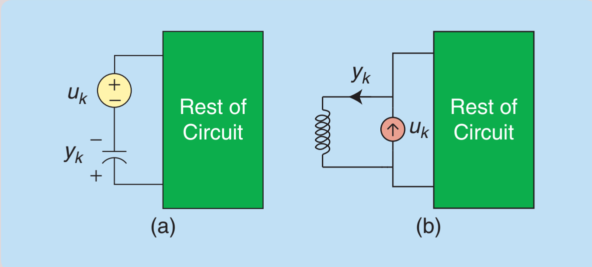

FIGURE S4 A resistor, inductor, capacitor (RLC) electrical circuit, where each input is a voltage or current source. (a) Voltage source with series capacitor voltage and (b) current source with parallel inductor current. Also, each output corresponds to the voltage across a capacitor in series with a voltage source or the current through an inductor in parallel with a current source. This circuit is described by $P(s)$, the transfer function matrix from the vector of inputs to the vector of outputs. The transfer function matrix $P(s)$ is negative-imaginary. That is, the transfer function matrix $P(s)$ has no poles in the closed right half of the complex plane and satisfies the condition $\jmath(P(\jmath\omega) - P^{\mathrm{T}}(-\jmath\omega)) \geq 0$ for all $\omega \geq 0$.

$$\Im_{\mathrm{H}}[P(\jmath\omega)] = -\frac{1}{2}\jmath(P(\jmath\omega) - P^{*}(\jmath\omega))$$

of the frequency response function matrix $P(\jmath\omega)$ satisfies

$$\Im_{\mathrm{H}}[P(\jmath\omega)] = -\omega\sum_{i=1}^{\infty}\frac{\kappa_i}{(\omega_i^2-\omega^2)^2+\omega^2\kappa_i^2}\psi_i\psi_i^{\mathrm{T}} \leq 0 \qquad (9)$$

for all $\omega \geq 0$. That is, the frequency response function matrix for the transfer function matrix (8) has a negative-semidefinite Hermitian-imaginary part for all $\omega \geq 0$. We thus refer to the transfer function matrix $P(s)$ in (8) as negative imaginary. A formal definition follows.

Definition 3

The square transfer function matrix $P(s)$ is *negative-imaginary (NI)* if the following conditions are satisfied:

1) All of the poles of $P(s)$ lie in the OLHP.
2) For all $\omega \geq 0$,

$$\jmath[P(\jmath\omega) - P^{*}(\jmath\omega)] \geq 0. \qquad (10)$$

A linear time-invariant system is NI if its transfer function matrix is NI.

A discussion of negative-imaginary transfer functions arising in electrical circuits is given in "Applications to Electrical Circuits."

In the SISO case, a transfer function is negative imaginary if and only if it has no poles in the closed right half of the complex plane (CRHP) and its phase is in the interval $[-\pi, 0]$ at all frequencies that do not correspond to imaginary-axis poles or zeros. Consequently, the positive-frequency Nyquist plot of a SISO negative-imaginary transfer function lies below the real axis as shown in Figure 1. Hence, a negative-imaginary transfer function can be viewed as a positive-real transfer function rotated clockwise by 90° in the Nyquist plane.

Velocity sensors can be used in negative-velocity feedback control, whereas position sensors can be used in positive-position feedback [1], [7], [8], [11]–[14]. Indeed, positive-real theory and negative-imaginary theory [9], [10] achieve internal stability by a process referred to as *phase stabilization*, since instability is avoided by ensuring

Velocity sensors can be used in negative-velocity feedback control, whereas position sensors can be used in positive-position feedback.

appropriate restrictions on the phase of the corresponding open-loop systems. *Gain stabilization*, which is based on the small-gain theorem [19], guarantees robust stability when the magnitude of the loop transfer function is less than unity at all frequencies. As in positive-real analysis, robust stability of negative-imaginary systems [9], [10] does not require the magnitude of the loop transfer function to be less than unity at all frequencies to guarantee stability. To present results on the robust stability of positive-position feedback and related control schemes, we now define MIMO strictly negative-imaginary systems.

Definition 4

The square transfer function matrix $P(s)$ is *strictly negative-imaginary* (*SNI*) if the following conditions are satisfied:

1) All of the poles of $P(s)$ lie in the OLHP.
2) For all $\omega > 0$,

$$\jmath[P(\jmath\omega) - P^*(\jmath\omega)] > 0. \tag{11}$$

A linear time-invariant system is SNI if its transfer function matrix is SNI.

Lemma 1

If the $m \times m$ transfer function matrix $P_1(s)$ is NI, respectively, SNI, and the $m \times m$ transfer function matrix $P_2(s)$ is NI, then

$$P(s) = P_1(s) + P_2(s) \tag{12}$$

is NI, respectively, SNI.

Proof

This result follows directly from definitions 3 and 4. ■

Theorem 2

Consider the NI transfer function matrices $M(s)$ and $N(s)$, and suppose that the positive-feedback interconnection shown in Figure 2 is internally stable. Then the corresponding $2m \times 2m$ closed-loop transfer function matrix

$$T(s) = \begin{bmatrix} M(s)(I - N(s)M(s))^{-1} & M(s)(I - N(s)M(s))^{-1}N(s) \\ N(s)(I - M(s)N(s))^{-1}M(s) & N(s)(I - M(s)N(s))^{-1} \end{bmatrix} \tag{13}$$

is NI. Furthermore, if, in addition, either $M(s)$ or $N(s)$ is SNI, then (13) is SNI.

Proof

The internal stability of the positive feedback interconnection shown in Figure 2 implies that $T(s)$ is asymptotically stable. Given $\omega \geq 0$, $w_1 \in \mathbb{C}^m$, and $w_2 \in \mathbb{C}^m$, define

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = T(\jmath\omega)\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}.$$

Letting $u_1 = w_1 + y_2$ and $u_2 = w_2 + y_1$, it follows from the positive feedback interconnection that $y_1 = M(\jmath\omega)u_1$ and $y_2 = N(\jmath\omega)u_2$. Furthermore, using the fact that $M(s)$ and $N(s)$ are NI, it follows that

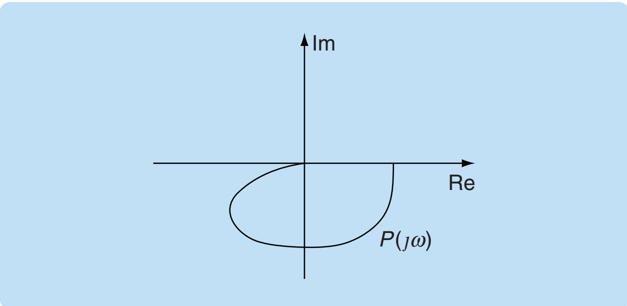

FIGURE 1 Positive-frequency Nyquist plot of a negative-imaginary system. A single-input, single-output negative-imaginary transfer function has no poles in the closed right half of the complex plane and has a frequency response with negative imaginary part for all frequencies. Consequently, the Nyquist plot for $\omega > 0$ is contained in the lower half of the complex plane.

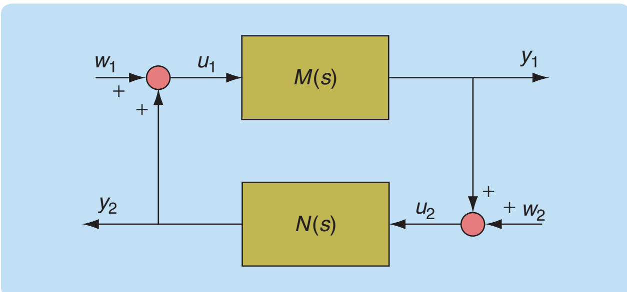

FIGURE 2 A positive feedback interconnection. The transfer functions $M(s)$ and $N(s)$ are interconnected by positive feedback. The stability of this feedback interconnection can be guaranteed by using, for example, Theorem 13. The relevant stability result depends on the properties of $M(s)$ and $N(s)$.

