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Brief paper

# Simultaneous optimization of performance weights and a controller in mixed-$\mu$ synthesis[☆]

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ABSTRACT

In this paper, we investigate the problem of synthesizing performance weights and a controller that maximize the level of robust performance for a plant subject to both complex and real uncertainties. In particular, in this work the formulation proposed in Lanzon (2005a) and Lanzon and Cantoni (2003) is manipulated in order to include the possibility to handle both real and complex uncertainties. Additionally, we introduce a novel solution algorithm that presents performance and computational advantages with respect to those described in Lanzon (2005a) and Lanzon and Cantoni (2003).



## 1. Introduction

Over the past decade, modern robust control theory has revolutionized multi-variable controller design. In particular, $\mu$-synthesis (Packard & Doyle, 1993; Young, 2001) has been widely applied to design complex multi-input–multi-output control systems with a guaranteed level of robust stability and performance (Balas, Doyle, Glover, Packard, & Smith, 1998; Braatz & Morari, 1992; Buschek & Calise, 1997; Skogestad & Postlethwaite, 1998). In general, this technique first requires the specification of weighting functions to reflect desired performance and robustness requirements. Then the control synthesis is recast into a weighted optimization problem to find a controller that attempts to achieve the level of robust stability and performance required by the specified weights.

The definition of appropriate weighting functions is a crucial phase of the control system design in the $\mu$ framework. These weighting functions must usually reflect different desired properties of the closed loop system that can be also in conflict with each other. For this reason there is no commonly accepted procedure that allows for a general design of weighting functions (Jovik & Lennartson, 0000; Stoughton, 0000). In practice, the typical control design comes down to an iterative procedure that first requires the definition of weighting functions, then the synthesis of the controller and finally a robust stability and performance test. In case the resulting controller does not achieve the required robustness margins, an additional iteration is required.

Papers (Lanzon, 2005a; Lanzon & Cantoni, 2003) extend skewed-$\mu$ (see Fan and Tits (1992)) ideas to recast the selection of appropriate weights into an optimization problem.[2] In the optimization technique proposed in Lanzon (2005a) and Lanzon and Cantoni (2003), the user is required to specify the uncertain plant set and the optimization directionality, i.e. functions that qualitatively reflect desired performance objectives over frequency. Then the optimization technique automatically synthesizes both the weighting functions and the controller resolving also the possible inconsistency between the qualitative desired specifications. It is demonstrated in Lanzon (2005a) and Lanzon and Cantoni (2003) through a series of examples that the specification of the directionality functions is much easier than the direct specification of performance weights, and that the proposed optimization technique greatly simplifies the often tedious trial and error process needed in a typical $\mu$ design. However, this technique has been developed to handle only complex structured singular value problems.

The work presented here has a twofold objective. On one side, it attempts to reformulate the optimization based control synthesis

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2 Related weight optimization work appeared in Anderson, Lanzon, Dehghani, and Bombois (2009), Lanzon (2005b) and Osinuga, Patra, and Lanzon (2010) in the $\mathcal{H}_\infty$ loop-shaping framework.

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technique introduced in Lanzon (2005a) and Lanzon and Cantoni (2003) to include the possibility of accounting for both complex and real uncertainty. This will generalize the robust performance design associated with the previously developed techniques thus being able to handle also real parametric uncertainties in addition to complex perturbations. On the other side, it introduces a novel solution algorithm that presents performance and computational advantages with respect to those respectively introduced in Lanzon (2005a) and Lanzon and Cantoni (2003). Note that the synthesis procedure proposed in this work does not represent an improvement over the classical $\mu$-synthesis technique with respect to issues such as the non-convexity of the associated optimization problem. Rather, it provides an alternative scheme that automatically performs the trade-off between performance and robustness, simultaneously designing both the controller and the performance weights.

*Notation*. Let $\overline{\mathbb{R}}_+$ denote the non negative real numbers, $\overline{\mathbb{C}}_+$ the closed right half complex plane and $\mathbb{C}^{m\times n}$ the set of complex matrices of dimension $m \times n$. The maximum singular value of a matrix $A \in \mathbb{C}^{m\times n}$ is denoted by $\overline{\sigma}(A)$. $A^T$ (resp. $A^*$) is the transpose (resp. complex conjugate transpose) of $A \in \mathbb{C}^{m\times n}$ and $\|A\|_F$ denotes the Frobenius norm of the matrix $A$. The $k \times k$ identity matrix and zero matrix are denoted by $I_k$, and $O_k$, respectively and $\otimes$ denotes the Kronecker product. A real rational matrix function $\Sigma(s)$ of a complex variable $s$ is such that $\Sigma(s) \in \mathcal{RH}_\infty$ if it is bounded and analytic in the open complex right half plane. The adjoint system of $\Sigma(s)$ is defined by $\Sigma^\sim(s) = \Sigma(-s)^T$. The $\|\cdot\|_\infty$ norm of an $m\times n$ matrix function $\Sigma(s)$ is defined by $\|\Sigma\|_\infty := \sup_\omega \overline{\sigma}(\Sigma(j\omega))$. Finally, diag$[A, B]$ with $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{p\times q}$ denotes the $(m+p) \times (n+q)$ block diagonal complex matrix composed of $A$ and $B$.

