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Brief paper

Strictly negative imaginary state feedback control with a prescribed degree of stability☆

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ABSTRACT

This paper presents conditions for the synthesis of a strictly negative imaginary closed-loop system with a prescribed degree of stability under the assumption of full state feedback. A perturbation method is used to ensure the closed-loop system has both the strict negative imaginary property and a prescribed degree of stability. This approach involves the real Schur decomposition of a matrix followed by the solution to two Lyapunov equations. Also, we present and clarify the perturbation properties of strictly negative imaginary systems.

1. Introduction

Negative imaginary (NI) systems theory is a rapidly growing topic originally motivated by the study of linear mechanical systems with collocated force inputs and position outputs (Lanzon & Petersen, 2008; Petersen & Lanzon, 2010). Since its inception, NI systems theory has been applied to many domains (Bhikkaji & Moheimani, 2009; Das, Pota, & Petersen, 2014; Mabrok, Kallapur, Petersen, & Lanzon, 2014b; Petersen, 2015; Tran, Garratt, & Petersen, 2017; Wang, Lanzon, & Petersen, 2015). Also, a discrete-time notion of NI systems has been explored in Ferrante, Lanzon, and Ntogramatzidis (2016, 2017), Liu and Xiong (2017). One of the major motivations for the study of NI systems is their robust stability properties. The earliest results considering the robust stability of NI and strictly negative imaginary (SNI) systems were presented in Petersen and Lanzon (2010) and Song, Lanzon, Patra, and Petersen (2010). Subsequently, the stability properties of NI and SNI systems have motivated feedback controller synthesis results with the aim of creating a closed-loop system with the NI or SNI property. In this paper, we focus on the state feedback control problem; see Dannatt and Petersen (2019), Kurawa, Bhowmick, and Lanzon (2019) and Salcan-Reyes and Lanzon (2019) and the references within for results concerning the output feedback control problem. The earliest of the NI state feedback results used a linear matrix inequality (LMI) approach to synthesize a controller that rendered the closed-loop system NI (Lanzon & Petersen, 2008; Patra & Lanzon, 2011). Following this, Mabrok, Kallapur, Petersen, and Lanzon (2012a) drew from the H_∞ literature (See Petersen, 1989; Petersen, Anderson, & Jonckheere, 1991) and offers an algebraic Riccati equation (ARE) approach to the NI state feedback control problem. The papers (Mabrok, Kallapur, Petersen, & Lanzon, 2012b, 2015) then modified the approach of Mabrok et al. (2012a) using a perturbation applied to the system matrix of the open-loop system. The perturbation was used to give closed-loop asymptotic stability. However, the result of Mabrok et al. (2012a) could not guarantee the closed loop system was NI. To address this, Dannatt and Petersen (2018) exploited the work in Ferrante and Ntogramatzidis (2013) to show that a transfer function preserves the NI property after a positive perturbation. Also, using an ARE approach to synthesis (Salcan-Reyes & Lanzon, 2018, 2019) presented sufficient conditions for the design of a controller that solves the NI state feedback control problem. Considering the strong strictly negative imaginary (SSNI) control problem, Salcan-Reyes, Lanzon, and Petersen (2018) presented necessary and sufficient conditions for state feedback controller synthesis using an LMI based

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approach. Alternatively, Bhowmick and Patra (2017) devised a state observer-based control scheme that resulted in an SSNI closed-loop system. More recently, Mabrok (2019) proposed a novel approach to the SNI synthesis problem using nonlinear optimization.

At this time, necessary conditions for SNI controller synthesis remain an open problem. This paper will address this with separate necessary and sufficient conditions for the solution to the SNI synthesis problem via static state feedback. Our sufficient conditions generalize and extend the work of Mabrok et al. (2012b, 2015) and Dannatt and Petersen (2018) by removing an assumption on minimality, while guaranteeing the SNI property of the closed-loop system along with a prescribed degree of stability. In addition, we show that our sufficient conditions for the existence of a suitable controller are also necessary when the closed-loop system is minimal.

In order to present our main results on state feedback controller synthesis, we introduce new definitions for NI and SNI state space realizations. Also, we define SNI transfer functions and realizations with a prescribed degree of stability. A new ARE based NI lemma is given that does not assume minimality and relaxes the constraints on the system matrix found in Song, Lanzon, Patra, and Petersen (2012). Conditions for a perturbed state space realization to have the NI or SNI property are derived. We show that an NI system with a transfer function that is not degenerate is SNI after any positive perturbation of the system matrix. Further, we show that the set of SNI transfer functions is not an open set in the space of real rational proper transfer functions. Sufficient conditions are derived for the solution to the state feedback SNI synthesis problem. We also show that these conditions are necessary under the assumption of a minimal closed-loop system. An advantage of our method is that it does not require an ad-hoc check for the closed-loop SNI property and it allows for a prescribed degree of stability in the closed-loop system.

2. Definitions

Let $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers respectively. The notation $\text{Im}[G(j\omega)]$ refers to the imaginary component of the frequency response $G(j\omega)$. Analogously, $\text{Re}[G(j\omega)]$ refers to the real component of $G(j\omega)$. C^* refers to the complex conjugate transpose of a matrix or vector C. The notation $\mathbb{C}_{\leq 0}$ refers to the closed left half of the complex plane and $\mathbb{C}_{<0}$ refers to the open left half of the complex plane. $\sigma(A)$ refers to the spectrum of a matrix A. Also, $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ denotes the state space model

$$\dot{x} = Ax + Bu,$$
$$z = Cx + Du.$$

All definitions to follow are restricted to systems with transfer functions that are real, rational and proper.