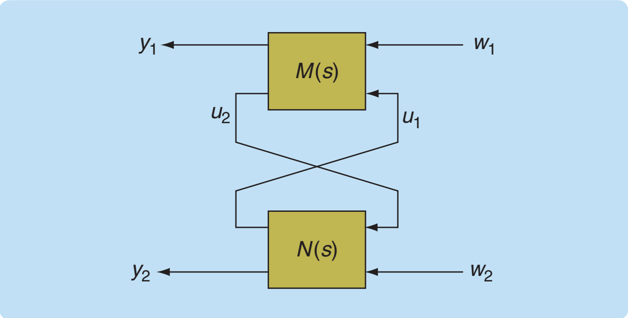

FIGURE 3 A Redheffer star product feedback interconnection. If this feedback interconnection is internally stable and the transfer function matrices $M(s)$ and $N(s)$ are negative imaginary, then $T(s)$, the closed-loop transfer function matrix from $[w_1^T\ w_2^T]^T$ to $[y_1^T\ y_2^T]^T$, is negative imaginary. Furthermore, if, in addition, either $M(s)$ or $N(s)$ is strictly negative imaginary, then $T(s)$ is strictly negative imaginary.

$$\begin{aligned}
&\jmath[w_1^*\ \ w_2^*][T(\jmath\omega) - T^*(\jmath\omega)]\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \\
&\quad = \jmath[w_1^*\ \ w_2^*]\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} - \jmath[y_1^*\ \ y_2^*]\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \\
&\quad = \jmath[u_1^* - y_2^*\ \ u_2^* - y_1^*]\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} - \jmath[y_1^*\ \ y_2^*]\begin{bmatrix} u_1 - y_2 \\ u_2 - y_1 \end{bmatrix} \\
&\quad = \jmath(u_1^* y_1 + u_2^* y_2) - \jmath(y_1^* u_1 + y_2^* u_2) \\
&\quad = \jmath(u_1^* M(\jmath\omega) u_1 - u_1^* M(\jmath\omega)^* u_1) \\
&\qquad + \jmath(u_2^* N(\jmath\omega) u_2 - u_2^* N(\jmath\omega)^* u_2) \\
&\quad \geq 0.
\end{aligned}$$

Since $\omega \geq 0$, $w_1 \in \mathbb{C}^m$, and $w_2 \in \mathbb{C}^m$ are arbitrary, it follows that

$$\jmath[T(\jmath\omega) - T(\jmath\omega)^*] \geq 0$$

for all $\omega \geq 0$ and hence, $T(s)$ is NI. The SNI result follows using similar arguments. ■

Theorem 3

Consider the $2m \times 2m$ NI transfer function matrices

$$M(s) = \begin{bmatrix} M_{11}(s) & M_{12}(s) \\ M_{21}(s) & M_{22}(s) \end{bmatrix}, \quad N(s) = \begin{bmatrix} N_{11}(s) & N_{12}(s) \\ N_{21}(s) & N_{22}(s) \end{bmatrix},$$

and suppose that the feedback interconnection shown in Figure 3 is internally stable. Then the corresponding $2m \times 2m$ closed-loop transfer function matrix

$$T(s) = \begin{bmatrix} M_{11}(s) + M_{12}(s)(I - N_{11}(s)M_{22}(s))^{-1}N_{11}(s)M_{21}(s) & M_{12}(s)(I - N_{11}(s)M_{22}(s))^{-1}N_{12}(s) \\ N_{21}(s)(I - M_{22}(s)N_{11}(s))^{-1}M_{21}(s) & N_{22}(s) + N_{21}(s)(I - M_{22}(s)N_{11}(s))^{-1}M_{22}(s)N_{12}(s) \end{bmatrix} \tag{14}$$

is NI. Furthermore, if in addition, either $M(s)$ or $N(s)$ is SNI, then (14) is SNI.

Proof

The internal stability of the feedback interconnection shown in Figure 3 implies that $T(s)$ is asymptotically stable. Given $\omega \geq 0$, $w_1 \in \mathbb{C}^m$, and $w_2 \in \mathbb{C}^m$, define

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = T(\jmath\omega)\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}.$$

Letting u_1 and u_2 be defined as in the equation at the bottom of the page, it follows from the feedback interconnection shown in Figure 3 that

$$\begin{bmatrix} y_1 \\ u_2 \end{bmatrix} = M(\jmath\omega)\begin{bmatrix} w_1 \\ u_1 \end{bmatrix}, \quad \begin{bmatrix} u_1 \\ y_2 \end{bmatrix} = N(\jmath\omega)\begin{bmatrix} u_2 \\ w_2 \end{bmatrix}. \tag{15}$$

Furthermore, using (15) and the fact that $M(s)$ and $N(s)$ are NI, it follows that

$$\begin{aligned}
&\jmath[w_1^*\ \ w_2^*][T(\jmath\omega) - T^*(\jmath\omega)]\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \\
&\quad = \jmath[w_1^*\ \ w_2^*]\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} - \jmath[y_1^*\ \ y_2^*]\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \\
&\quad = \jmath\left([w_1^*\ \ u_1^*]\begin{bmatrix} y_1 \\ u_2 \end{bmatrix} - [y_1^*\ \ u_2^*]\begin{bmatrix} w_1 \\ u_1 \end{bmatrix}\right) \\
&\qquad + \jmath\left([u_2^*\ \ w_2^*]\begin{bmatrix} u_1 \\ y_2 \end{bmatrix} - [u_1^*\ \ y_2^*]\begin{bmatrix} u_2 \\ w_2 \end{bmatrix}\right) \\
&\quad = \jmath\left([w_1^*\ \ u_1^*]M(\jmath\omega)\begin{bmatrix} w_1 \\ u_1 \end{bmatrix} - [w_1^*\ \ u_1^*]M(\jmath\omega)^*\begin{bmatrix} w_1 \\ u_1 \end{bmatrix}\right) \\
&\qquad + \jmath\left([u_2^*\ \ w_2^*]N(\jmath\omega)\begin{bmatrix} u_2 \\ w_2 \end{bmatrix} - [u_2^*\ \ w_2^*]N(\jmath\omega)^*\begin{bmatrix} u_2 \\ w_2 \end{bmatrix}\right) \\
&\quad \geq 0.
\end{aligned}$$

Since $\omega \geq 0$, $w_1 \in \mathbb{C}^m$, and $w_2 \in \mathbb{C}^m$ are arbitrary, it follows that

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} (I - N_{11}(s)M_{22}(s))^{-1}N_{11}(s)M_{21}(s) & (I - N_{11}(s)M_{22}(s))^{-1}N_{12}(s) \\ (I - M_{22}(s)N_{11}(s))^{-1}M_{21}(s) & (I - M_{22}(s)N_{11}(s))^{-1}M_{22}(s)N_{12}(s) \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix},$$

Positive-position feedback can be regarded as one of the last areas of classical control theory to be encompassed by modern control theory.

$$\jmath[T(\jmath\omega) - T(\jmath\omega)^*] \geq 0$$

for all $\omega \geq 0$ and hence, $T(s)$ is NI. The SNI result follows using similar arguments. ■

Underlying the stability properties of positive-position feedback is the observation that the transfer function matrix of a lightly damped flexible structure with colocated force actuators and position sensors is NI. Indeed, note that all poles of

$$P_i(s) = \frac{1}{s^2 + \kappa_i s + \omega_i^2}\psi_i\psi_i^{\mathrm{T}}$$

in the transfer function matrix (8) lie in the OLHP. Also, for all $\omega \geq 0$,

$$\begin{aligned}\jmath[P_i(\jmath\omega) - P_i^*(\jmath\omega)] &= \Im_{\mathrm{H}}(P_i(\jmath\omega))\\ &= \frac{2\kappa_i\omega}{(\omega_i^2-\omega^2)^2+\kappa_i^2\omega^2}\psi_i\psi_i^{\mathrm{T}} \geq 0.\end{aligned}$$

Hence, it follows from Definition 3 that each $P_i(s)$ is NI. Therefore, it follows from Lemma 1 that the transfer function matrix (8) is NI.

THE NEGATIVE-IMAGINARY LEMMA

The following theorem, which is proved in [10] and [21], provides a state-space characterization of NI systems in terms of a pair of linear matrix inequalities (LMIs). This result is analogous to the positive-real lemma [19], [20] and thus is referred to as the *negative-imaginary lemma*.

Theorem 4

Consider the minimal state-space system

$$\dot{x} = Ax + Bu, \tag{16}$$

$$y = Cx + Du, \tag{17}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, and $D \in \mathbb{R}^{m\times m}$. The system (16), (17) is NI if and only if A has no eigenvalues on the imaginary axis, D is symmetric, and there exists a positive-definite matrix $Y \in \mathbb{R}^{n\times n}$ satisfying

$$AY + YA^{\mathrm{T}} \leq 0, \tag{18}$$

$$B + AYC^{\mathrm{T}} = 0. \tag{19}$$

In Theorem 4 it follows from the Lyapunov inequality (18), the positive definiteness of Y, and the assumption that A has no eigenvalues on the imaginary axis that the matrix A is asymptotically stable [22, Corollary 11.8.1].

Corollary 5

Consider the minimal state-space system (16), (17), where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, and $D \in \mathbb{R}^{m\times m}$. The system (16), (17) is SNI if and only if the following conditions are satisfied:

1) A has no eigenvalues on the imaginary axis.
2) D is symmetric.
3) There exists a positive-definite matrix $Y \in \mathbb{R}^{n\times n}$ such that (18) and (19) are satisfied.
4) The transfer function matrix $M(s) = C(sI - A)^{-1}B + D$ is such that $M(s) - M^{\mathrm{T}}(-s)$ has no transmission zeros on the imaginary axis except possibly at $s = 0$.

Proof

Assuming conditions 1)–3), it follows from Theorem 4 that (16), (17) is NI. Now suppose that (16), (17) is not SNI. Then using definitions 3 and 4, it follows that there exist $\omega > 0$ and a nonzero vector $u \in \mathbb{C}^m$ such that

$$\jmath u^*[M(\jmath\omega) - M^*(\jmath\omega)]u = 0.$$

Thus, $M(s) - M^{\mathrm{T}}(-s)$ has a transmission zero at $s = \jmath\omega$, which contradicts condition 4). Hence (16), (17) is SNI.