## 2. Problem statement

The general control synthesis set up in robust control theory is depicted in Fig. 1 (a), where $\Sigma(s)$ is the generalized plant, partitioned consistently with the interconnection, and $\Delta(s)$ represents a stable structured perturbation with $r$ inputs and $r$ outputs. The structure of the perturbation is defined by the set

$$\begin{aligned}\mathbf{\Delta} := \Big\{&\Delta = \mathrm{diag}[I_{n_1} \otimes \Delta_1, \ldots, I_{n_g} \otimes \Delta_g, I_{n_{g+1}} \otimes \Delta_{g+1},\\ &\ldots, I_{n_{g+d}} \otimes \Delta_{g+d}] : \Delta_i = \Delta_i^T \in \mathbb{R}^{k_i\times k_i}\ \forall i \in \{1,\ldots,g\}\\ &\text{and } \Delta_i \in \mathbb{C}^{k_i\times k_i}\ \forall i \in \{g+1,\ldots,g+d\}\Big\} \qquad (1)\end{aligned}$$

where $\sum_{i=1}^{g+d} n_i k_i = r$. $\Delta(s)$ is assumed to belong to the set $\mathbf{B\Delta} := \{\Delta(s) \in \mathcal{RH}_\infty : \Delta(s_0) \in \mathbf{\Delta}\forall s_0 \in \overline{\mathbb{C}}_+, \|\Delta\|_\infty \le 1\}$. $K(s)$ is a controller with $q$ inputs and $p$ outputs belonging to the set of controllers that internally stabilize the generalized plant $\Sigma$ (denoted by $\mathcal{K}$). The system is subject to $n$ exogenous disturbances and the performance is measured in terms of the $n$ error signals. The closed loop requirements for the system performances are included in the design by means of the diagonal frequency dependent performance weights matrix $W(s) \in \mathcal{W} := \{\mathrm{diag}_{i=1}^n[w_i(s)] : w_i(s), w_i(s)^{-1} \in \mathcal{RH}_\infty\}$. It is well known (see Zhou and Doyle (1999)) that the closed loop system in Fig. 1 (a) achieves robust performance in the presence of uncertainty if the following condition holds

$$\sup_\omega \mu_{\mathbf{\Delta}_T}[\mathrm{diag}[I_r, W(j\omega)]\mathcal{F}_l(\Sigma, K)(j\omega)] < 1, \qquad (2)$$

where $\mu$ denotes the structured singular value and $\mathbf{\Delta}_T := \{\Delta_T = \mathrm{diag}(\Delta, \Delta_p) : \Delta_p \in \mathbb{C}^{n\times n}, \Delta \in \mathbf{\Delta}\}$ denotes the augmented uncertainty structure introduced to consider the robust performance problem (see Fig. 1(b)). Following the idea in Lanzon (2005a) and Lanzon and Cantoni (2003), in this paper we will address the following problem:

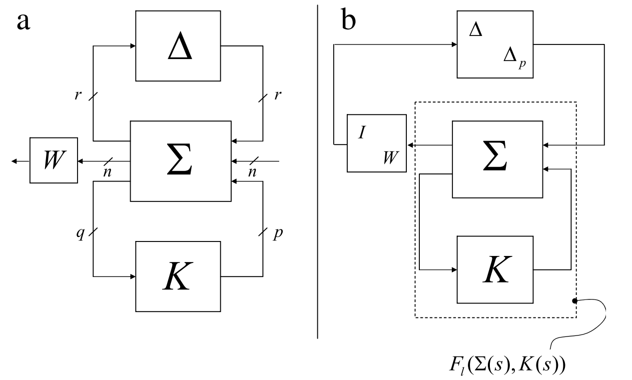



**Fig. 1.** Generalized block interconnection for synthesis and analysis.

$$\max_{W\in\mathcal{W}} J(W) \quad \text{subject to}$$
$$\min_{K\in\mathcal{K}} \sup_\omega \mu_{\mathbf{\Delta}_T}[\mathrm{diag}[I_r, W(j\omega)]\mathcal{F}_l(\Sigma, K)(j\omega)] < 1 \qquad (3)$$

for some cost function $J(\cdot)$. In Section 3, optimization problem (3) is manipulated to allow for the definition of an efficient solution algorithm.

## 3. Performance weight and controller optimization

The objective function in (3) must be able to capture the performance preferences of the design that, in common practice, are handled by penalizing each output of the closed loop system with a weight, $w_i(j\omega)$, whose magnitude reflects the inverse of the desired specification.