Definition 1 (*Lanzon & Petersen, 2008; Mabrok, Kallapur, Petersen, & Lanzon, 2014a; Petersen & Lanzon, 2010*)**.** A square transfer function matrix $G(s)$ is NI if the following conditions are satisfied:

(1) $G(s)$ has no pole in $\text{Re}[s] > 0$;
(2) For all $\omega \geq 0$ such that jw is not a pole of $G(s)$, $j(G(j\omega) - G(j\omega)^*) \geq 0$;
(3) If $s = j\omega_0$ with $\omega_0 > 0$ is a pole of $G(s)$, then it is a simple pole and the residual matrix $K = \lim_{s \to j\omega_0}(s - j\omega_0)jG(s)$ is positive semidefinite Hermitian;
(4) If $s = 0$ is a pole of $G(s)$, then it is either a simple pole or a double pole. If it is a double pole, then, $\lim_{s\to 0} s^2 G(s) \geq 0$.

Definition 2. A state space realization $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is NI if the following conditions are satisfied:

(1) The corresponding transfer function matrix $G(s) = C(sI - A)^{-1}B + D$ is NI;
(2) $\sigma(A) \subset \mathbb{C}_{\leq 0}$.

Definition 3 (*Dannatt & Petersen, 2017; Lanzon & Petersen, 2008; Petersen & Lanzon, 2010*)**.** A square transfer function matrix $G(s)$ is SNI if the following conditions are satisfied:

(1) $G(s)$ has no poles in $\text{Re}[s] \geq 0$;
(2) For all $\omega > 0$ such that $j\omega$ is not a pole of $G(s)$, $j(G(j\omega) - G(j\omega)^*) > 0$.

Definition 4. A state space realization $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is SNI if the following conditions are satisfied:

(1) The corresponding transfer function matrix $G(s) = C(sI - A)^{-1}B + D$ is SNI;
(2) A is Hurwitz.

3. Preliminary results

The following example motivates some of the results to follow. Consider a non-minimal state space realization $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ defined by the matrices

$$A = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \end{bmatrix}, D = 0 \tag{1}$$

and corresponding transfer function $G(s) = C(sI - A)^{-1}B + D = \frac{2}{s}$. The transfer function $G(s)$ is NI via Definition 1 with a pole at the origin. However, the corresponding state space realization also has an unstable, unobservable pole at $s = 1$. This state space realization is not NI via Definition 2.

We now present some existing and new results for checking if a state space realization $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ has the NI or SNI property. These results are referred to as NI or SNI lemmas. In this paper, we are often concerned with NI and SNI lemmas that do not assume minimality. An NI lemma without an assumption of minimality is found in Song et al. (2012) and uses an LMI approach under the assumption of a non-singular system matrix A. This assumption is relaxed in Salcan-Reyes and Lanzon (2018) which gives necessary and sufficient conditions for the transfer function matrix of a given state space realization to be NI but allows for unstable unobservable modes; e.g., consider the state space realization (1). The following NI lemma provides an ARE based alternative to the NI lemmas found in Salcan-Reyes and Lanzon (2018) and Song et al. (2012). Here, we relax the non-singular system matrix condition and do not allow unstable unobservable modes. We will use this lemma in the proof of our main result.

Lemma 5. *Let $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be a given state space realization with $R = CB + B^T C^T > 0$. Suppose $D = D^T$ and there exists a real $P = P^T \geq 0$ such that*

$$PA + A^T P + (CA - B^T P)^T R^{-1}(CA - B^T P) = 0 \tag{2}$$

with $\sigma(A - BR^{-1}(CA - B^T P)) \subset \mathbb{C}_{\leq 0}$. Then, the state space realization is NI.

Proof. Suppose there exists a $P = P^T \geq 0$ such that $\sigma(A - BR^{-1}(CA - B^T P)) \subset \mathbb{C}_{\leq 0}$ and the ARE (2) is satisfied. It follows from Condition 1 of Lemma 2 in Salcan-Reyes and Lanzon (2018) that the transfer function $G(s)$ corresponding to this realization is NI. We will now show that $\sigma(A-BR^{-1}(CA-B^TP))\subset\mathbb{C}_{\leq 0}$ implies $\sigma(A)\subset\mathbb{C}_{\leq 0}$. Let $W=R^{-\frac{1}{2}}(CA-B^TP)$. Then, (2) implies

$$\begin{aligned}0&=PA+A^TP+(CA-B^TP)^TR^{-1}(CA-B^TP)\\&=PA+A^TP+W^TW.\end{aligned}\qquad(3)$$

Let $\lambda\in\mathbb{C}$ be an eigenvalue of A with corresponding eigenvector $y\neq 0$. Then $Ay=\lambda y$ and $y^*A^T=\lambda^*y^*$. If we pre-multiply (3) by y^* and post multiply by y then

$$\begin{aligned}0&=y^*(PA+A^TP+W^TW)y\\&=(\lambda+\lambda^*)y^*Py+y^*W^TWy.\end{aligned}\qquad(4)$$

Note that $P\geq 0$ by assumption. When considering (4) there are two cases of interest:

Case 1, $y^*Py=0$. In this case, (4) implies $Wy=0$. Hence, $(A-BR^{-\frac{1}{2}}W)y=Ay=\lambda y$. However, all of the eigenvalues of $(A-BR^{-\frac{1}{2}}W)$ are in $\mathbb{C}_{\leq 0}$ and thus $\lambda\in\mathbb{C}_{\leq 0}$;

Case 2, $y^*Py>0$. In this case, (4) implies $(\lambda+\lambda^*)\leq 0$; i.e., $\lambda\in\mathbb{C}_{\leq 0}$.