Conversely, suppose that (16), (17) is SNI. Then, (16), (17) is NI, and Theorem 4 implies that conditions 1)–3) are satisfied. Also, it follows from Definition 4 that

$$\jmath[M(\jmath\omega) - M^*(\jmath\omega)] > 0$$

for all $\omega > 0$. Therefore $M(s) - M^{\mathrm{T}}(-s)$ has no transmission zeros on the imaginary axis except possibly at $s = 0$, and thus condition 4) is satisfied. ■

To illustrate Theorem 4 and Corollary 5, consider the system

$$\dot{x} = -x + u, \tag{20}$$

$$y = x \tag{21}$$

with transfer function

$$M(s) = \frac{1}{s+1}. \tag{22}$$

The positive-frequency Nyquist plot of (22) given in Figure 4 shows that (20), (21) is both SNI and strictly positive real.

Applying Theorem 4 with $A = -1$, $B = 1$, $C = 1$, and $D = 0$, condition (19) can be satisfied by choosing $Y = -B/(AC) = 1 > 0$. Then, $AY + YA^T = -2 < 0$. It now follows from Theorem 4 that (20), (21) is NI. Also, note that

$$M(s) - M^T(-s) = \frac{1}{s+1} - \frac{1}{-s+1} = \frac{2s}{s^2-1}$$

has no zeros on the imaginary axis except at $s = 0$. It then follows from Corollary 5 that (20), (21) is SNI.

Now consider the transfer function

$$M(s) = \frac{2s^2 + s + 1}{(s^2 + 2s + 5)(s+1)(2s+1)}. \tag{23}$$

The positive-frequency Nyquist plot of $M(\jmath\omega)$ in Figure 5 shows that $\Im[M(\jmath\omega)] \le 0$ for all $\omega \ge 0$, and thus $M(s)$ is NI. However, Figure 5 shows that there exists $\omega > 0$ such that $\Im[M(\jmath\omega)] = 0$, and thus $M(s)$ is not SNI. Now consider the minimal realization (16), (17) of (23) given by

$$A = \begin{bmatrix} -3.5 & -8.5 & -8.5 & -2.5 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2.5 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \tag{24}$$

$$C = [0 \quad 0 \quad 0 \quad 1], \qquad D = 0. \tag{25}$$

To construct a matrix Y satisfying the assumptions of Theorem 4, note that the assumptions of Theorem 4 are equivalent to the requirement that the matrix A have no eigenvalues on the imaginary axis and

$$\begin{bmatrix} AY + YA^T & B + AYC^T \\ B^T + CYA^T & 0 \end{bmatrix} \le 0, \quad Y > 0.$$

Using LMI software [23], we obtain

$$Y = \begin{bmatrix} 100.375 & -36.75 & 2.5 & 3 \\ -36.75 & 18.5 & -3 & -1 \\ 2.5 & -3 & 1 & 0 \\ 3 & -1 & 0 & 0.2 \end{bmatrix} > 0.$$

Therefore Theorem 4 implies that (16), (17), (24), (25) is NI.

Now to determine whether (16), (17), (24), (25) is SNI, note that

$$\begin{aligned} M(s) - M^T(-s) &= \frac{2s^2 + s + 1}{(s^2 + 2s + 5)(s+1)(2s+1)} \\ &\quad - \frac{2s^2 - s + 1}{(s^2 - 2s + 5)(-s+1)(-2s+1)} \\ &= \frac{-24(s^2+1)^2}{4s^8 + 19s^6 + 71s^4 - 119s^2 + 25}, \end{aligned}$$

has a double zero at $s = \jmath$. Consequently, (16), (17), (24), (25) is not SNI.

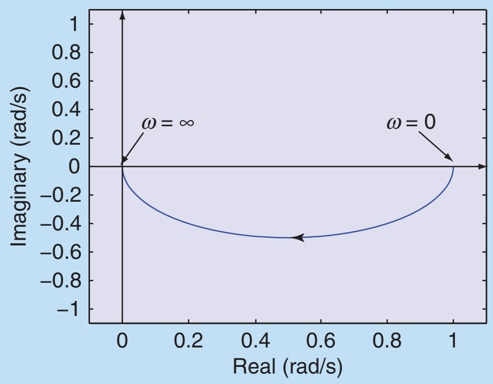

FIGURE 4 Positive-frequency Nyquist plot of the transfer function $M(s) = 1/(s+1)$. The imaginary part of $M(\jmath\omega)$ is negative for all $\omega > 0$, and thus $M(s)$ is strictly negative imaginary.

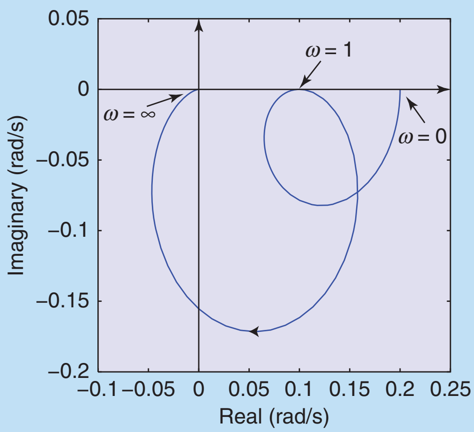

FIGURE 5 Positive-frequency Nyquist plot of the transfer function $M(s) = (2s^2 + s + 1)/((s^2 + 2s + 5)(s+1)(2s+1))$. This Nyquist plot shows that the imaginary part of $M(\jmath\omega)$ is negative for all $\omega \ge 0$ except $\omega = 0$ and $\omega = 1$, where the imaginary part of $M(\jmath\omega)$ is zero. Thus $M(s)$ is negative imaginary, but not strictly negative imaginary.

TWO STRICT NEGATIVE-IMAGINARY LEMMAS

The following theorems give sufficient conditions for the SNI property.

Theorem 6

Consider the minimal state-space system (16), (17), where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, and $D \in \mathbb{R}^{m\times m}$. Suppose the following conditions are satisfied:

1) All eigenvalues of A are in OLHP.
2) D is symmetric.

An alternative approach to negative-velocity feedback is *positive-position feedback*, where position sensors are used in place of velocity sensors.

3) There exist a positive-definite matrix $\widetilde{Y} \in \mathbb{R}^{n\times n}$ and positive numbers α, ε such that $-\alpha$ is not an eigenvalue of A and the matrices

$$\widetilde{A} = \begin{bmatrix} A & 0 \\ 0 & -\alpha I \end{bmatrix}, \ \widetilde{B} = \begin{bmatrix} B \\ \varepsilon I \end{bmatrix}, \ \widetilde{C} = [C \quad -I] \tag{26}$$

satisfy

$$\widetilde{A}\widetilde{Y} + \widetilde{Y}\widetilde{A}^{\mathrm{T}} \le 0$$

and

$$\widetilde{B} + \widetilde{A}\widetilde{Y}\widetilde{C}^{\mathrm{T}} = 0.$$

Then (16), (17) is SNI.

The proof of Theorem 6 requires the following lemma.

Lemma 7

Let $\varepsilon > 0$ and $\alpha > 0$. Then the transfer function matrix

$$M(s) = \frac{\varepsilon}{s+\alpha} I \tag{27}$$

is SNI.

Proof

Let the transfer function matrix (27) have minimal state-space realization

$$\dot{x} = -\alpha x + \varepsilon u, \tag{28}$$
$$y = x. \tag{29}$$

Theorem 4 and Corollary 5 can be applied to (28), (29) with $A = -\alpha I$, $B = \varepsilon I$, $C = I$, and $D = 0$. Setting $Y = (\varepsilon/\alpha) I > 0$, it follows that $AY + YA^{\mathrm{T}} = -2\varepsilon I < 0$ and $B + AYC^{\mathrm{T}} = \varepsilon I - (\alpha\varepsilon/\alpha) I = 0$. Hence, Theorem 4 implies that (28), (29) is NI. Furthermore,

$$M(s) - M^{\mathrm{T}}(-s) = \frac{\varepsilon}{s+\alpha} I - \frac{\varepsilon}{-s+\alpha} I = \frac{2\varepsilon s}{s^2 - \alpha^2} I.$$

Thus, $M(s) - M^{\mathrm{T}}(-s)$ has no purely imaginary transmission zeros except possibly at $s = 0$. Hence, it follows from Corollary 5 that (28), (29) is SNI. ∎

Proof of Theorem 6

Let $\hat{M}(s)$ be the transfer function matrix of (16), (17). Since $s = -\alpha$ is not a pole of $\hat{M}(s)$, a minimal state-space realization of the transfer function matrix $M_1(s) = \hat{M}(s) - \varepsilon I/(s+\alpha)$ is

$$\dot{x}_1 = Ax_1 + Bu,$$
$$\dot{x}_2 = -\alpha x_2 + \varepsilon u,$$
$$y = Cx_1 - x_2 + Du.$$

Let

$$\widetilde{A} = \begin{bmatrix} A & 0 \\ 0 & -\alpha I \end{bmatrix}, \ \widetilde{B} = \begin{bmatrix} B \\ \varepsilon I \end{bmatrix}, \ \widetilde{C} = [C \quad -I], \ \widetilde{D} = D.$$

Assuming conditions 1)–3), it follows from Theorem 4 that $M_1(s)$ is NI. Then Lemmas 1 and 7 imply that $\hat{M}(s) = M_1(s) + \varepsilon I/(s+\alpha)$ is SNI. ∎

To illustrate Theorem 6, we consider lightly damped flexible structures with force actuators and position sensors. An *integral resonant controller* [13], [14] has the form

$$C(s) = [sI + \Gamma\Phi]^{-1}\Gamma, \tag{30}$$

where Γ and Φ are positive-definite matrices. In the SISO case [13], integral resonant controllers are derived by first adding a direct feedthough to a resonant system with a colocated force actuator and position sensor. Then, application of integral feedback leads to damping of the resonant poles. Combining the direct feedthrough with the integral feedback leads to a SISO controller of the form (30). In [14], this class of SISO controllers is generalized to MIMO controllers of the form (30).