In this work, the following objective function

$$J(W) = \frac{1}{\int_{\log_{10}\omega_L}^{\log_{10}\omega_H} \sum_{i=1}^{n} \frac{1}{|w_i(j\omega)/\upsilon_i(j\omega)|^2} \mathrm{d}(\log_{10}\omega)} \qquad (4)$$

first introduced in Lanzon (2005a), is used in the optimization problem (3). In Eq. (4), $[\omega_L, \omega_H]$ is the frequency range where the performance requirements are defined and $\upsilon_i(s)$ are $n$ given stable minimum phase transfer functions dubbed optimization directionalities to be defined by the user. As explained in Lanzon (2005a), $J(W)$ is a cumulative measure of the frequency dependent size of $W(j\omega)$ scaled by the optimization directionalities $\upsilon_i(j\omega)$. Such functions can be specified in order to qualitatively direct the maximization where desired. Defining an optimization directionality matrix as $\Upsilon(j\omega) := \mathrm{diag}(\upsilon_1(j\omega), \ldots, \upsilon_n(j\omega))$, then (similar to Lanzon (2005a)) the cost function in (3) can be defined as:

$$J(W) = \frac{1}{\|\Upsilon W^{-1}\|^2_{[\omega_L,\omega_H]}},$$

where $\|X\|_{[\omega_L,\omega_H]} := \sqrt{\int_{\log_{10}\omega_L}^{\log_{10}\omega_H} \|X(j\omega)\|_F^2 \mathrm{d}(\log_{10}\omega)}$.

Note that only the argument of the optimization is of interest. Therefore the maximization of $J(W) = \frac{1}{\|\Upsilon W^{-1}\|^2_{[\omega_L,\omega_H]}}$ will be replaced by the minimization of the inverse of the cost function in order to formulate a convex optimization problem as will be seen in the following.

In order to define an efficient solution algorithm, the robust performance constraint in (3) written in terms of $\mu_{\mathbf{\Delta}_T}$ will be replaced with a convex upper bound on $\mu$. The definition of such an upper bound involves matrix scalings $G$ and $D$ allowed to vary in sets $\tilde{\mathcal{D}}$ and $\tilde{\mathcal{G}}$ that depend on the structure of the perturbation matrix, i.e.:

$$\tilde{\mathcal{D}} = \{D = \mathrm{diag}[D_1 \otimes I_{k_1}, \ldots, D_g \otimes I_{k_g}, D_{g+1} \otimes I_{k_{g+1}}, \ldots, D_{g+d} \otimes I_{k_{g+d}}, I_n] : 0 < D_i = D_i^* \in \mathbb{C}^{n_i \times n_i}\ \forall i \in \{1, \ldots, g+d\}\}$$

$$\tilde{\mathcal{G}} = \{G = \mathrm{diag}[G_1 \otimes I_{k_1}, \ldots, G_g \otimes I_{k_g}, 0_{k_{g+1}}, \ldots, 0_{k_{g+d}}, 0_n] : G_i = G_i^* \in \mathbb{C}^{n_i \times n_i}\ \forall i \in \{1, \ldots, g\}\}. \quad (5)$$

Note that the last entry in each element of $\tilde{\mathcal{D}}$ is normalized to unity. A convenient upper bound on $\mu$ when both real and complex uncertainties are present is then given in the following lemma from Zhou and Doyle (1999).

**Lemma 1** (Zhou and Doyle (1999)). *Let $M \in \mathbb{C}^{r\times r}$ and $\Delta \in \mathbf{\Delta}_T$. Then if $\exists\, D \in \tilde{\mathcal{D}}, G \in \tilde{\mathcal{G}}, \beta > 0$ and $\gamma \in [0, 1]$ such that*

$$\overline{\sigma}\left(\left(\frac{DMD^{-1}}{\beta} - jG\right)\left(I + G^2\right)^{-\frac{1}{2}}\right) \le \gamma \quad (6)$$

*then $\mu_{\mathbf{\Delta}_T}(M) \le \gamma\beta$.*

Proposition 1 given in the following provides an equivalent reformulation of the upper bound on $\mu_{\mathbf{\Delta}_T}[\mathcal{F}_l(\Sigma, K)(j\omega)^T \mathrm{diag}[I_r, W(j\omega)]]$ that, since $\mu_{\mathbf{\Delta}_T}(M) = \mu_{\mathbf{\Delta}_T}(M^T)$, is equivalent to the constraint appearing in the optimization problem (3).