Since λ was an arbitrary eigenvalue of A, we have $\sigma(A)\subset\mathbb{C}_{\leq 0}$. Therefore, the state space realization is NI. □

Remark 6. Note that Lemma 5 only gives a sufficient condition for a state space realization to be NI. It is straightforward to show that the state space realization defined by (1) is NI but the ARE condition in Lemma 5 is not satisfied.

The following lemma gives conditions for the SNI property of a state space realization which is not necessarily minimal.

Lemma 7 (*Salcan-Reyes & Lanzon, 2019*). *Let* $\left[\begin{array}{c|c}A&B\\\hline C&D\end{array}\right]$ *be a given state space realization with* $R=CB+B^TC^T>0$. *Then, the following statements are equivalent:*

(1) The state space realization $\left[\begin{array}{c|c}A&B\\\hline C&D\end{array}\right]$ *is SNI;*

(2) $D=D^T$, *A is Hurwitz and there exists a real* $P=P^T\geq 0$ *such that*

$$PA+A^TP+(CA-B^TP)^TR^{-1}(CA-B^TP)=0\qquad(5)$$

with $\sigma(A-BR^{-1}(CA-B^TP))\subset\mathbb{C}_{<0}\cup\{0\}$.

The following theorem provides motivation for our Lyapunov equation approach to controller equation synthesis, as opposed to an ARE approach such as in Salcan-Reyes and Lanzon (2018, 2019).

Theorem 8. *Consider a state space realization* $\left[\begin{array}{c|c}A&B\\\hline C&D\end{array}\right]$ *such that* $R=CB+B^TC^T$ *is non-singular. If there exists a matrix* $P=P^T$ *which solves (5) then the matrix* $(A-BR^{-1}(CA-B^TP))$ *will always be singular.*

Proof. Suppose R is invertible and a solution $P=P^T$ exists that satisfies (5). The Hamiltonian matrix associated with (5) is given by

$$H=\begin{bmatrix}A-BR^{-1}CA&BR^{-1}B^T\\-A^TC^TR^{-1}CA&-A^T+A^TC^TR^{-1}B^T\end{bmatrix};\qquad(6)$$

e.g., see equations (13.1) and (13.2) in Zhou, Doyle, and Glover (1996). It is straightforward to verify that $(A-BR^{-1}(CA-B^TP))$ is singular if and only if H is singular. Also, we can write

$$H=\begin{bmatrix}I&0\\0&-A^T\end{bmatrix}(I-V_1(V_2^TV_1)^{-1}V_2^T)\begin{bmatrix}A&0\\0&I\end{bmatrix}\qquad(7)$$

where $V_1^T:=[B^T\quad -C]$, and $V_2^T:=[C\quad -B^T]$. Hence if $K:=(I-V_1(V_2^TV_1)^{-1}V_2^T)$ is singular, then H is singular. However, $KV_1=V_1-V_1=0$ (where $V_1\neq 0$ follows from the assumption that $R=V_2^TV_1$ is invertible.). Thus, H is singular and therefore $(A-BR^{-1}(CA-B^TP))$ must be singular. □

The proof of Theorem 8 shows that the Hamiltonian matrix associated with the ARE (5) will always have eigenvalues at the origin. This is significant since singular Hamiltonians may correspond to AREs which are computationally difficult to solve (Bini, Iannazzo, & Meini, 2012). Furthermore, it highlights the fact that the ARE (2) cannot have a stabilizing solution; e.g., see Zhou et al. (1996).

The following result is known as the Maximum Modulus Theorem (e.g., see Zhou et al., 1996) and is used in the proof of our main result.

Lemma 9. *If* $f(s):\mathbb{C}\to\mathbb{C}$ *is defined and continuous on a closed-bounded set S and analytic on the interior of S, then the maximum of* $|f(s)|$ *on S is attained on the boundary of S; i.e.,* $\max_{s\in S}|f(s)|=\max_{s\in\delta S}|f(s)|$ *where* δS *denotes the boundary of S.*

4. Perturbation of NI systems

The perturbation of PR systems is well understood (See Chapters 2 and 3 in Lozano, Brogliato, Egeland, & Maschke, 2000). Although a direct mapping between PR and NI systems exists, there is no such mapping between strictly positive real (SPR) and SNI systems. A consequence of this is that perturbation results for SPR systems do not readily translate to NI systems. This section outlines new perturbation results for NI systems.

The following lemma is a generalization of the corresponding transfer function results in Ferrante et al. (2016) and Ferrante and Ntogramatzidis (2013). It applies to state space realizations with a symmetric transfer function matrix $G(s)$; i.e., $G(s)=G(s)^T$.

Lemma 10. *A state space realization* $\left[\begin{array}{c|c}A&B\\\hline C&D\end{array}\right]$ *with symmetric transfer function matrix* $G(s)=C(sI-A)^{-1}B+D$ *is NI if and only if* $\sigma(A)\subset\mathbb{C}_{\leq 0}$ *and the inequality*

$$j(G(s)-G(s)^*)\geq 0\qquad(8)$$

is satisfied for all $s=j\omega+\epsilon$ *which is not a pole of* $G(s)$ *where* $\omega\geq 0$, $\epsilon\geq 0$.

Proof. Suppose the state space realization s$\left[\begin{array}{c|c}A&B\\\hline C&D\end{array}\right]$ with symmetric transfer function $G(s)$ is NI. Hence, it satisfies the conditions of Definition 2. Then $\sigma(A)\subset\mathbb{C}_{\leq 0}$ is automatically satisfied. Also, using Lemma 3.1 of Ferrante and Ntogramatzidis (2013), it follows that (8) is satisfied.