Integral resonant controllers provide integral force feedback [1], which refers to control that uses position actuators, force sensors, and integral feedback. In [1], integral feedback is modified by moving the integrator pole slightly to the left in the complex plane to alleviate actuator saturation. A SISO controller transfer function of the form (30) results from this process.

Theorem 8

The transfer function matrix (30) with Γ positive definite and Φ positive definite is SNI.

Proof

Consider the minimal state-space realization of (30) given by

$$\dot{x} = -\Gamma\Phi x + \Gamma u,$$
$$y = x.$$

Let $\varepsilon > 0$ and $\alpha > 0$ be such that $-\alpha$ is not an eigenvalue of $-\Gamma\Phi$. The corresponding matrices in (26) are

$$\widetilde{A} = \begin{bmatrix} -\Gamma\Phi & 0 \\ 0 & -\alpha I \end{bmatrix}, \ \widetilde{B} = \begin{bmatrix} \Gamma \\ \varepsilon I \end{bmatrix}, \ \widetilde{C} = [I \quad -I], \ \widetilde{D} = 0.$$

What Is Finsler's Theorem?

Finsler's theorem, which is used in the proof of Lemma 8, is summarized in the following lemma [S3].

LEMMA S2

Let M and N be real symmetric matrices such that M is positive semidefinite and $x^{\mathrm{T}}Nx \geq 0$ for all real x such that $Mx = 0$. Then there exists $\bar{\tau} > 0$ such that $N + \tau M \geq 0$ for all $\tau \geq \bar{\tau}$.

To illustrate Finsler's theorem, let $M = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $N = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. All nonzero x such that $Mx = 0$ are given by $x = \begin{bmatrix} 0 \\ \alpha \end{bmatrix}$, where $\alpha \in \mathbb{R}$ is nonzero. Then $x^{\mathrm{T}}Nx = \alpha^2 > 0$. It now follows from Finsler's theorem that there exists $\bar{\tau} > 0$ such that $N + \tau M = \begin{bmatrix} \tau - 1 & 0 \\ 0 & 1 \end{bmatrix} \geq 0$ for all $\tau \geq \bar{\tau}$. In this example, $\bar{\tau} = 1$.

REFERENCE

[S3] F. Uhlig, "A recurring theorem about pairs of quadratic forms and extensions: A survey," *Linear Algebra Its Applicat.*, vol. 25, pp. 219–237, 1979.

Also, let

$$\tilde{Y} = \begin{bmatrix} \Phi^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \varepsilon \begin{bmatrix} \left(\frac{1}{\alpha} + 1\right) I & \left(\frac{1}{\alpha} + 1\right) I \\ \left(\frac{1}{\alpha} + 1\right) I & I \end{bmatrix}.$$

Thus,

$$\tilde{B} + \tilde{A}\tilde{Y}\tilde{C}^{\mathrm{T}} = 0. \qquad (31)$$

Furthermore, note that

$$\begin{bmatrix} \Phi^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

is positive semidefinite, and

$$\begin{bmatrix} \left(\frac{1}{\alpha} + 1\right) I & \left(\frac{1}{\alpha} + 1\right) I \\ \left(\frac{1}{\alpha} + 1\right) I & I \end{bmatrix}$$

is positive definite. Hence, $\tilde{Y} > 0$.

Using the definitions of $\tilde{A}$ and $\tilde{Y}$, it follows that

$$\tilde{A}\tilde{Y} + \tilde{Y}\tilde{A}^{\mathrm{T}} = \begin{bmatrix} -\Gamma & 0 \\ 0 & 0 \end{bmatrix} + \varepsilon \begin{bmatrix} -\left(\frac{1}{\alpha} + 1\right)(\Gamma\Phi + \Phi\Gamma) & -\left(\frac{1}{\alpha} + 1\right)(\Gamma\Phi + \alpha I) \\ -\left(\frac{1}{\alpha} + 1\right)(\Gamma\Phi + \alpha I)^{\mathrm{T}} & -2(\alpha + 1) I \end{bmatrix}.$$

Furthermore, the matrix

$$\begin{bmatrix} \Gamma & 0 \\ 0 & 0 \end{bmatrix}$$

is positive semidefinite. For every nonzero vector of the form $x = [0 \ \ x_2^{\mathrm{T}}]^{\mathrm{T}}$, we have

$$\begin{bmatrix} 0 \\ x_2 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \left(\frac{1}{\alpha} + 1\right)(\Gamma\Phi + \Phi\Gamma) & \left(\frac{1}{\alpha} + 1\right)(\Gamma\Phi + \alpha I) \\ \left(\frac{1}{\alpha} + 1\right)(\Gamma\Phi + \alpha I)^{\mathrm{T}} & 2(\alpha + 1) I \end{bmatrix} \times \begin{bmatrix} 0 \\ x_2 \end{bmatrix} > 0.$$

Hence, it follows using Finsler's theorem (see "What Is Finsler's Theorem?"), Lemma S2, that there exists $\bar{\tau} > 0$ such that

$$\begin{bmatrix} \left(\frac{1}{\alpha} + 1\right)(\Gamma\Phi + \Phi\Gamma) & \left(\frac{1}{\alpha} + 1\right)(\Gamma\Phi + \alpha I) \\ \left(\frac{1}{\alpha} + 1\right)(\Gamma\Phi + \alpha I)^{\mathrm{T}} & 2(\alpha + 1) I \end{bmatrix} + \tau \begin{bmatrix} \Gamma & 0 \\ 0 & 0 \end{bmatrix} \geq 0$$

for all $\tau \geq \bar{\tau}$. Let $\hat{\varepsilon} = \bar{\tau}^{-1} > 0$. Consequently, choosing $\varepsilon \leq \hat{\varepsilon}$ implies

$$\tilde{A}\tilde{Y} + \tilde{Y}\tilde{A}^{\mathrm{T}} \leq 0. \qquad (32)$$

Combining (31) and (32), it follows that conditions 1)–3) of Theorem 6 are satisfied, and, therefore, the transfer function (30) is SNI. ■

Theorem 9

Consider the minimal state-space system (16), (17), where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{m \times n}$, and $D \in \mathbb{R}^{m \times m}$. Suppose the following conditions are satisfied:

1) All of the eigenvalues of A are in OLHP.
2) D is symmetric.
3) There exist a positive-definite matrix $\tilde{Y} \in \mathbb{R}^{n \times n}$ and positive numbers ε, α, and β such that $\alpha \neq \beta$, $-\alpha$, $-\beta$ are not eigenvalues of A, and the matrices

$$\tilde{A} = \begin{bmatrix} A & 0 & 0 \\ 0 & -\alpha I & 0 \\ 0 & 0 & -\beta I \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B \\ \varepsilon I \\ \varepsilon I \end{bmatrix}, \quad \tilde{C} = [C \quad -I \quad -I]$$

satisfy

$$\tilde{A}\tilde{Y} + \tilde{Y}\tilde{A}^{\mathrm{T}} \leq 0$$

Integral resonant controllers provide integral force feedback, which refers to control that uses position actuators, force sensors, and integral feedback.

and

$$\tilde{B} + \tilde{A}\tilde{Y}\tilde{C}^{\mathrm{T}} = 0.$$

Then (16), (17) is SNI.

The proof of Theorem 6 requires the following lemmas.

Lemma 10

Let $\varepsilon > 0$, $\alpha > 0$, and $\beta > 0$. Then the transfer function

$$M(s) = \frac{\varepsilon}{(s+\alpha)(s+\beta)} \tag{33}$$

is SNI.

Proof

The transfer function (33) has a minimal state-space realization

$$\dot{x} = Ax + Bu, \tag{34}$$
$$y = Cx, \tag{35}$$

where

$$A = \begin{bmatrix} -\alpha & 0 \\ 0 & -\beta \end{bmatrix},\quad B = \begin{bmatrix} \varepsilon \\ \varepsilon \end{bmatrix},\quad C = [1 \quad 1].$$

Applying Theorem 4 and Corollary 5 to (34), (35), and setting

$$Y = \begin{bmatrix} \frac{\varepsilon}{\alpha} & 0 \\ 0 & \frac{\varepsilon}{\beta} \end{bmatrix} > 0,$$

it follows that $AY + YA^{\mathrm{T}} = -2\varepsilon I < 0$ and $B + AYC^{\mathrm{T}} = 0$. Hence, Theorem 4 implies that (34), (35) is NI. Furthermore, for (34), (35), $M(s) - M^{\mathrm{T}}(-s)$ is given by

$$M(s) - M^{\mathrm{T}}(-s) = \frac{\varepsilon}{(s+\alpha)(s+\beta)} - \frac{\varepsilon}{(-s+\alpha)(-s+\beta)}$$
$$= \frac{2\varepsilon(\alpha+\beta)s}{s^2(\alpha+\beta)^2 - (s^2+\alpha\beta)^2}.$$

Since $M(s) - M^{\mathrm{T}}(-s)$ has no imaginary transmission zeros except at $s = 0$, it follows from Corollary 5 that (34), (35) is SNI. ■

Lemma 11

If $M(s)$ is an SISO SNI transfer function, then the transfer function matrix $M(s)I$ is SNI.