**Proposition 1.** *Given a closed loop system $\mathcal{F}_l(\Sigma, K) \in \mathcal{RH}_\infty$ and performance weights $W \in \mathcal{W}$, let $M_\omega = \mathcal{F}_l(\Sigma, K)(j\omega)^T$ and $W_\omega := \mathrm{diag}[I_r, W(j\omega)]$. Then, $\forall\omega$, $\exists\, D_\omega \in \tilde{\mathcal{D}}, G_\omega \in \tilde{\mathcal{G}}, \gamma_\omega \in [0, 1]$ and $\beta_\omega > 0$ such that*

$$\overline{\sigma}\left(\left(\frac{D_\omega M_\omega W_\omega D_\omega^{-1}}{\beta_\omega} - jG_\omega\right)\left(I + G_\omega^2\right)^{-\frac{1}{2}}\right) \le \gamma_\omega \quad (7)$$

*if and only if $\exists\, \tilde{D}_\omega \in \tilde{\mathcal{D}}, \tilde{G}_\omega \in \tilde{\mathcal{G}}, \gamma_\omega \in [0, 1]$ and $\beta_\omega > 0$ such that*

$$\Omega(M_\omega, \tilde{G}_\omega, \tilde{D}_\omega, \gamma_\omega, \beta_\omega) := \begin{bmatrix} M_\omega^*\tilde{D}_\omega M_\omega + j(\tilde{G}_\omega M_\omega - M_\omega^*\tilde{G}_\omega) - (\gamma_\omega\beta_\omega)^2\tilde{D}_\omega & \sqrt{1-\gamma_\omega^2}\tilde{G}_\omega \\ \sqrt{1-\gamma_\omega^2}\tilde{G}_\omega & -\tilde{D}_\omega \end{bmatrix} \le \begin{bmatrix} 0_r & 0 & 0 \\ 0 & (\beta_\omega\gamma_\omega)^2(\hat{W}_\omega - I_n) & 0 \\ 0 & 0 & 0_{r+n} \end{bmatrix} \quad (8)$$

*where $\hat{W}_\omega := [W(j\omega)^* W(j\omega)]^{-1}$.*

**Proof.**

$$\overline{\sigma}\left(\left(\frac{D_\omega M_\omega W_\omega D_\omega^{-1}}{\beta_\omega} - jG_\omega\right)\left(I + G_\omega^2\right)^{-\frac{1}{2}}\right) \le \gamma_\omega$$
$$\Leftrightarrow \left(\frac{D_\omega M_\omega W_\omega D_\omega^{-1}}{\beta_\omega} - jG_\omega\right)^*\left(\frac{D_\omega M_\omega W_\omega D_\omega^{-1}}{\beta_\omega} - jG_\omega\right) \le \gamma_\omega^2(I + G_\omega^2)$$
$$\Leftrightarrow (M_\omega W_\omega)^*\tilde{D}_\omega M_\omega W_\omega + j\tilde{G}_\omega M_\omega W_\omega - j(M_\omega W_\omega)^*\tilde{G}_\omega \le (\beta_\omega\gamma_\omega)^2\tilde{D}_\omega + (\gamma_\omega^2 - 1)\tilde{G}_\omega\tilde{D}_\omega^{-1}\tilde{G}_\omega.$$

By virtue of the Schur complement lemma, the last matrix inequality is equivalent to

$$\Omega(\overline{M}_\omega, \tilde{G}_\omega, \tilde{D}_\omega, \gamma_\omega, \beta_\omega) \le 0 \quad (9)$$

where $\overline{M}_\omega = M_\omega W_\omega$. Finally, (8) follows by pre-and post-multiplying the first row and column of the left hand side of (9) by $W_\omega^{-*}$ and $W_\omega^{-1}$, respectively and then adding $\mathrm{diag}[\mathrm{diag}[0_r, (\beta_\omega\gamma_\omega)^2(\hat{W}_\omega - I_n)], 0_{r+n}]$ to both sides of the resulting inequality. □

By virtue of Lemma 1 and Proposition 1, the constraint in optimization problem (3) can be replaced by the constraint in (8) which is quasi-convex in $\tilde{D}_\omega$, $\tilde{G}_\omega$ and $\gamma_\omega^2$ (or $\beta_\omega^2$) for all $\omega$.

The following proposition introduces a relation between $\beta_\omega$ and $\gamma_\omega$ both appearing in Eq. (8). Such a relation will be useful in the remainder of this paper.

**Proposition 2.** *Given $\tilde{D}_\omega \in \tilde{\mathcal{D}}$, $\tilde{G}_\omega \in \tilde{\mathcal{G}}$ and $\overline{M}_\omega = \mathcal{F}_l(\Sigma, K)(j\omega)^T \mathrm{diag}[I_r, W(j\omega)]$, let*

$$\beta_{\omega_*} = \min_{\beta_\omega > 0}\left\{\beta_\omega : \overline{M}_\omega^*\tilde{D}_\omega\overline{M}_\omega + j(\tilde{G}_\omega\overline{M}_\omega - \overline{M}_\omega^*\tilde{G}_\omega) - \beta_\omega^2\tilde{D}_\omega \le 0\right\} \quad \forall\omega. \quad (10)$$