Conversely suppose $\sigma(A)\subset\mathbb{C}_{\leq 0}$ and (8) is satisfied. $G(s)$ is clearly analytic in $\mathrm{Re}[s]>0$ and since $G(s)$ is symmetric, proper, real and rational, it satisfies the conditions of Lemma 3.1 of Ferrante and Ntogramatzidis (2013). Thus $G(s)$ is NI and the corresponding realization is NI according to Definition 2. □

Definition 11. A transfer function matrix $G(s)$ is degenerate if there exists a non-zero vector v such that the function $f(s)=v^*(j(G(s)-G(s)^*))v$ is identically zero.

The following theorem relates Lemma 10 to perturbations in the system matrix of a not necessarily minimal state space realization. Specifically, we show that any positive perturbation of an NI state space realization with a symmetric non-degenerate transfer function will result in an SNI system.

Theorem 12. *If a given state space realization* $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ *is NI with a symmetric and non-degenerate transfer function* $G(s)$*, then the perturbed state space realization* $\left[\begin{array}{c|c} A-\epsilon I & B \\ \hline C & D \end{array}\right]$ *will be SNI for all* $\epsilon > 0$.

Proof. Consider a state space realization $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ with a symmetric transfer function matrix $G(s) = C(sI - A)^{-1}B + D$ that is non-degenerate. Now let $\epsilon > 0$ be given. The corresponding state space model of the perturbed system is given by

$$\dot{x} = (A - \epsilon I)x + Bu,$$
$$y = Cx + Du \tag{9}$$

with transfer function

$$G_\epsilon(s) = C(sI - A + \epsilon I)^{-1}B + D = G(s + \epsilon). \tag{10}$$

If $\sigma(A) \subset \mathbb{C}_{\leq 0}$ then $\sigma(A - \epsilon I) \subset \mathbb{C}_{<0}$ for $\epsilon > 0$. We now show that after the perturbation, $G_\epsilon(s)$ is NI. Since $G(s)$ is assumed to be NI, it follows from Lemma 10 that $j(G_\epsilon(j\omega) - G_\epsilon(j\omega)^*) = j(G(\epsilon + j\omega) - G(\epsilon + j\omega)^*) \geq 0$ for all $\omega \geq 0$. Also, $G_\epsilon(s)$ is analytic in $\mathbb{C}_{-\epsilon} = \{s \in \mathbb{C} : \mathrm{Re}[s] > -\epsilon\}$. Hence, $G_\epsilon(s)$ is analytic in $\mathrm{Re}[s] > -\alpha$ for some $\alpha > 0$. Thus, $G_\epsilon(j\omega)$ satisfies the conditions of Definition 1 and is NI. It remains to show that $G_\epsilon(s)$ is SNI. Suppose that there exists a finite $s_0 = j\omega_0 \in \{\mathbb{C}_{-\epsilon} : \mathrm{Im}[s] > 0\}$ and a nonzero vector v such that $v^*(j(G_\epsilon(s_0) - G_\epsilon(s_0)^*))v = 0$.

Since $G_\epsilon(s)$ is analytic in $\mathbb{C}_{-\epsilon}$, it follows that the function $f_\epsilon(s) = v^*(j(G_\epsilon(s) - G_\epsilon(s)^*))v$ is harmonic within the same domain. Now, choose an $\epsilon_s < \epsilon$ such that $\mathrm{Re}[s_0] > -\epsilon_s$ and an arbitrary $M > 0$. We define the compact set $\tilde{\mathbb{C}}_{-\epsilon_s} = \{s \in \mathbb{C} : -\epsilon_s \leq \mathrm{Re}[s] \leq M \text{ and } 0 \leq \mathrm{Im}[s] \leq M\}$ (see Fig. 1). This set is such that $\tilde{\mathbb{C}}_{-\epsilon_s} \subset \mathbb{C}_{-\epsilon}$. Since $\tilde{\mathbb{C}}_{-\epsilon_s}$ is a nonempty compact subset of $\mathbb{C}_{-\epsilon}$, it follows from Lemma 9 that the function $f_\epsilon(s)$, which is harmonic on $\tilde{\mathbb{C}}_{-\epsilon_s}$, attains its maximum and minimum on the boundary of $\tilde{\mathbb{C}}_{-\epsilon_s}$. Also, as $\tilde{\mathbb{C}}_{-\epsilon_s}$ is connected, the function $f_\epsilon(s)$ can only have a local maxima or minima on $\tilde{\mathbb{C}}_{-\epsilon_s}$ if it is constant.

Now by taking M sufficiently large, s_0 will be in the interior of $\tilde{\mathbb{C}}_{-\epsilon_s}$. Remembering that we assumed that $f_\epsilon(s_0) = 0$, it follows from the connectedness of $\tilde{\mathbb{C}}_{-\epsilon_s}$ that $f_\epsilon(s) = f_\epsilon(s_0) = 0$ for all s. From this it follows that $f_0(s) = f(s) = v^*(j(G(s) - G(s)^*))v = 0$ for all s. This contradicts the assumption that $G(s)$ is non-degenerate. Therefore $f_\epsilon(j\omega) > 0$ for all $\omega \in (0, \infty)$. Since v was arbitrary, we can now conclude $j(G_\epsilon(jw) - G_\epsilon(jw)^*) > 0$ for all $\omega \in (0, \infty)$. Also, $G_\epsilon(s)$ is analytic in $\mathrm{Re}[s] \geq 0$. Therefore, according to Definition 4, the state space realization is SNI. □

