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Brief paper

Weight optimisation in $\mathscr{H}_\infty$ loop-shaping☆

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Abstract

An optimisation problem is introduced which integrates a number of steps of the standard $\mathscr{H}_\infty$ loop-shaping design procedure. This optimisation problem maximises the robust stability margin over allowable loop-shaping weights while ensuring that the resulting loop-shape lives in a pre-defined region that characterises the desired performance specifications. The resulting algorithm thus considerably facilitates the design of loop-shaping weights by giving the designer an indication of what is attainable and hence allows the designer to concentrate on more fundamental design issues than simply finding weights that achieve a desired loop-shape.

Keywords: $\mathscr{H}_\infty$ loop-shaping; Weight synthesis; Performance optimisation; Robust performance; Robust control

1. Introduction

A well-known method for the design of robust multiple-input multiple-output (MIMO) feedback controllers is the so-called "$\mathscr{H}_\infty$ loop-shaping design procedure" proposed by McFarlane and Glover (1992). This method has been successfully used in a variety of applications (see Papageorgiou & Glover, 1999b and references therein). A full tutorial on how to design robust controllers using the $\mathscr{H}_\infty$ loop-shaping design procedure can be found in McFarlane and Glover (1990) and Papageorgiou and Glover (1999b).

Even though designers usually obtain good loop-shaping weights and controllers using their engineering insight and intuition, it is well recognised in the practicing community that the design of loop-shaping weights W_1 and W_2 to achieve a desired loop-shape is not always straightforward,[1] especially for plants with strong cross-coupling (Hyde, 1995). This is because it is not always clear how each element in the weights affects the singular values of the scaled nominal plant P and the complexity of this relationship considerably increases when non-diagonal weights need to be used. Also, the ad hoc techniques (or trial and error methods) typically employed can be fairly time-consuming and can never be guaranteed to yield the best possible results.

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[1] Similar observations have also been made with respect to the design of good *performance weights* in the standard μ-synthesis framework. See Lanzon and Richards (1999), Lanzon and Cantoni (2000, 2003) for a detailed discussion and an optimisation problem that facilitates the selection of performance weights in that framework. See also Lanzon, Bombois, and Anderson (2003) for a thorough study of the effects of small weight adjustments in $\mathscr{H}_\infty$-control.

The length of this iterative process strongly depends on the experience of the designer with loop-shaping concepts and on the cross-coupling present in the plant. Furthermore, lack of design experience with loop-shaping concepts may lead to a designed loop-shape that does not achieve a sufficiently large robust stability margin $b_{\text{opt}}(P_s)$, where $P_s = W_2 P W_1$ is the shaped plant. In this case, the designer has to first determine the factors in the designed loop-shape that are giving rise to a small $b_{\text{opt}}(P_s)$ and then understand how to modify these factors (without compromising the specifications) in order to increase the robust stability margin. This may not be obvious and the designer may have to iterate between the selection of loop-shaping weights and the evaluation of $b_{\text{opt}}(P_s)$ several times before a sufficiently large value of $b_{\text{opt}}(P_s)$ is achieved. For instance, the designer must ensure that the loop-gain is large around the frequencies and in the directions of open-loop unstable poles, small around the frequencies and in the directions of open-loop unstable zeros and that the loop-gain does not roll-off at a high rate around cross-over. If any one of these is violated, a small $b_{\text{opt}}(P_s)$ is obtained.

The problem of designing good loop-shaping weights is in fact more complicated than simply due to the issues just described above, since the designer might also want to ensure that some of the closed-loop transfer function matrices appearing in Zhou, Doyle, and Glover (1996, Theorem 18.11) are small in appropriate frequency ranges besides achieving the desired loop-shape. This can be achieved by ensuring that the corresponding upper bounds of these closed-loop transfer function matrices, given in the same theorem, are small in the required frequency ranges. However, these upper bounds depend on the condition number and the maximum/minimum singular values of the loop-shaping weights, and the standard $\mathscr{H}_\infty$ loop-shaping design procedure gives no direct easy handle on these variables. Thus, the designer can only check the size of these closed-loop transfer function matrices at the end of the design cycle and if they are not sufficiently small, the whole design must be repeated from the beginning.

Consequently, in this paper, I will attempt to streamline the standard $\mathscr{H}_\infty$ loop-shaping design procedure by combining a number of steps into one optimisation problem so that the design procedure can be made even more systematic and hence even easier to use in application. The proposed optimisation problem maximises the robust stability margin over loop-shaping weights subject to constraints which ensure that the loop-shape and the singular values/condition numbers of the weights lie in pre-specified regions. Thus, loop-shaping weights, which can be required to have a diagonal or a non-diagonal structure, and a robustly stabilising controller are simultaneously synthesised by one algorithm in a systematic way. This algorithm enables the designer to quickly get a feel of what performance is achievable, determine whether non-diagonal weights would be beneficial and easily understand the tradeoffs involved in the particular problem at hand.