*Given also $\epsilon \ge 0$, let $\gamma_{\omega_*}$ be the solution of the following optimization problem*

$$\min_{\gamma_\omega \in [0,1]} \gamma_\omega \quad \text{such that } \Omega(\overline{M}_\omega, \tilde{G}_\omega, \tilde{D}_\omega, \gamma_\omega, \beta_{\omega_*}(1+\epsilon)) \le 0. \quad (11)$$

*Then $\gamma_{\omega_*} \in [\frac{1}{1+\epsilon}, 1]$.*

**Proof.** First note that $\forall\omega$ $\gamma_{\omega_*} \le 1$ is obvious via (9) and (10). Then suppose $\forall\omega$ $\exists\kappa_\omega \in [0, 1)$ such that $\gamma_{\omega_*} = \frac{\kappa_\omega}{1+\epsilon}$. Then (11) implies

$$\overline{M}_\omega^*\tilde{D}_\omega\overline{M}_\omega + j(\tilde{G}_\omega\overline{M}_\omega - \overline{M}_\omega^*\tilde{G}_\omega) - (\beta_\omega\kappa_\omega)^2\tilde{D}_\omega \le 0$$

which contradicts (10). □

### 3.1. Performance weight optimization problem

The formulation of the constraint in (8) shows the additional advantage that, if $K$ (i.e. $M_\omega$) is held fixed, the search space is characterized by a set of LMI constraints each one containing only the decision variables $\tilde{D}_\omega$, $\tilde{G}_\omega$, $\hat{W}_\omega$, $\beta_\omega$ and $\gamma_\omega$ valid at a given frequency $\omega$. Under this assumption, the minimization of the integral required to compute $\|\Upsilon W^{-1}\|^2_{[\omega_L,\omega_H]}$ in (12) is equivalent to the minimization of the integrand on the continuum of frequencies. Therefore, if $K$ (i.e. $M_\omega$) is held fixed, the cost function in (12) can be replaced at each $\omega$ by $\|\Upsilon(j\omega)W(j\omega)^{-1}\|_F^2 = \mathrm{trace}(\hat{\Upsilon}_\omega\hat{W}_\omega)$ where we define the diagonal positive matrix $\hat{\Upsilon}_\omega := \frac{\Upsilon(j\omega)^*\Upsilon(j\omega)}{\omega}$ noting that the division by $\omega$ corresponds to a change of variable between $\omega$ and $\log_{10}\omega$ necessary to take account of the logarithmic scale appearing in the definition of $\|\cdot\|_{[\omega_L,\omega_H]}$.

These final manipulations lead to the formulation of the following optimization problem.

*Performance weight and controller optimization problem:*

For all $\omega$ in $[\omega_L, \omega_H]$

$$\min_{\hat{W}_\omega} \mathrm{trace}(\hat{\Upsilon}_\omega\hat{W}_\omega)$$

such that $\exists\tilde{D}_\omega \in \tilde{\mathcal{D}}$, $\tilde{G}_\omega \in \tilde{\mathcal{G}}$ and $\beta_\omega > 0$ and $\gamma_\omega \in [0, 1]$ satisfying

$$\Omega(M_\omega, \tilde{G}_\omega, \tilde{D}_\omega, \gamma_\omega, \beta_\omega) \le \begin{bmatrix} 0_r & 0 & 0 \\ 0 & (\beta_\omega\gamma_\omega)^2(\hat{W}_\omega - I_n) & 0 \\ 0 & 0 & 0_{r+n} \end{bmatrix}. \quad (12)$$

Optimization problem (12) shows the important feature that the search space and the cost function are both convex in the decision variables provided that $K$ (i.e. $M_\omega$) is held fixed. This allows for the definition of an efficient numerical solution algorithm in which the controller $K$ and performance and scaling matrices $D, G, W$ are alternatively held fixed and convex optimization subproblems are iteratively solved.

As a final remark, note that optimization problem (12) can be regarded as a generalization of the skewed-$\mu$ one treated in Fan and Tits (1992) in which the output channels are not weighted over frequency.

## 4. Solution algorithm

In this section, we introduce an algorithm to search for optimized values of $K_*$, $\tilde{D}_{\omega*}$, $\tilde{G}_{\omega*}$, $\hat{W}_{\omega*}$ that solve optimization problem (12).

In the following, a detailed description of the proposed solution algorithm is given.

*Inputs to the algorithm:*

- Generalized plant $\Sigma$ partitioned consistently with Fig. 1 i.e.

$$\Sigma(s) = \left[\begin{array}{c|cc|c} A & B_1 & B_2 & B_3 \\ \hline C_1 & D_{11} & D_{12} & D_{13} \\ C_2 & D_{21} & D_{22} & D_{23} \\ \hline C_3 & D_{31} & D_{32} & D_{33} \end{array}\right]$$

  with $(A, B_3)$ stabilizable, $(A, C_3)$ detectable and $D_{33} = 0$ (without loss of generality),
- Optimization directionality matrix $\Upsilon(j\omega)$ and a frequency range $[\omega_L, \omega_H]$ where maximization of the performance weight is required.