Remark 13. Note that Theorem 12 does not hold in the MIMO case unless the symmetry of $G(s)$ is assumed. To illustrate this, consider the following non-symmetric MIMO state space realization:

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

with non-symmetric transfer function matrix

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & -\frac{s}{(s+1)^2} \\ 0 & \frac{1}{s+1} \end{bmatrix}.$$

Here, A is Hurwitz and $G(s)$ is SNI according to Definition 3 if $j[G(j\omega) - G(j\omega)^*] > 0$ for $\omega \in (0, \infty)$. It is straightforward to verify that this property is satisfied here. Perturbing $G(s)$ results in the transfer function

$$G_\epsilon(j\omega) = G(j\omega + \epsilon) = \begin{bmatrix} \frac{1+\epsilon-j\omega}{(1+\epsilon)^2+\omega^2} & \frac{-(2+\epsilon)\omega^2-\epsilon(1+\epsilon)^2+\omega(\omega^2+2\epsilon(1+\epsilon)-(1+\epsilon)^2)j}{((1+\epsilon)^2-\omega^2)^2+4(1+\epsilon)^2\omega^2} \\ 0 & \frac{1+\epsilon-j\omega}{(1+\epsilon)^2+\omega^2} \end{bmatrix}.$$

The corresponding quantity $M(j\omega) = j[G_\epsilon(j\omega) - G_\epsilon(j\omega)^*]$ has the form

$$M(j\omega) = \begin{bmatrix} \frac{2\omega}{(1+\epsilon)^2+\omega^2} & \frac{-\alpha(\omega)-\beta(\omega)j}{\zeta(\omega)} \\ \frac{-\alpha(\omega)+\beta(\omega)j}{\zeta(\omega)} & \frac{2\omega}{(1+\epsilon)^2+\omega^2} \end{bmatrix}$$

where

$$\alpha(\omega) = \omega(\omega^2 + 2\epsilon(1+\epsilon) - (1+\epsilon)^2),$$
$$\beta(\omega) = ((2+\epsilon)\omega^2 + \epsilon(1+\epsilon)^2),$$
$$\zeta(\omega) = ((1+\epsilon)^2 - \omega^2)^2 + 4(1+\epsilon)^2\omega^2.$$

$M(j\omega)$ satisfies Condition (2) of Definition 3 if and only if its leading principle minors are positive for all $\omega \in (0, \infty)$. We can immediately see this is satisfied for the $(1, 1)$ block of $M(j\omega)$. Therefore, only the following minor which is the determinant of $M(j\omega)$ needs to be analyzed:

$$det(M(j\omega)) = \frac{4\omega^2}{((1+\epsilon)^2 + \omega^2)^2} - \frac{\alpha(\omega)^2 + \beta(\omega)^2}{\zeta(\omega)^2}. \tag{11}$$

Note that if $det(M(j\omega))$ is strictly negative at $\omega = 0$, it follows from continuity that it cannot be positive for all $\omega > 0$. When $\epsilon = 0$, (11) yields

$$det(M(j\omega)) = 4\omega^{10} + 16\omega^8 + 24\omega^6 + 16\omega^4 + 4\omega^2 > 0.$$

This is clearly positive-definite for all $\omega \in (0, \infty)$. Therefore, $G(s)$ is SNI for $\epsilon = 0$. When $\omega = 0$, (11) yields

$$det(M(0)) = -\epsilon^2(1+\epsilon)^8 < 0 \quad \text{for} \quad \epsilon > 0.$$

Therefore $G(s)$ is not NI for any $\epsilon > 0$. Thus, this example shows that Theorem 12 does not generalize to non-symmetric MIMO systems.

Definition 14. A symmetric transfer function matrix $G(s)$ is said to be SNI with degree of stability $\epsilon > 0$ if $G(s - \epsilon)$ is SNI.

Definition 15. A state space realization $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ with symmetric, non-degenerate transfer function matrix $G(s)$ is said to be SNI with degree of stability $\epsilon > 0$ if $G(s - \epsilon)$ is SNI and $\sigma(A) \subset \{s \in \mathbb{C} : \mathrm{Re}(s) \leq -\epsilon\}$.

Remark 16. The set of SNI transfer functions is not open in the space of real, rational and proper transfer functions. To illustrate this, consider the SISO transfer function $G(s) = \frac{s+1}{(s+2)^2}$, which has imaginary component

$$\mathrm{Im}[G(j\omega)] = \frac{-\omega^3}{16\omega^2 + (4 - \omega^2)^2} < 0 \qquad \forall \omega \in (0, \infty).$$

Hence, $G(s)$ is SNI. Now consider the perturbed transfer function $G_\epsilon(s) = G(s + \epsilon)$ with imaginary component

$$\mathrm{Im}[G_\epsilon(j\omega)] = \frac{-2\epsilon - \epsilon^2 - \omega^2}{((2+\epsilon)^2 - \omega)^2 + 4\omega^2(2+\epsilon)^2}.$$

It follows that $\mathrm{Im}[G_\epsilon(j\omega)]$ is positive when $-2\epsilon - \epsilon^2 - \omega^2 > 0$. Thus, $G_\epsilon(s)$ does not have the NI property for any $\epsilon \in (-2, 0)$. Therefore, the set of SNI transfer functions is not open.