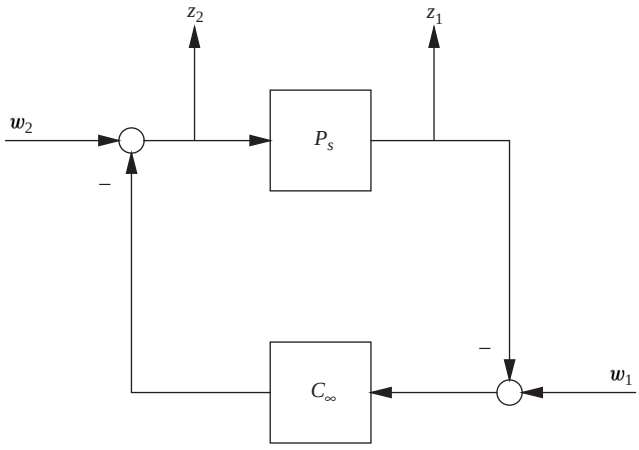

Fig. 1. Feedback interconnection.

2. Preliminaries

Let $\mathbb{R}_+$ denote the strictly positive real numbers. Let the singular values of a matrix $A \in \mathbb{C}^{n\times m}$ be denoted by $\sigma_i(A)$, with the largest (resp. smallest) singular value denoted by $\bar{\sigma}(A)$ (resp. $\underline{\sigma}(A)$). Also, let $\kappa(A)$ denote the condition number of $A \in \mathbb{C}^{n\times m}$ defined as $\kappa(A) = \bar{\sigma}(A)/\underline{\sigma}(A)$. Furthermore, let A^{T} (resp. A^*) denote the transpose (resp. complex conjugate transpose) of $A \in \mathbb{C}^{n\times m}$ and $G^{\sim}(s)$ denote the adjoint of a real rational (denoted by $\mathscr{R}$) system $G(s)$ defined by $G^{\sim}(s) = G(-s)^{\mathrm{T}}$. Let $\mathscr{L}_\infty$ denote the space of functions bounded on $\mathrm{j}\mathbb{R}$ including ∞, $\mathscr{H}_\infty$ denote the space of functions bounded and analytic in the open right half complex plane, and the same function spaces with prefix $\mathscr{R}$ denoting their real-rational subspaces. Denote also the space of functions that are units in $\mathscr{RH}_\infty$ by $\mathscr{GH}_\infty$ (i.e. $f \in \mathscr{GH}_\infty \Leftrightarrow f,\ f^{-1} \in \mathscr{RH}_\infty$).

Also, let the feedback interconnection of $P_s = W_2 P W_1$ and $C_\infty = W_1^{-1} C W_2^{-1}$ shown in Fig. 1 be denoted by $[P_s, C_\infty]$. This interconnection is said to be *internally stable* if it is well-posed and each of the four transfer functions mapping inputs to outputs belongs to $\mathscr{RH}_\infty$. Denote by $\mathscr{C}(P_s)$ the set of all internally stabilising controllers for plant P_s, that is $\mathscr{C}(P_s) := \{C_\infty : [P_s, C_\infty] \text{ is internally stable}\}$. Furthermore, given a plant P_s and a controller C_∞, the *robust stability margin* $b(P_s, C_\infty)$ is given by

$$b(P_s, C_\infty) := \left\| \begin{bmatrix} P_s \\ I \end{bmatrix} (I - C_\infty P_s)^{-1} [-C_\infty \ \ I] \right\|_\infty^{-1}$$

if $[P_s, C_\infty]$ is internally stable and by $b(P_s, C_\infty) := 0$ otherwise. Then, the largest value of the robust stability margin is defined by $b_{\text{opt}}(P_s) := \sup_{C_\infty} b(P_s, C_\infty)$. It is shown in Glover and McFarlane (1989) that $b_{\text{opt}}(P_s) \leqslant 1$ for any P_s.