*Algorithm*:

- *Step* 0 *(Initialization):* Set $i = 0$, then design a robustly stabilizing controller $K_i$ for the truncated generalized system $\hat{\Sigma}(s)$ which corresponds to deleted performance channels and deleted exogenous disturbances in $\Sigma(s)$. Let $\beta_i$ be a number in the interval $[\sup_\omega \mu_{\mathbf{\Delta}}(\mathcal{F}_l(\hat{\Sigma}, K_i)(j\omega)), 1)$ which represents an achieved level of robust stability. Fix the minimum number of iterations desired for convergence as $N$. Define

$$\epsilon := \left(\frac{1}{\beta_i}\right)^{\frac{1}{N}} - 1. \tag{13}$$

  Choose a frequency grid between $\omega_L$ and $\omega_H$ dense enough to ensure that all the expected changes in the performance weights matrix $W(j\omega)$ are captured.
- *Step* 1: Set $i = i + 1$. Set in optimization problem (12) $\forall\omega$ $M_\omega = \mathcal{F}_l(\Sigma, K_i)(j\omega)$, $\beta_\omega = \beta_i$ and $\gamma_\omega = 1$. Solve the resulting optimization problem on the defined frequency grid with decision variables $\tilde{D}_\omega$, $\tilde{G}_\omega$ and $\hat{W}_\omega$. Denote the optimized performance weights with $\hat{W}_{\omega*}$.
- *Step* 2: If $i + 1 < N$, set $\beta_{i+1} = (1 + \epsilon)\beta_i$ else set $\beta_{i+1} = 1$. Then compute optimal pointwise $\tilde{D}_\omega$ and $\tilde{G}_\omega$ scaling matrices by solving the following quasi-convex optimization problem on the defined frequency grid:

$$\begin{aligned} &\min_{\tilde{D}_\omega \in \tilde{\mathcal{D}}, \tilde{G}_\omega \in \tilde{\mathcal{G}}} \gamma_\omega \quad \text{subject to the constraint in (12)} \\ &\text{with } \forall\omega \hat{W}_\omega = \hat{W}_{\omega*}, \qquad M_\omega = \mathcal{F}_l(\Sigma, K_i)(j\omega) \\ &\quad\text{and} \quad \beta_\omega = \beta_{i+1}. \end{aligned} \tag{14}$$

  By virtue of Proposition 2, the solution to the above problem can be efficiently computed by means of a bisection search on $\gamma_\omega$ in the interval $[\frac{1}{1+\epsilon}, 1]$. Denote the pointwise minimizing arguments and the pointwise solution with $\tilde{D}_{\omega*}$, $\tilde{G}_{\omega*}$ and $\gamma_{\omega*}$ respectively.
- *Step* 3: Find stable minimum-phase transfer function matrices $W(s)$ and $D(s)$ so that $W(j\omega)^{\sim}W(j\omega) \approx \hat{W}_{\omega*}^{-1}$ and $D(j\omega)^{\sim}D(j\omega) \approx \tilde{D}_{\omega*}$ and find a transfer function matrix such that $G(j\omega) \approx j\frac{1}{\beta_i}D(j\omega)^{-*}\tilde{G}_{\omega*}D(j\omega)^{-1}$. Such transfer function matrices can be computed exploiting the techniques described in Young (1993). Then find normalized right coprime factors $G_N(s)$, $G_M(s)$ of $G(s)$ (see Zhou and Doyle (1999)) and build the augmented generalized plant:

$$\begin{aligned} \Sigma_{DGW} = {} & \mathrm{diag}[D(s), W(s), I_p]\frac{\Sigma(s)}{\beta_i}\mathrm{diag}[D^{-1}(s)G_M(s), I_{n+q}] \\ & - \mathrm{diag}[G_N(s), 0_{n+q}]. \end{aligned}$$

- *Step* 4: Find the controller $K_{i+1}$ that minimizes $\|\mathcal{F}_l(\Sigma_{DGW}, K_{i+1})\|_\infty$ via $\mathcal{H}_\infty$ synthesis. Note that the new closed loop system is such that $\|\mathcal{F}_l(\Sigma_{DGW}, K_{i+1})\|_\infty \le \gamma$. If $\gamma < 1$, then, return to Step 1. If $\gamma \ge 1$, a controller that reduces the peak of the structured singular value over frequency cannot be found, exit.