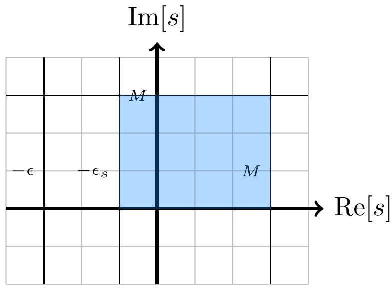

Fig. 1. Complex plane showing the shaded region $\tilde{\mathbb{C}}_{-\epsilon_s}$. This region is bounded by a choice of $\epsilon_s < \epsilon$ and an arbitrary $M > 0$. M is chosen to be sufficiently large such that a point $s_0 = j\omega_0 \in \{\mathbb{C}_{-\epsilon} : \mathrm{Im}[s] > 0\}$ will be in the interior of $\mathbb{C}_{-\epsilon_s}$.

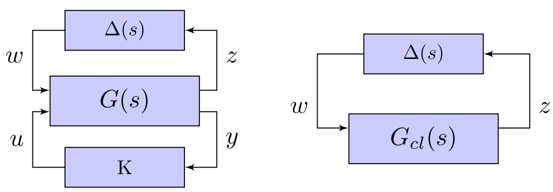

Fig. 2. Positive feedback interconnection of an NI plant uncertainty $\Delta(s)$ and the closed-loop SNI transfer function $G_{cl}(s) = C_1(sI - A - B_2K)^{-1}B_1$.

5. SNI state feedback controller synthesis

Consider the state space representation of a linear uncertain system given by

$$\dot{x} = Ax + B_1 w + B_2 u,$$
$$z = C_1 x \tag{12}$$

where $A \in \mathbb{R}^{n\times n}$, $B_1 \in \mathbb{R}^{n\times r}$, $B_2 \in \mathbb{R}^{n\times r}$, $C_1 \in \mathbb{R}^{r\times n}$. This system is connected to an uncertain transfer function $\Delta(s)$ with minimal state space realization

$$\dot{x}_\Delta = A_\Delta x_\Delta + B_\Delta z,$$
$$w = C_\Delta x_\Delta + D_\Delta z. \tag{13}$$

We will assume that this system satisfies the assumptions

A1. The matrix C_1B_2 is non-singular;
A2. $R = C_1B_1 + B_1^TC_1^T > 0$.

If we apply a state feedback control law $u = Kx$ to this system, the corresponding closed-loop uncertain system has state space representation

$$\dot{x} = (A + B_2K)x + B_1 w,$$
$$z = C_1 x \tag{14}$$

with corresponding closed-loop transfer function $G_{cl}(s) = C_1(sI - A - B_2K)^{-1}B_1$ (see Fig. 2).

In order to design a robust control law $u = Kx$ for this system, first consider a real Schur transformation applied to the system.

5.1. Schur decomposition

Let the constant $\epsilon > 0$ defining the required stability margin be given. We begin by applying a Schur decomposition to the matrix $A_m = A + \epsilon I - B_2(C_1B_2)^{-1}C_1(A + \epsilon I)$ and then apply the corresponding transformation to the rest of the system as follows:

$$A_f = U^TA_mU = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \tag{15a}$$

$$B_f = U^T(B_2(C_1B_2)^{-1} - B_1R^{-1}) = \begin{bmatrix} B_{f1} \\ B_{f2} \end{bmatrix}, \tag{15b}$$

$$\tilde{B_1} = U^TB_1 = \begin{bmatrix} B_{11} \\ B_{22} \end{bmatrix}, \tag{15c}$$

where A_{11} has all of its eigenvalues in the closed left half of the complex plane and A_{22} is an anti-stable matrix; i.e., $\sigma(A) \subset \{s : \mathrm{Re}[s] > 0\}$. Here, U is an orthogonal matrix obtained through the real Schur transformation; see Section 5.4 of Bernstein (2009).

The following theorem, which is our main result, extends Theorem 4 in Mabrok et al. (2015); see also Dannatt and Petersen (2018) and Mabrok et al. (2012a, 2012b). In Mabrok et al. (2015), the perturbation approach was not shown to preserve the NI property. We show that in fact, the perturbation approach leads to the closed-loop SNI property. Also, in Mabrok et al. (2015) the closed-loop system is assumed minimal. This assumption is relaxed in our result using Lemma 5.

Theorem 17. *Consider the uncertain system (12) with $r = 1$ satisfying assumptions A1–A2. For a given $\epsilon > 0$, there exists a static state feedback matrix K such that the closed-loop system (14) is SNI with degree of stability ϵ if there exist matrices $T \geq 0$ and $S \geq 0$ such that*

$$-A_{22}T - TA_{22}^T + B_{f2}RB_{f2}^T = 0, \tag{16}$$
$$-A_{22}S - SA_{22}^T + B_{22}R^{-1}B_{22}^T = 0, \tag{17}$$
$$T - S > 0, \tag{18}$$

where A_{22}, B_{f2} and B_{22} are obtained from the Schur decomposition (15). Moreover, if the conditions (16)–(18) are satisfied, then the required state feedback controller matrix K is given by

$$K = (C_1B_2)^{-1}(B_1^TP - C_1A - \epsilon C_1 - R(B_2^TC_1^T)^{-1}B_2^TP), \tag{19}$$

where $P = UP_fU^T$ and $P_f = \begin{bmatrix} 0 & 0 \\ 0 & (T-S)^{-1} \end{bmatrix} \geq 0$. Here, U is the orthogonal matrix obtained through the Schur transformation (15).