3. Formulating an optimisation problem

Before proposing the optimisation problem of interest that addresses the difficulties mentioned in the introduction, first assume that *the scaled nominal plant* $P \in \mathscr{R}^{m\times n}$ *is such that* $m \geqslant n$. This assumption incurs no loss of generality but considerably simplifies notation. If the plant has strictly fewer outputs than inputs (i.e. $m < n$), then a dual problem to that shown in Fig. 1 would be considered. Then, define the following set for compactness of notation

$$\Xi(\alpha, \beta, \zeta) := \{W \in \mathscr{GH}_\infty : \kappa(W(\mathrm{j}\omega)) < \zeta(\omega)\ \forall\omega, \quad \alpha(\omega) < \sigma_i(W(\mathrm{j}\omega)) < \beta(\omega)\ \forall i, \omega\}$$

for some given continuous functions $\alpha, \beta, \zeta : \mathbb{R} \to \mathbb{R}_+$ that satisfy $\zeta(\omega) > 1\ \forall\omega$ and $\beta(\omega) > \alpha(\omega)\ \forall\omega$.

Now, the following optimisation problem can be introduced:

$$\max_{\substack{W_1\in\Xi(\underline{w}_1,\overline{w}_1,k_1)\\ W_2\in\Xi(\underline{w}_2,\overline{w}_2,k_2)}} b_{\mathrm{opt}}(P_\mathrm{s}) \quad \text{subject to} \quad \underline{s}(\omega) < \sigma_i(P_\mathrm{s}(\mathrm{j}\omega)) < \bar{s}(\omega) \quad \forall i, \omega, \tag{1}$$

where the scaled nominal plant P is given and satisfies the above assumption, and $\underline{s}$, $\bar{s}$, $\underline{w}_i$, $\overline{w}_i$ and $k_i (i = 1, 2)$ are continuous real-valued positive functions of the variable ω specified by the designer such that: (i) the frequency functions $\underline{s}(\omega)$ and $\bar{s}(\omega)$ are boundaries for an allowable loop-shape, (ii) the frequency functions $\underline{\omega}_i(\omega)$ and $\bar{\omega}_i(\omega)$ delimit the allowable region for the singular values of loop-shaping weight $W_i(\mathrm{j}\omega)(i = 1, 2)$, (iii) the frequency function $k_i(\omega)$ bounds the condition number of loop-shaping weight $W_i(\mathrm{j}\omega)$ $(i = 1, 2)$. The posed optimisation problem does not depend on any particular controller in that it seeks to maximise the optimal robust stability margin.

Standard results such as in Vinnicombe (2000, Theorem 1.20) illustrate the importance of the robust stability margin by linking it to the size of the largest $\mathscr{H}_\infty$ (or $\mathscr{L}_\infty$)-norm bounded perturbation of the normalised coprime factors of the shaped plant P_s for which closed-loop stability is maintained. Consequently, maximising the robust stability margin will effectively increase the robustness of the feedback interconnection to perturbations of the plant. Besides stability robustness, one typically wishes a certain level of robust performance and this is usually specified as the gain of certain closed-loop transfer functions being smaller than certain values in particular frequency regions. Zhou et al. (1996, Theorem 18.11) gives upper bounds on the gains of all closed-loop transfer function matrices between inputs and outputs at any point in the feedback loop. These upper bounds are expressed in an $\mathscr{H}_\infty$ loop-shaping setting in terms of the robust stability margin, the loop-shape and the singular values/condition number of the weights. Consequently, the sizes of these quantities are limited via the bounding functions $\underline{s}$, $\bar{s}$, $\underline{w}_i$, $\overline{w}_i$ and k_i $(i = 1, 2)$ in optimisation problem (1) to give the desired closed-loop transfer function matrix gains.

It is worth noting that it is not a requirement of the new method that specification of the loop-shaping weights W_1 and W_2 be left entirely up to the optimisation procedure. If the designer knows that a certain level of performance is achievable and compatible with the robustness requirements, then the corresponding weights can be absorbed into the plant P. The optimisation-based synthesis above can then be thought of as a systematic mechanism for 'tuning' the weights and hence the design.

The restriction in optimisation problem (1) that the loop-shaping weights need to be units in $\mathscr{RH}_\infty$ does not limit generality for a practicing engineer, as any strictly proper weight or any weight containing an integrator can be approximated by an element in $\mathscr{GH}_\infty$. Also, if one insists on wanting W_1 to be strictly proper and/or W_2 to have a pure integrator, for example, then these requirements can be absorbed into the plant P, as described in the preceding paragraph, and regard the above posed optimisation problem as a systematic mechanism for 'tuning' the specified weights.

3.1. Manipulating the optimisation problem

The objective of this subsection is to massage optimisation problem (1) into a form that is better suited for a subsequent algorithm while ensuring that each reformulation yields the same optimal arguments and that the optimal costs are easily related between reformulations. Thus, in this sense, each reformulation does not introduce conservatism.