Proof

This result follows directly from Definition 4. ■

Proof of Theorem 9

Let $\hat{M}(s)$ be the transfer function matrix of (16), (17). Since neither $s = -\alpha$ nor $s = -\beta$ is a pole of $\hat{M}(s)$, a minimal state-space realization of $M_1(s) = \hat{M}(s) - (\varepsilon/(s+\alpha)(s+\beta))I$ is

$$\dot{x}_1 = Ax_1 + Bu,$$
$$\dot{x}_2 = -\alpha x_2 + \varepsilon u,$$
$$\dot{x}_3 = -\beta x_3 + \varepsilon u,$$
$$y = Cx_1 - x_2 - x_3 + Du.$$

Let

$$\tilde{A} = \begin{bmatrix} A & 0 & 0 \\ 0 & -\alpha I & 0 \\ 0 & 0 & -\beta I \end{bmatrix},\quad \tilde{B} = \begin{bmatrix} B \\ \varepsilon I \\ \varepsilon I \end{bmatrix},$$

$$\tilde{C} = [C \quad -I \quad -I],\quad \tilde{D} = D.$$

Assuming conditions 1)–3), it follows from Theorem 4 that $M_1(s)$ is NI. Finally, Lemmas 1, 10, and 11 imply that $\hat{M}(s) = M_1(s) + (\varepsilon/(s+\alpha)(s+\beta))I$ is SNI. ■

ROBUST STABILITY OF NEGATIVE-IMAGINARY CONTROL SYSTEMS

We now present a result given by Theorem 13 below that guarantees the robustness and stability of control systems involving the positive-feedback interconnection of an NI system and an SNI system. This positive-feedback interconnection is illustrated in Figure 2. The result is analogous to the passivity theorem given in "What Is Positive-Real and Passivity Theory?" concerning the negative-feedback interconnection of a positive-real system and a strictly positive-real system.

Theorem 13 guarantees the internal stability of the positive-feedback interconnection of two systems through phase stabilization, as opposed to gain stabilization in the small-gain theorem. In phase stabilization the gains of the systems can be arbitrarily large, but the phase of the loop transfer function needs to be such that the critical Nyquist point is not encircled by the Nyquist plot. In the passivity theorem given in "What Is Positive-Real and Passivity Theory?," negative feedback is used, and thus the Nyquist point is at $s = -1 + \jmath 0$. Then the cascade of two positive-real systems gives a loop transfer function whose phase is in the interval $(-\pi, \pi)$. Hence, the Nyquist plot excludes the negative real axis. In NI systems, positive feedback

This article describes properties of a class of systems termed NI systems using ideas from classical control theory.

interconnection is used and thus the Nyquist point is $s = 1 + \jmath 0$. This alternative Nyquist point is required since an NI system has a phase lag in the interval $(-\pi, 0)$ and thus two NI systems in cascade have a phase lag in the interval $(-2\pi, 0)$. That is, the Nyquist plot excludes the positive-real axis.

The following lemma is required to state the result given in Theorem 13.

Lemma 12

Let $M(s)$ be an NI transfer function matrix. Then $M(\infty)$ and $M(0)$ are symmetric, and

$$M(0) - M(\infty) \geq 0. \tag{36}$$

Also, let $N(s)$ be an SNI transfer function matrix. Then $N(\infty)$ and $N(0)$ are symmetric, and

$$N(0) - N(\infty) > 0. \tag{37}$$

If, in addition, $N(\infty)$ is positive semidefinite, then $N(0)$ is positive definite and all of the eigenvalues of the matrix $M(0)N(0)$ are real.

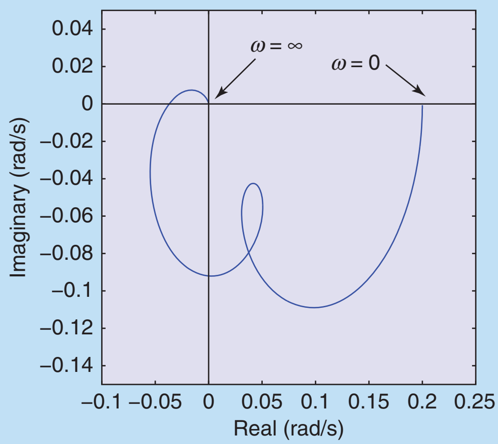

FIGURE 6 Positive-frequency Nyquist plot of the loop transfer function $L(s) = N(s)M(s)$ corresponding to the positive-feedback interconnection of $M(s) = 1/(s+1)$ and $N(s) = (2s^2 + s + 1)/((s^2 + 2s + 5)(s + 1)(2s + 1))$. Here $M(s)$ is strictly negative imaginary, and $N(s)$ is negative imaginary. Since $M(s)$ and $N(s)$ both have no poles in the closed right half of the complex plane and the Nyquist plot does not encircle the critical point $s = 1 + \jmath 0$, it follows from the Nyquist stability criterion that the positive feedback interconnection of $M(s)$ and $N(s)$ is internally stable.

Proof

See [10]. ■

Theorem 13

Consider the NI transfer function matrix $M(s)$ and the SNI transfer function matrix $N(s)$, and suppose that $M(\infty)N(\infty) = 0$ and $N(\infty) \geq 0$. Then, the positive-feedback interconnection of $M(s)$ and $N(s)$ is internally stable if and only if

$$\lambda_{\max}(M(0)N(0)) < 1. \tag{38}$$

Proof

See [10]. ■

In the MIMO case, the proof of Theorem 13 given in [10] uses Theorem 4. In the SISO case, the sufficiency part of Theorem 13 follows directly from Nyquist arguments and thus has an intuitive interpretation. For example, consider

$$M(s) = \frac{1}{s+1}, \tag{39}$$

whose positive-frequency Nyquist plot is shown in Figure 4. Also consider

$$N(s) = \frac{2s^2 + s + 1}{(s^2 + 2s + 5)(s + 1)(2s + 1)}, \tag{40}$$

whose positive-frequency Nyquist plot is shown in Figure 5. Figure 4 shows that $N(s)$ is SNI, whereas Figure 5 shows that $M(s)$ is NI but not SNI. The positive-frequency Nyquist plot of the corresponding loop transfer function $L(s) = N(s)M(s)$ is shown in Figure 6. Since both $N(s)$ and $M(s)$ have no poles in the CRHP and the Nyquist plot of $L(s)$ does not encircle the critical point $s = 1 + \jmath 0$, it follows that the positive-feedback interconnection of $M(s)$ and $N(s)$ is internally stable. A similar Nyquist argument is mentioned in [8] as a justification for the stability of SISO positive-position feedback systems. Furthermore, a condition equivalent to (38) is required in the result of [16].

Consider $M(s)$ and $N(s)$ as in Theorem 13 in the SISO case. Since $N(s)$ is SNI, it follows that $\angle N(\jmath\omega) \in (-\pi, 0)$ for all $\omega > 0$. Furthermore, since $M(s)$ is NI, it follows that $\angle M(\jmath\omega) \in [-\pi, 0]$ for all $\omega \geq 0$ such that $M(\jmath\omega) \neq 0$. Hence, $L(s) = M(s)N(s)$ satisfies $\angle L(\jmath\omega) \in (-2\pi, 0)$ for all $\omega > 0$ such that $L(\jmath\omega) \neq 0$. Thus the Nyquist plot of $L(\jmath\omega)$ can intersect the positive-real axis only at $\omega = 0$ since at infinite frequency $M(\infty)N(\infty) = 0$. Thus, the Nyquist plot of $L(j\omega)$ does not encircle the critical point $s = 1 + j0$ if $M(0)N(0) < 1$. Hence, in the SISO case, the sufficiency part of Theorem 13 follows from the Nyquist test.

A discussion on how rigid-body modes can be handled using Theorem 13 is given in "How Are Rigid-Body Modes Handled?"

NEGATIVE-IMAGINARY FEEDBACK CONTROLLERS

We now apply Theorem 13 to NI feedback control systems in the case where one of the blocks in the feedback connection shown in Figure 2 corresponds to the plant, while the other block corresponds to the controller. This situation is shown in Figure 7.

Since flexible structures with colocated force actuators and position sensors are typically SNI, Theorem 13 implies that NI controllers guarantee closed-loop internal stability if the dc gain condition (38) is satisfied. Indeed, many schemes considered for controlling flexible structures rely on controllers that are NI. These schemes include positive-position feedback [1], [7], [8], [24], resonant feedback control [11], [12], and integral resonant control [13], [14]. We now consider each of these control schemes in more detail.