Proof. Let $\epsilon > 0$ be given and suppose there exist matrices $T, S \geq 0$ satisfying (16)–(18). Subtracting (17) from (16) gives the following Lyapunov equation

$$A_{22}X + XA_{22}^T - B_{22}R^{-1}B_{22}^T + B_{f2}RB_{f2}^T = 0 \tag{20}$$

where $X = T - S > 0$. Let $P_1 = X^{-1} > 0$. Pre and post multiplying (20) by X yields the ARE

$$P_1A_{22} + A_{22}^TP_1 - P_1B_{22}R^{-1}B_{22}^TP_1 + P_1B_{f2}RB_{f2}^TP_1 = 0. \tag{21}$$

It follows that the ARE

$$P_fA_f + A_f^TP_f - P_f\tilde{B_1}R^{-1}\tilde{B_1}^TP_f + P_fB_fRB_f^TP_f = 0 \tag{22}$$

has a solution $P_f = \begin{bmatrix} 0 & 0 \\ 0 & P_1 \end{bmatrix} \geq 0$, where A_f, B_f and $\tilde{B_1}$ are defined as in (15). After rearranging and collecting terms and using the Schur decomposition (15), (22) is equivalent to

$$P\tilde{A} + \tilde{A}^TP + PB_1R^{-1}B_1^TP + Q = 0 \tag{23}$$

where

$$\tilde{A} = A_{cl} - B_1R^{-1}C_1A_{cl}, \quad R = C_1B_1 + B_1^TC_1^T, \quad Q = A_{cl}^TC_1^TR^{-1}C_1A_{cl}.$$

Here $A_{cl} = A + \epsilon I + B_2K$ is the perturbed system matrix of the closed-loop system (14). We now show that $\sigma(A_{cl} - BR^{-1}(CA_{cl} - B^TP)) \subset \mathbb{C}_{\leq 0}$. After substituting in our choice of K we see that

$$\begin{aligned}&\sigma(A_{cl} - BR^{-1}(CA_{cl} - B^TP)) \\ =&\sigma(A_f + (B_fRB_f^T - \tilde{B}_1R^{-1}\tilde{B}_1^T)P_f) \\ =&\sigma(\begin{bmatrix} A_{11} & \star \\ 0 & A_{22} + B_{f2}RB_{f2}^TP_1 - B_{22}R^{-1}B_{22}^TP_1 \end{bmatrix}).\end{aligned} \quad (24)$$

Here, $\star$ denotes a matrix element which is not relevant to our argument. Note $\sigma(A_{11}) \subset \mathbb{C}_{\leq 0}$ by assumption so we need only concern ourselves with the (2, 2) block of the matrix in (24). To that end, note that if we post-multiply (20) by P_1 we obtain

$$A_{22} + P_1^{-1}A_{22}^TP_1 - (B_{22}R^{-1}B_{22}^T - B_{f2}RB_{f2}^T)P_1 = 0. \quad (25)$$

Therefore $\sigma(A_{22} + B_{f2}RB_{f2}^TP_1 - B_{22}R^{-1}B_{22}^TP_1) = \sigma(-A_{22}) \subset \mathbb{C}_{<0}$. Thus, $\sigma(A_{cl} - BR^{-1}(CA_{cl} - B^TP)) \subset \mathbb{C}_{\leq 0}$. It then follows from Lemma 5 that the perturbed closed-loop state space realization is NI with $\sigma(A_{cl}) \subset \mathbb{C}_{\leq 0}$. Theorem 12 then implies that the actual closed-loop system corresponding to the unperturbed system will have all its poles shifted by an amount ϵ to the left in the complex plane. Therefore, the closed-loop system is SNI with degree of stability ϵ. □

Remark 18. Theorem 17 assumes that $r = 1$ in (12) which means that the closed-loop system is SISO. This assumption is used in order to guarantee the symmetry of the closed-loop transfer function. For the case of a MIMO closed-loop transfer function matrix, symmetry would first need to be confirmed before the perturbation approach could be applied.

Remark 19. When considering a feasible value of ϵ we note that we can choose ϵ within a set $\mathcal{S} = \{\epsilon : 0 < \epsilon < \epsilon_x\}$ such that $T - S > 0$ for all $\epsilon \in \mathcal{S}$. The size of ϵ_x depends on the parameters of a given system and currently needs to be found numerically.

Note that by choosing $\epsilon = 0$, Theorem 17 generalizes the main result of Mabrok et al. (2015) and will synthesize an NI closed-loop system with a pole at the origin.

General necessary conditions for the existence of a state feedback control that renders the closed-loop system SNI with degree of stability ϵ remains an open problem. However, if the closed-loop system is assumed to be minimal and SNI with degree of stability ϵ, then the following result can be established.

Theorem 20. *Consider the uncertain system (12). For a given $\epsilon > 0$, if there exists a static state feedback matrix K such that the closed-loop system (14) is minimal and SNI with degree of stability ϵ then there exist matrices $T \geq 0$ and $S \geq 0$ such that*

$$-A_{22}T - TA_{22}^T + B_{f2}RB_{f2}^T = 0, \quad (26)$$

$$-A_{22}S - SA_{22}^T + B_{22}R^{-1}B_{22}^T = 0, \quad (27)$$

$$T - S > 0, \quad (28)$$

where A_{22}, B_{f2} and B_{22} are obtained from the Schur decomposition (15).