First note that optimisation problem (1) can be rewritten as

$$\min_{\substack{W_1\in\Xi(\underline{w}_1,\overline{w}_1,k_1)\\ W_2\in\Xi(\underline{w}_2,\overline{w}_2,k_2)\\ C_\infty\in\mathscr{C}(P_\mathrm{s})}} \left\| \begin{bmatrix} P_\mathrm{s} \\ I \end{bmatrix} (I - C_\infty P_\mathrm{s})^{-1} [-C_\infty \ \ I] \right\|_\infty \quad \text{subject to} \quad \underline{s}(\omega) < \sigma_i(P_\mathrm{s}(\mathrm{j}\omega)) < \bar{s}(\omega) \quad \forall i, \omega,$$

and then using $P_\mathrm{s} = W_2 P W_1$, $C = W_1 C_\infty W_2$ and noting that $W_1 \in \Xi(\underline{w}_1, \overline{w}_1, k_1)$ if and only if $W_1^{-1} \in \Xi(1/\overline{w}_1, 1/\underline{w}_1, k_1)$, it can be also rearranged as

$$\min_{\substack{W_1^{-1}\in\Xi(\frac{1}{\overline{w}_1},\frac{1}{\underline{w}_1},k_1)\\ W_2\in\Xi(\underline{w}_2,\overline{w}_2,k_2)\\ C\in\mathscr{C}(P)}} \left\| \begin{bmatrix} W_2 & 0 \\ 0 & W_1^{-1} \end{bmatrix} \begin{bmatrix} 0 & P \\ 0 & I \end{bmatrix} \begin{bmatrix} I & P \\ C & I \end{bmatrix}^{-1} \times \begin{bmatrix} W_2^{-1} & 0 \\ 0 & W_1 \end{bmatrix} \right\|_\infty \quad \text{subject to} \quad \underline{s}(\omega) < \sigma_i(W_2(\mathrm{j}\omega) P(\mathrm{j}\omega) W_1(\mathrm{j}\omega)) < \bar{s}(\omega) \quad \forall i, \omega.$$

Now, note that the set $\Xi(\alpha, \beta, \zeta)$ defined at the beginning of Section 3 can be also characterised by

$$\Xi(\alpha, \beta, \zeta) = \{W \in \mathscr{GH}_\infty : \forall\omega\ \exists \underline{\xi}_\omega, \bar{\xi}_\omega \in \mathbb{R} \text{ satisfying } \underline{\xi}_\omega I < W(\mathrm{j}\omega)^* W(\mathrm{j}\omega) < \bar{\xi}_\omega I,\ \bar{\xi}_\omega < \zeta(\omega)^2 \underline{\xi}_\omega,\ \alpha(\omega)^2 < \underline{\xi}_\omega,\ \bar{\xi}_\omega < \beta(\omega)^2\}.$$

Dropping the dependence on $(j\omega)$ in the interest of clarity, the above optimisation problem can be restated as:

$$\begin{array}{ll}\text{Minimise} & \gamma^2 \\ \text{subject to} & \exists C \in \mathscr{C}(P),\ W_1, W_2 \text{ satisfying}\end{array}$$

(a) $W_1^{-1} \in \Xi(1/\overline{w}_1, 1/\underline{w}_1, k_1)$, $W_2 \in \Xi(\underline{w}_2, \overline{w}_2, k_2)$,

(b) $$\begin{bmatrix} 0 & P \\ 0 & I \end{bmatrix}^* \begin{bmatrix} W_2^* W_2 & 0 \\ 0 & W_1^{-*} W_1^{-1} \end{bmatrix} \begin{bmatrix} 0 & P \\ 0 & I \end{bmatrix} \leqslant \gamma^2 \begin{bmatrix} I & P \\ C & I \end{bmatrix}^* \begin{bmatrix} W_2^* W_2 & 0 \\ 0 & W_1^{-*} W_1^{-1} \end{bmatrix} \begin{bmatrix} I & P \\ C & I \end{bmatrix} \quad \forall \omega,$$

(c) $$\underline{s}(\omega)^2 (W_1^{-*} W_1^{-1}) < P^* (W_2^* W_2) P \quad \forall \omega,$$

$$\bar{s}(\omega)^2 (W_1^{-*} W_1^{-1}) > P^* (W_2^* W_2) P \quad \forall \omega.$$