As a final remark, note that in the proposed solution algorithm, pointwise in frequency approximations are used in the first phase (Steps 1 and 2) whereas state space methods are exploited in the second phase (Step 4). This mixed solution approach combines the strengths of the solution algorithms in Lanzon (2005a), which only uses frequency approximations, and Lanzon and Cantoni (2003) where state-space techniques are mainly exploited. In particular, in this work, the controller that achieves a maximized level of robust performance is searched over the entire set of internally stabilizing controllers $\mathcal{K}$ whereas in Lanzon (2005a) the search space is restricted to a parametrization of $\mathcal{K}$. Additionally, the controller synthesis step is recast here into a conventional $\mathcal{H}_\infty$ design of a scaled plant that only requires the solution of two Riccati equations at each iteration. This is much more computationally efficient with respect to the algorithm proposed in Lanzon and Cantoni (2003) where the controller is synthesized together with the performance weights by iteratively solving a linear optimization problem subject to an LMI constraint.

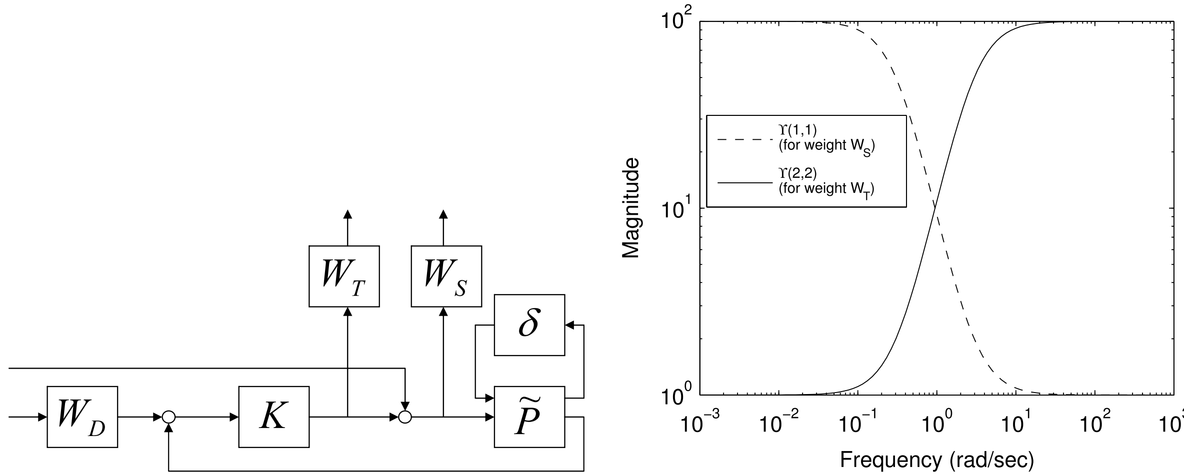
## 5. Numerical example

In this section, we present a simple numerical example to illustrate the algorithm proposed in Section 4. The example comprises two different test cases that highlight how the design algorithm is capable to resolve the possible inconsistencies between the closed loop specifications in different situations of interest. Let us consider the uncertain plant set

$$P = \frac{\kappa(s - z)}{s^2} \tag{15}$$

where we set $\kappa = 5(1 + 0.25\delta)$, $\delta \in [-1, 1]$. The location of the right half plane zero $z$ is set alternatively to $z = 10$ rad/s and to $z = 0.5$ rad/s in the two different test cases. We consider as design set up the typical $S/T$ mixed shaping scheme shown in Fig. 2(a) (where $\tilde{P}$ is the generalized plant obtained after the real parametric uncertainty is extracted from the plant in (15)) with an additional input (associated to input weight $W_D$) to fulfill the assumptions needed for the $\mathcal{H}_\infty$ design. $W_S$ and $W_T$ denote the weighting functions that shape the sensitivity and complementary sensitivity functions, respectively. These weighting functions will be automatically designed by the algorithm together with an internally stabilizing controller. $W_D$ is chosen as static and small so that its influence on the $\mathcal{H}_\infty$ norm of the whole system is negligible. The optimization directionalities chosen in this example are shown in Fig. 2(b). We require the optimization algorithm to maximize with the same preference the magnitude of $W_S$ at low frequencies and the magnitude of $W_T$ at high frequencies. Moreover, both the directionality functions present a reduction of one order of magnitude with respect to their maximum values at the frequency of 1 rad/s that can therefore be considered as the desired bandwidth of the closed loop system. Note that specifying an absolute size of each entry of $\Upsilon(j\omega)$ is not required but only the shape across frequency and the relative magnitudes of the optimization directionalities are relevant as they only appear in the cost function of optimization problem (12). We set $N = 4$, so that convergence of the algorithm is attained at least after 4 iterations in both the two test cases.

In both examples, the algorithm converged after 4 iterations. The cost $J$ associated with the posed optimization problem took




(a) Design set up.

(b) Desired directionality for the optimization.