Proof. Suppose there exists a control law $u = Kx$ such that the closed-loop system

$$\dot{x} = (A + \epsilon I + B_2K)x + B_1w,$$
$$z = C_1x,$$

is minimal and SNI with degree of stability $\epsilon > 0$. It follows from Theorem 3 in Mabrok et al. (2015) that there exists a $P = P^T > 0$ that satisfies

$$P\tilde{A} + \tilde{A}^TP + PB_1R^{-1}B_1^TP + Q = 0, \quad (29)$$

where

$$\tilde{A} = A_\epsilon + B_2K - B_1R^{-1}C_1A_\epsilon - B_1R^{-1}C_1B_2K,$$
$$A_\epsilon = A + \epsilon I, \quad R = C_1B_1 + B_1^TC_1^T > 0,$$
$$Q = (A_\epsilon + B_2K)^TC_1^TR^{-1}C_1(A_\epsilon + B_2K).$$

It follows using straightforward algebraic manipulations that (29) is equivalent to

$$PA_m + A_m^TP + P(B_1R^{-1}B_1^T - B_mRB_m^T)P + (K^TM^T + N^T)(MK + N) = 0,$$

where

$$A_m = A_\epsilon - B_2(C_1B_2)^{-1}C_1A_\epsilon,$$
$$B_m = B_2(C_1B_2)^{-1} - B_1R^{-1}, \quad M = R^{-\frac{1}{2}}C_1B_2,$$
$$N = R^{-\frac{1}{2}}C_1A_\epsilon - R^{-\frac{1}{2}}B_1^TP + R^{\frac{1}{2}}(B_2^TC_1^T)^{-1}B_2^TP.$$

The matrix $(K^TM^T + N^T)(MK + N)$ is positive semi-definite and therefore it follows that

$$PA_m + A_m^TP - PB_mRB_m^TP + PB_1R^{-1}B_1^TP \leq 0. \quad (30)$$

Now let $X = P^{-1} = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} > 0$, where X is partitioned to be compatible with the Schur decomposition (15). We pre and post multiply (30) by X to obtain

$$A_mX + XA_m^T - B_mRB_m^T + B_1R^{-1}B_1^T \leq 0. \quad (31)$$

If we apply the Schur decomposition (15) to (31), it follows that

$$U^T(A_mX + XA_m^T - B_mRB_m^T + B_1R^{-1}B_1^T)U \leq 0.$$

The (2,2) block of this inequality gives

$$A_{22}X_{22} + X_{22}A_{22}^T - B_{f2}RB_{f2}^T + B_{22}R^{-1}B_{22}^T \leq 0.$$

Hence, there exists a $Q = Q^T \geq 0$ such that

$$(-A_{22})X_{22} + X_{22}(-A_{22})^T + B_{f2}RB_{f2}^T - B_{22}R^{-1}B_{22}^T - Q = 0.$$

Let $\tilde{X} - X_{22} = \int_0^\infty e^{-A_{22}^Tt}Qe^{-A_{22}t}dt \geq 0$. $\tilde{X}$ satisfies the Lyapunov equation

$$(-A_{22})\tilde{X} + \tilde{X}(-A_{22})^T + B_{f2}RB_{f2}^T - B_{22}R^{-1}B_{22}^T = 0.$$

Now let $T \geq 0$ and $S \geq 0$ be defined by

$$T = \int_0^\infty e^{-A_{22}^Tt}B_{f2}RB_{f2}^Te^{-A_{22}t}dt,$$
$$S = \int_0^\infty e^{-A_{22}^Tt}B_{22}R^{-1}B_{22}^Te^{-A_{22}t}dt.$$

T and S satisfy the Lyapunov equations

$$(-A_{22})T + (-A_{22})^TT + B_{f2}RB_{f2}^T = 0,$$
$$(-A_{22})S + (-A_{22})^TS + B_{22}R^{-1}B_{22}^T = 0,$$

which we recognize as Eqs. (26) and (27). Furthermore,

$$\begin{aligned}\tilde{X} &= \int_0^\infty e^{-A_{22}^Tt}\left(B_{f2}RB_{f2}^T - B_{22}R^{-1}B_{22}^T\right)e^{-A_{22}t}dt \\ &= T - S.\end{aligned}$$

Hence $T - S \geq X_{22} > 0$ as required. □

6. Illustrative example

Consider an uncertain system of the form (12) where:

$$A = \begin{bmatrix} -1 & 0 & -1 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}, B_1 = \begin{bmatrix} 2 \\ 1 \\ 0.5 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

$C_1 = \begin{bmatrix} 2 & 2 & -1 \end{bmatrix}$.

The example given in Mabrok et al. (2012b) considers this system with a perturbation of $\epsilon = 0.3$ in order to move the poles away from the origin to achieve a stable closed-loop NI system. It follows from Theorem 17 that any $\epsilon > 0$ will actually result in a SNI closed-loop system provided that the condition $T > S$ is satisfied. We choose $\epsilon = 1$.

The solutions to the Lyapunov equations (16) and (17) are $T = 0.027$ and $S = 0.012$ which implies that $X = 0.015$. We can use this to solve for $P = UP_f U^T \geq 0$ and the controller gain matrix is then given by $K = \begin{bmatrix} -1 & -17.99 & -35.77 \end{bmatrix}$. The closed-loop transfer function formed using this state feedback controller is given by

$$G_{cl}(s) = \frac{5.5s^2 + 16.21s + 11.85}{s^3 + 16.99s^2 + 35.77s + 19.78}.$$

$G_{cl}(s)$ is SNI with real poles located at -1.0, -1.35 and -14.64 in the complex plane. In this case, the closed loop state space realization (14) is minimal and so these poles correspond to the eigenvalues of the closed loop system matrix $A_{cl} = A + B_2K$ which is Hurwitz.

7. Conclusions

In this paper, we have developed conditions for synthesizing a state feedback controller that guarantees a strictly negative imaginary closed-loop system with a prescribed degree of stability. This synthesis approach relies on the solution to two Lyapunov equations. In addition, we show that our sufficient condition for the existence of a suitable controller is also necessary when the closed-loop transfer function is minimal. Future work will extend the necessity result to remove this minimality assumption. Currently no SNI synthesis results have been published for the discrete-time case. However, the discrete-time mappings in Ferrante et al. (2017) may be useful in extending the results of this paper to discrete-time systems.

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