Positive-Position Feedback

In the SISO case, a positive-position feedback controller is a controller of the form

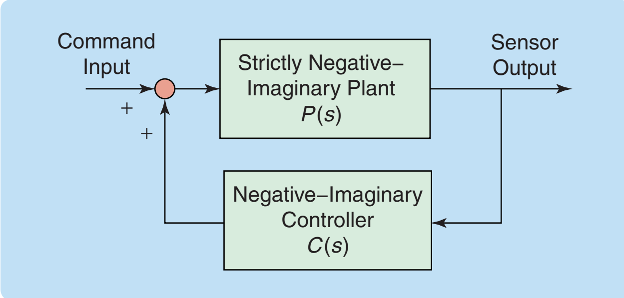

FIGURE 7 Negative-imaginary feedback control system. If the plant transfer function matrix $P(s)$ is strictly negative imaginary and the controller transfer function matrix $C(s)$ is negative imaginary, then the closed-loop system is internally stable if and only if the dc gain condition $\lambda_{\max}[P(0)C(0)] < 1$ is satisfied.

$$C(s) = \sum_{i=1}^{M} \frac{k_i}{s^2 + 2\zeta_i\omega_i s + \omega_i^2}, \tag{41}$$

where $\omega_i > 0$, $\zeta_i > 0$, and $k_i > 0$ for $i = 1, 2, \ldots, M$. Using Nyquist arguments, the SISO transfer function $C(s) = k/(s^2 + 2\zeta\omega s + \omega^2)$, where $\omega, \zeta, k > 0$, is SNI. Consequently, it follows from Lemma 1 that (41) is SNI. Furthermore, this result can be extended to the MIMO case to show that the transfer function matrix

$$C(s) = K^{\mathrm{T}}(s^2 I + Ds + \Omega)^{-1}K, \tag{42}$$

How Are Rigid-Body Modes Handled?

Output feedback control methods rely on output signal information measured through sensors to asymptotically stabilize all the internal states of a system. In the case of a system that has unobservable modes that are not asymptotically stable, output feedback control cannot asymptotically stabilize the system. Systems with rigid-body modes, which are characterized by a zero natural frequency, are an example of systems that cannot be asymptotically stabilized by velocity feedback alone, and position feedback is essential [S4, pp. 333–336]. Under velocity feedback alone, systems with rigid-body modes can come to rest at a position other than the origin of the state space. The unobservability of the position states corresponding to the rigid-body modes from the velocity outputs are the cause of this problem [S4, pp. 333–336].

As a result of this problem, rigid-body modes need special consideration in passivity approaches. Typically a position feedback is applied before using the passivity theorem, which is given in "What Is Positive-Real and Passivity Theory?" Position feedback is applied in an inner loop before applying velocity feedback on the outer loop. This technique converts the rigid-body modes into vibrational modes, which renders the corresponding position states observable from the velocity outputs of the system.

Now consider positive-position control of systems with rigid-body modes. The definitions of NI and SNI systems given in definitions 3 and 4 require that NI and SNI systems have no poles at the origin. Hence, theorems 4 and 13 cannot directly handle rigid-body modes. However, a similar technique to velocity feedback involving a position-feedback inner loop can be used on NI systems that have rigid-body modes. This position feedback inner loop is used to convert the rigid-body modes into vibrational modes. Then the result of [21], which generalizes Theorem 13 to allow for modes on the imaginary axis except at the origin, can be applied to guarantee internal stability of the overall feedback system. Thus, the resulting inner feedback loop consists of unity feedback and proportional feedforward control to convert the rigid-body modes to vibrational modes. Then, positive-position feedback is applied in the outer loop. An advantage in this case relative to velocity feedback is that a position sensor output is already available.

REFERENCE

[S4] L. Meirovitch, *Dynamics and Control of Structures*. New York: Wiley, 1990.

where $D > 0$ and $\Omega > 0$, is SNI [9]. A MIMO positive-position feedback controller is a controller of the form (42), while a *positive-position feedback system* is a control system for a flexible structure with colocated force actuators and position sensors with a controller of the form (42) [1], [7], [8], [24].

The Nyquist proof of Theorem 13 justifies the use of positive-position feedback in the SISO case. That is, since the positive-position feedback controller (41) is SNI, its phase is in the interval $(-\pi, 0)$ for all $\omega > 0$. Furthermore, since the flexible structure plant is NI, its phase is in the interval $[-\pi, 0]$ for all $\omega \geq 0$ such that $\jmath\omega$ is not a zero. Hence, the phase of the loop transfer function is in the interval $(-2\pi, 0)$ for all $\omega > 0$ such that $\jmath\omega$ is not a zero. This fact, together with the strict properness of the controller (41), implies that the Nyquist plot of the loop transfer function can intersect the positive-real axis at only the frequency $\omega = 0$. Thus, the Nyquist plot of the loop transfer function does not encircle the critical point $s = 1 + \jmath 0$ if the dc value of the loop transfer function is strictly less than unity.

Resonant Control

We now consider the exactly proper SISO SNI controller

$$C(s) = \sum_{i=1}^{M} \frac{-k_i s^2}{s^2 + 2\zeta_i \omega_i s + \omega_i^2}, \tag{43}$$

where $\omega_i > 0$, $\zeta_i > 0$, and $k_i > 0$ for $i = 1, 2, \ldots, M$. The controller (43) can be implemented as the positive-position feedback controller (41) using an acceleration sensor rather than a position sensor. Alternatively, (43) can be implemented as the positive-real feedback controller

$$\overline{C}(s) = \sum_{i=1}^{M} \frac{k_i s}{s^2 + 2\zeta_i \omega_i s + \omega_i^2},$$

where $\omega_i > 0$, $\zeta_i > 0$, and $k_i > 0$ for $i = 1, 2, \ldots, M$, using a velocity sensor rather than a position sensor. To see that (43) defines an NI controller, we rewrite (43) as $C(s) = -s^2 \widetilde{C}(s)$, where $\widetilde{C}(s)$ is a SISO positive-position feedback controller of the form defined in (41). If $s = \jmath\omega$ and $\omega > 0$, then $-s^2 = \omega^2 > 0$. Therefore, since $\widetilde{C}(s)$ is SNI, $C(s)$ is SNI.

Next consider the SISO SNI controller

$$C(s) = \sum_{i=1}^{M} \frac{-k_i s(s + 2\zeta_i \omega_i)}{s^2 + 2\zeta_i \omega_i s + \omega_i^2}, \tag{44}$$

where $\omega_i > 0$, $\zeta_i > 0$, and $k_i > 0$ for $i = 1, 2, \ldots, M$. Application of (44) is described in [11] and [12]. By writing

$$\frac{-k_i s(s + 2\zeta_i \omega_i)}{s^2 + 2\zeta_i \omega_i s + \omega_i^2} = -k_i + \frac{k_i \omega_i^2}{s^2 + 2\zeta_i \omega_i s + \omega_i^2} \tag{45}$$

for each i, it follows that the controller (44) is SNI. This result follows from the fact that the first term on the right side of (45) has zero imaginary part, and the second term on the right side of (45) is SNI as in the case of the positive-position feedback controller (41). Using these facts, it follows from Lemma 1 that the controller (44) is SNI.

The SNI controllers (43) and (44) can be extended to the MIMO case to obtain the MIMO SNI controller

$$C(s) = \sum_{i=1}^{M} \frac{-s^2}{s^2 + 2\zeta_i \omega_i s + \omega_i^2} \alpha_i \alpha_i^{\mathrm{T}} \tag{46}$$

and

$$C(s) = \sum_{i=1}^{M} \frac{-s(s + 2\zeta_i \omega_i)}{s^2 + 2\zeta_i \omega_i s + \omega_i^2} \beta_i \beta_i^{\mathrm{T}}, \tag{47}$$

where α_i and β_i are $m \times 1$ vectors [24]. Control systems for flexible structures with colocated force actuators and position sensors using controllers of the form (46), (47) are *resonant control systems* [11], [12].

Integral Resonant Control

Theorem 8 shows that MIMO transfer function matrices of the form

$$C(s) = [sI + \Gamma\Phi]^{-1}\Gamma$$

are SNI. Here, Γ is a positive-definite matrix and Φ is a positive-definite matrix. The use of a controller of this form when applied to a flexible structure with force actuators and position sensors is referred to as *integral resonant control*, or integral force control [1], [13], [14].

To illustrate Theorem 13 and integral resonant control, consider a SISO integral resonant control system where the plant is a flexible structure with colocated force actuation and position measurement. The plant is assumed to have the transfer function

$$P(s) = \sum_{k=1}^{10} \frac{1}{s^2 + 2s + 10^4 k^2}. \tag{48}$$

Now consider this system controlled with the SISO integral resonant controller

$$C(s) = \frac{\Gamma}{s + \Gamma\Phi}, \tag{49}$$

where $\Gamma > 0$ and $\Phi > 0$. It follows from Theorem 8 that (49) is SNI. Using Theorem 13, it follows that the closed-loop system is internally stable if the dc gain condition (38) is satisfied. The dc value of the plant transfer function is $P(0) = \sum_{k=1}^{10} 1/10^4 k^2 = 1.5498 \times 10^{-4}$, while the dc value of the controller transfer function is $C(0) = 1/\Phi$. By choosing $\Phi = 1.2 \times P(0) = 1.8597 \times 10^{-4}$, the condition $\lambda_{\max}[P(0)C(0)] < 1$ is satisfied. To choose the parameter $\Gamma > 0$, Figure 8 shows the root locus of the closed-loop poles for the feedback control system with plant $P(s)$ given by (48) and controller $C(s)$ given by (49) as the parameter $\Gamma > 0$ is varied. From this root locus diagram, the parameter Γ is chosen as $\Gamma = 9.6584 \times 10^5$ to maximize the damping of the first resonant mode.