**Fig. 2.** Input to the algorithm.

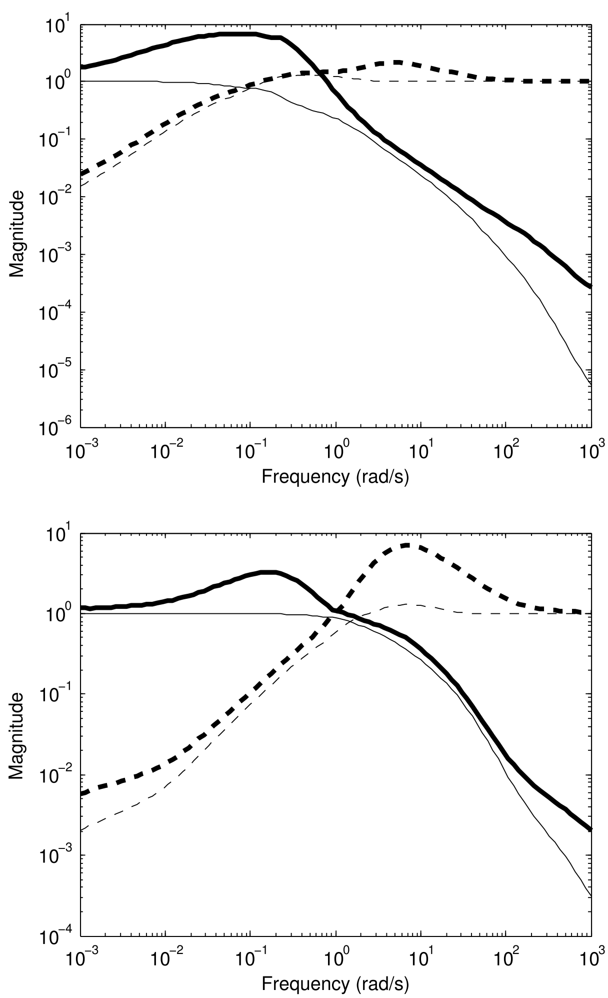



**Fig. 3.** Optimal $|W_S(j\omega)|^{-1}$ and $|W_T(j\omega)|^{-1}$ (bold lines) and nominal $S$ and $T$ (light lines) (Left figure: $z = 0.5$ rad/s, right figure $z = 10$ rad/s).

values in the sequence $\{1.2483; 0.1404; 0.0601; 0.0286\} \cdot 10^6$ and $\{5.0128; 0.4643; 0.2223; 0.1104\} \cdot 10^6$, respectively, which are monotonically decreasing as expected.

In Fig. 3(a) and (b), the performance weights in the output to the synthesis algorithm for both the two test cases are shown together with the sensitivity and complementary sensitivity functions of the nominal system. In both cases, at the end of the iterations the mixed $\mu$ computed for the closed loop system assumes a constant value of 1 across frequency, i.e. the sensitivity and complementary sensitivity of any plant in the uncertain plant set in (15) are below $|W_S(j\omega)|^{-1}$ and $|W_T(j\omega)|^{-1}$, respectively. With $z = 10$ rad/sec the open loop zero imposes no limitation upon the sensitivity properties of the system and, as expected, the desired bandwidth of 1 rad/s is achieved (see Fig. 3(b)). On the other hand, with $z = 0.5$ rad/s the right half plane zero of the plant lies within the required closed loop bandwidth that, therefore, may not be achievable (see for example Freudenberg and Looze (1985)). Hence, a trade off must be performed between desired bandwidth and limitation due to the plant dynamic. This is automatically accounted for by the algorithm being the optimal closed loop bandwidth approximately 0.1 rad/s (see Fig. 3(a)). Once again this demonstrates that, unlike the performance weights, it is impossible to specify incompatible optimization directionalities, because any inconsistency between the optimization directionalities and the robustness requirements or fundamental performance limitations, are resolved via the automatic selection of the performance weights performed by solving optimization problem (12). As a final remark, note that variations of the optimized closed loop bandwidth of the order of 10% have been observed in case different initial guess controllers (i.e. $K_i$ with $i = 0$) are used. This shows that the iterative algorithm proposed to solve optimization problem (12) is robust against modifications of the initialization parameters.

## 6. Conclusions

In this paper, we present a control synthesis technique that automatically performs an optimized trade off between achievable performance and limitations due to uncertainty or plant dynamics for plants subject to mixed real and complex uncertainty. With respect to the analogous techniques in Lanzon (2005a) and Lanzon and Cantoni (2003), the one presented here allows to handle also mixed complex and real uncertainties in the design (besides also a much improved iterative algorithm in terms of performance and computational speed). The performance of the design algorithm has been tested through a numerical example that showed how the proposed synthesis technique simplifies the direct design of appropriate performance weights and provides an indication of the achievable performance for a given uncertain plant set. Further research work will focus on the incorporation of a mixed $\mathcal{H}_2$-$\mathcal{H}_\infty$ constraint in the optimization-based synthesis technique so that the optimized weights will more precisely reflect optimized performance.

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