The damping of the resonant modes arising from the integral resonant feedback controller (49) is illustrated in Figure 9, which shows the open-loop frequency response of the plant from the actuator input to the sensor output. Also shown is the closed-loop frequency response from the command input to the sensor output when the integral resonant feedback controller $C(s) = (9.6584 \times 10^5)/(s + 179.6379)$ is applied as in Figure 7.

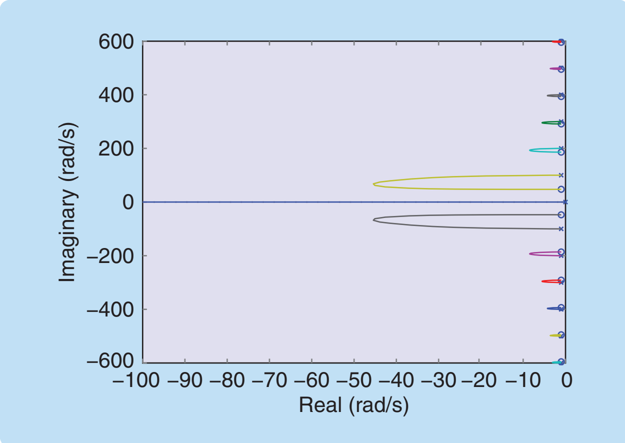

FIGURE 8 Root locus of the closed-loop poles of a control system consisting of a flexible structure plant and an integral resonant controller. Here, the plant transfer function is $P(s) = \sum_{k=1}^{10} 1/(s^2 + 2s + 10^4 k^2)$, and the integral resonant controller transfer function is $C(s) = \Gamma/(s + \Gamma\Phi)$. In this control system, both the plant and the controller are strictly negative imaginary. The root locus is obtained by varying the parameter $\Gamma > 0$ with $\Phi = 1.8597 \times 10^{-4}$ m/N.

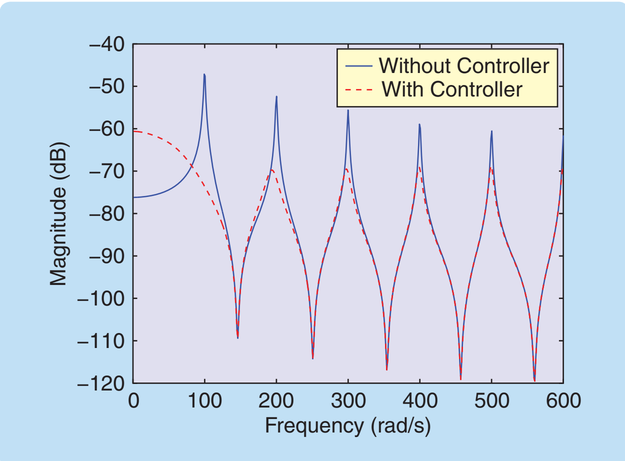

FIGURE 9 Open- and closed-loop frequency responses for a lightly damped flexible structure with an integral resonant feedback controller. Here the plant transfer function is $P(s) = \sum_{k=1}^{10} 1/(s^2 + 2s + 10^4 k^2)$, and the frequency response is taken from the command input to the sensor output. The closed loop uses the integral resonant feedback controller $C(s) = \Gamma/(s + \Gamma\Phi)$. The parameter $\Gamma = 9.6584 \times 10^5$ rad-N/(s-m) is chosen to provide adequate damping of the low-frequency resonant modes, and the parameter Φ is fixed at $\Phi = 1.8597 \times 10^{-4}$ m/N.

State-Feedback Controller Synthesis

An alternative approach to the direct use of Theorem 13 for establishing the closed-loop stability of a feedback control system is to design the controller to be robust against only a specific uncertainty structure as shown in Figure 10. In this case, it follows from Theorem 13 that if the plant uncertainty is known to be SNI, and if the feedback controller is constructed so that the nominal closed-loop system is NI and the dc gain condition is satisfied, then the resulting closed-loop uncertain system is guaranteed to be robustly stable [10]. We now present some further results on this problem when full state measurements are available using an LMI approach. The assumption of full state measurements means that there is a sensor available to measure each of the quantities that define a state variable in the state-space model of the nominal plant shown in Figure 10.

Consider the feedback control system in Figure 10 in the case that full state feedback is available. In this case, Theorem 4 can be used to synthesize a state-feedback control law such that the resulting closed-loop system is NI. Indeed, suppose the uncertain system shown in Figure 10 is described by the state equations

$$\dot{x} = Ax + B_1 w + B_2 u, \tag{50}$$
$$z = C_1 x, \tag{51}$$
$$w = \Delta(s) z, \tag{52}$$

where the uncertainty transfer function matrix $\Delta(s)$ is assumed to be SNI with $\lambda_{\max}(\Delta(0)) \le 1$ and $\Delta(\infty) \ge 0$. Applying the state-feedback control law $u = Kx$ yields the closed-loop uncertain system

$$\dot{x} = (A + B_2 K)x + B_1 w, \tag{53}$$
$$z = C_1 x, \tag{54}$$
$$w = \Delta(s) z. \tag{55}$$

The corresponding nominal closed-loop transfer function matrix is

$$G_{cl}(s) = C_1 (sI - A - B_2 K)^{-1} B_1. \tag{56}$$

Theorem 14

Consider the uncertain system (50)–(52) and suppose there exist matrices $Y > 0$, M, and a scalar $\varepsilon > 0$ such that

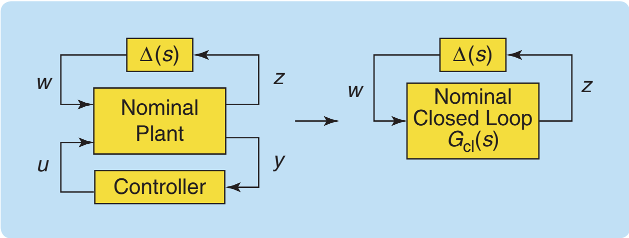

FIGURE 10 A feedback control system. The plant uncertainty $\Delta(s)$ is strictly negative imaginary and satisfies the dc gain condition $\sigma_{\max}(\Delta(0)) < \mu$ and $\Delta(\infty) \geq 0$. If the controller is chosen so that the nominal closed-loop transfer function matrix $G_{\rm cl}(s)$ is strictly proper, negative imaginary, and satisfies the dc gain condition $\sigma_{\max}(G_{\rm cl}(0)) \leq 1/\mu$, then the closed-loop system is robustly stable for all strictly negative imaginary uncertainty $\Delta(s)$ satisfying $\sigma_{\max}(\Delta(0)) < \mu$ and $\Delta(\infty) \geq 0$.

$$\begin{bmatrix} AY + YA^{\rm T} + B_2M + M^{\rm T}B_2^{\rm T} + \varepsilon I & B_1 + AYC_1^{\rm T} + B_2MC_1^{\rm T} \\ B_1^{\rm T} + C_1YA^{\rm T} + C_1M^{\rm T}B_2^{\rm T} & 0 \end{bmatrix} \leq 0, \quad (57)$$

$$C_1YC_1^T - I < 0, \quad (58)$$

$$Y > 0. \quad (59)$$

Then the state-feedback control law $u = MY^{-1}x$ is robustly stabilizing for the uncertain system (50)–(52).

Proof

Suppose the LMIs (57), (59) are satisfied and let

$$K = MY^{-1}.$$

Then, (57) implies

$$(A+B_2K)Y+Y(A+B_2K)^{\rm T} = AY+YA^{\rm T}+B_2M+M^{\rm T}B_2^{\rm T}+\varepsilon I \leq 0, \quad (60)$$

$$B_1 + (A+B_2K)YC_1^{\rm T} = B_1 + AYC_1^{\rm T} + B_2MC_1^{\rm T} = 0. \quad (61)$$

It follows from (60) that $A + B_2K$ has no eigenvalues on the imaginary axis. Furthermore, Theorem 4 implies that the closed-loop transfer function $G_{\rm cl}(s)$ (56) is NI.

We now show that the feedback system defined by (53)–(55), corresponding to the state feedback control law $u = MY^{-1}x$, satisfies the assumptions of Theorem 13. Since $G_{\rm cl}(s)$ is strictly proper, it follows that $G_{\rm cl}(\infty) = 0$ and hence $\Delta(\infty)G_{\rm cl}(\infty) = 0$. Also, it follows from (61) that

$$G_{\rm cl}(0) = -C_1(A+B_2K)^{-1}B_1 = C_1YC_1^T.$$

Therefore, the LMI (58) implies $G_{\rm cl}(0) < I$ and, since $G_{\rm cl}(s)$ is negative imaginary, it follows from Lemma 12 that $G_{\rm cl}(0) \geq G_{\rm cl}(\infty)$. Moreover, $G_{\rm cl}(\infty) = 0$ and thus $G_{\rm cl}(0) \geq 0$. Hence, $\lambda_{\max}(G_{\rm cl}(0)) = \sigma_{\max}(G_{\rm cl}(0))$, and consequently $\lambda_{\max}(G_{\rm cl}(0)) < 1$. Also, the assumptions on $\Delta(s)$ in (50)–(52) imply that $\Delta(\infty) \geq 0$ and $\lambda_{\max}(\Delta(0)) \leq 1$. From these conditions, it follows that $\lambda_{\max}(\Delta(0) G_{\rm cl}(0)) < 1$.

Thus, we have $\Delta(\infty)G_{\rm cl}(\infty) = 0$, $\Delta(\infty) \geq 0$, and $\lambda_{\max}(\Delta(0)G_{\rm cl}(0)) < 1$. Therefore, the assumptions of Theorem 13 are satisfied. Now Theorem 13 implies that the closed-loop system (50)–(52) with the state-feedback controller $u = MY^{-1}x$ is robustly stable. ■

An LMI State-Feedback Synthesis Example

To illustrate Theorem 14, consider the system shown in Figure 11, which includes a flexible structure. The force applied to the flexible structure is denoted by x_2, and the deflection of the structure at the same location is denoted y. The transfer function from x_2 to y is denoted $G(s)$. The flexible structure has a colocated force actuator and position sensor and the transfer function $G(s)$ is assumed to be SNI. It is desired to construct a state-feedback controller for this system, which is robust against unmodeled flexible dynamics. Indeed, to apply the method of Theorem 14 to this example, the transfer function $G(s)$ is replaced by a constant unity gain, and the resulting error is the SNI transfer function $\Delta(s) = G(s) - 1$. The transfer function $\Delta(s)$ is treated as an uncertainty in the system as shown in Figure 12. A state-space realization of this uncertain system is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w + \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} u,$$

$$z = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad w = \Delta(s)z.$$

Then, Theorem 14 can be applied with

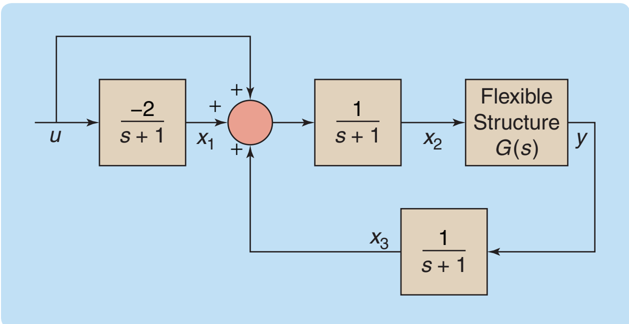

FIGURE 11 Control of a flexible structure system using a state-feedback linear matrix inequality (LMI) approach to robust controller design. This system includes the unknown flexible structure transfer function $G(s)$. In this system, the force applied to the structure is labeled x_2, and the deflection of the structure at the same location is labeled y. A state-feedback controller is to be designed for this system by replacing the flexible structure transfer function $G(s)$ by a unity gain and treating the resulting error $\Delta(s) = G(s) - 1$ as a strictly negative-imaginary uncertainty. The state-feedback controller gain matrix can be obtained by solving an LMI feasibility problem.

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}, \ B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \ B_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix},$$

$$C_1 = [0 \quad 1 \quad 0].$$

To apply Theorem 14, we choose $\varepsilon = 10^{-6}$. Then the LMIs (57)–(59) are solved using LMI software [23] to find the matrices Y and M as

$$Y = \begin{bmatrix} 3.9594 \times 10^9 & -2.0008 & -3.9594 \times 10^9 \\ -2.0008 & 0.72850 & 1.7293 \\ -3.9594 \times 10^9 & 1.7293 & 3.9594 \times 10^9 \end{bmatrix} > 0,$$

$$M = [-2.8122 \quad 1.0000 \quad 2.6260].$$

Therefore, using Theorem 14, the required state-feedback gain matrix K can be constructed as

$$K = MY^{-1} = [0.22927 \quad 1.4581 \quad 0.22927].$$

The Bode plot of the corresponding closed-loop transfer function from w to z, given by (56) is shown in Figure 13. From this Bode plot, it is seen that $G_{cl}(s)$ is SNI since

$$\angle G_{cl}(\jmath\omega) \in (-\pi, 0)$$

for all $\omega > 0$. Also, the Bode plot of Figure 13 shows that the magnitude of the dc value of $G_{cl}(s)$ is less than unity. Since the uncertainty transfer function $\Delta(s)$ in this example is SNI, it follows from Theorem 13 that if $|\Delta(0)| \le 1$, then the closed-loop system is internally stable.

In the above example, the nominal system is obtained by replacing the flexible structure transfer function $G(s)$ by a fixed unity gain. This gain can be regarded as an approximation of the dc value of the flexible structure transfer function $G(0)$. If the dc value of the flexible structure transfer function is known to be exactly unity, then it follows that $\Delta(0) = G(0) - 1 = 0$. In this case, the dc gain condition in Theorem 13 is automatically satisfied, and there is no need to require the LMI condition (58) in constructing the state-feedback controller. However, the current approach means that the dc value of the flexible structure transfer function does not have to be known exactly, and the control system is robust against uncertainty in $G(0)$.

CONCLUSIONS

This article describes properties of a class of systems termed NI systems using ideas from classical control theory. Connections to positive-real and passive systems are also given. It is also shown that the class of NI systems yields a robust stability analysis result, which broadly speaking can be captured by saying that if one system is negative imaginary and the other system is strictly negative imaginary, then a necessary and sufficient condition for internal stability of the positive-feedback interconnection of the two systems is that the dc loop gain is less than unity. This result provides a natural framework for the analysis of robust stability of lightly damped flexible structures with unmodeled dynamics. This result also captures, in a systematic framework, graphical design

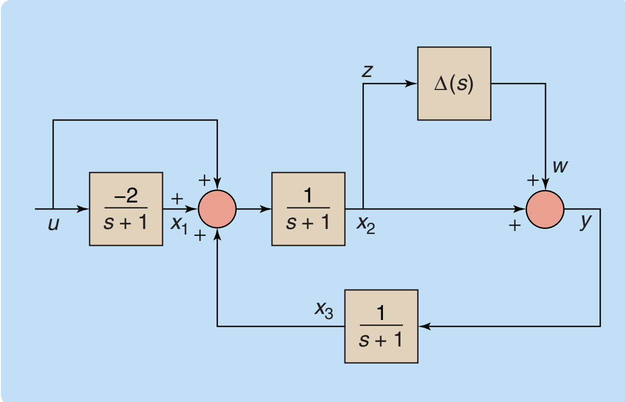

FIGURE 12 Control of an uncertain system using full-state-feedback control. The uncertain system is constructed from the system shown in Figure 11 by replacing the flexible structure transfer function $G(s)$ by $1 + \Delta(s)$, where $\Delta(s)$ is an uncertain but strictly negative-imaginary transfer function. The signal z is treated as an uncertainty output, and the signal w is treated as an uncertainty input.

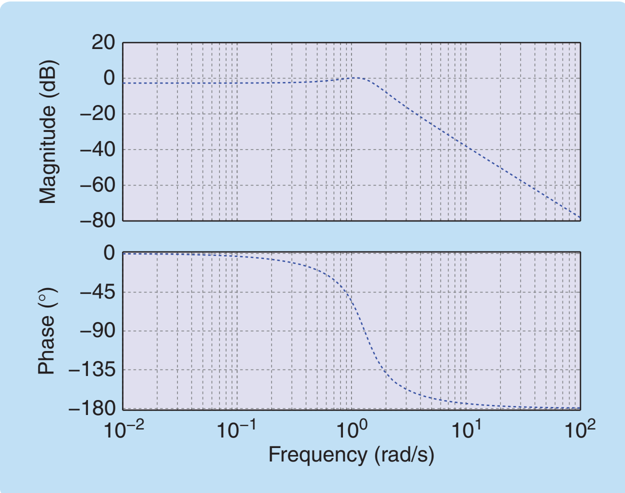

FIGURE 13 Bode plot of the closed-loop transfer function $G_{cl}(s)$ from the uncertainty input w to the uncertainty output z. This closed-loop system is obtained from the system shown in Figure 12 using a full-state-feedback controller obtained from Theorem 14. The fact that $\angle G_{cl}(\jmath\omega) \in (-\pi, 0)$ for all $\omega > 0$ implies that this transfer function is strictly negative imaginary. Also, since $G_{cl}(s)$ has no poles in CRHP and $|G_{cl}(0)| < 1$, it follows that the closed-loop uncertain system is internally stable for all uncertainties $\Delta(s)$ that are strictly negative imaginary and satisfy $|\Delta(0)| < 1$. The magnitude Bode plot shows that $|G_{cl}(0)| < 1$.

methods adopted in the 1980s by engineers related to positive-position feedback and similar techniques. This article further provides a full-state-feedback controller synthesis technique that achieves an NI closed-loop system. The use of this theory is similar to the use of passivity theory, and hence extends and complements existing passivity results [5], [6].

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