Now, let the set of real strictly positive diagonal matrices of dimension $n \times n$ be denoted by $\mathbf{\Lambda}_n := \{\text{diag}(x) : x \in \mathbb{R}_+^n\}$. When the loop-shaping weights are required to have a diagonal structure, the problem is easy because the frequency functions $W_1(j\omega)^{-*} W_1(j\omega)^{-1}$ and $W_2(j\omega)^* W_2(j\omega)$ reduce to simple strictly positive diagonal real matrices at each frequency ω, denoted by $\Lambda_{1\omega}$ and $\Lambda_{2\omega}$, respectively. Construction of a diagonal transfer function matrix W_1 (resp. W_2) in $\mathscr{GH}_\infty$ is always possible from $\Lambda_{1\omega} \in \mathbf{\Lambda}_n\ \forall\omega$ (resp. $\Lambda_{2\omega} \in \mathbf{\Lambda}_m\ \forall\omega$) by fitting stable minimum phase transfer functions to each magnitude function on the main diagonal of $\Lambda_{1\omega}$ (resp. $\Lambda_{2\omega}$). On the other hand, when the loop-shaping weights are required to have a non-diagonal structure, the problem is harder if approached directly because the frequency functions $W_1(j\omega)^{-*} W_1(j\omega)^{-1}$ and $W_2(j\omega)^* W_2(j\omega)$ are strictly positive non-diagonal complex hermitian matrices at each frequency ω. Luckily, the work of Papageorgiou and Glover (1997) can be used here to considerably simplify the calculations. As in that paper, let $\hat{U}(s)$ and $\hat{V}(s)$ be units in $\mathscr{RL}_\infty$ that approximately interpolate the frequency-by-frequency unitary matrices containing the left and right singular vectors of the scaled nominal plant P. A detailed discussion on how to construct $\hat{U}(s)$ and $\hat{V}(s)$ is given in Papageorgiou and Glover (1997) and Lanzon (2000). Then, it is possible to parameterise $W_1(j\omega)^{-*} W_1(j\omega)^{-1}$ by $\hat{V}(j\omega)^{-*} \Lambda_{1\omega} \hat{V}(j\omega)^{-1}$ for some $\Lambda_{1\omega} \in \mathbf{\Lambda}_n\ \forall\omega$, and $W_2(j\omega)^* W_2(j\omega)$ by $\hat{U}(j\omega) \Lambda_{2\omega} \hat{U}(j\omega)^*$ for some $\Lambda_{2\omega} \in \mathbf{\Lambda}_m$ $\forall\omega$ with very little restriction since the parameters in $\Lambda_{1\omega}$ and $\Lambda_{2\omega}$are able to directly influence the singular values of $P(j\omega)$. As in the diagonal weight case, note that given any $\Lambda_{1\omega} \in \mathbf{\Lambda}_n\ \forall\omega$ (resp. $\Lambda_{2\omega} \in \mathbf{\Lambda}_m\ \forall\omega$), it is always possible to construct a diagonal weight D_1 (resp. D_2) that belongs to $\mathscr{GH}_\infty$ and satisfies $D_1(j\omega)^{-*} D_1(j\omega)^{-1} = \Lambda_{1\omega}$ $\forall\omega$ (resp. $D_2(j\omega)^* D_2(j\omega) = \Lambda_{2\omega}\ \forall\omega$). Then, corresponding non-diagonal weights W_1 and W_2 in $\mathscr{GH}_\infty$ are obtained by solving the co-spectral factorisation $W_1 W_1^\sim = \hat{V} D_1 D_1^\sim \hat{V}^\sim$ and the spectral factorisation $W_2^\sim W_2 = \hat{U} D_2^\sim D_2 \hat{U}^\sim$.

Using the above parameterisations $W_2^* W_2 = \hat{U} \Lambda_{2\omega} \hat{U}^*$ and $W_1^{-*} W_1^{-1} = \hat{V}^{-*} \Lambda_{1\omega} \hat{V}^{-1}$, the previous optimisation problem can be rewritten as:

$$\begin{array}{ll}\text{Minimise} & \gamma^2 \\ \text{subject to} & \exists C \in \mathscr{C}(P) \text{ and } \forall\omega\ \Lambda_{1\omega} \in \mathbf{\Lambda}_n \text{ and } \Lambda_{2\omega} \in \mathbf{\Lambda}_m \\ & \text{satisfying}\end{array}$$

(a) $$\begin{bmatrix} 0 & P \\ 0 & I \end{bmatrix}^* \begin{bmatrix} \hat{U} \Lambda_{2\omega} \hat{U}^* & 0 \\ 0 & \hat{V}^{-*} \Lambda_{1\omega} \hat{V}^{-1} \end{bmatrix} \begin{bmatrix} 0 & P \\ 0 & I \end{bmatrix} \leqslant \gamma^2 \begin{bmatrix} I & P \\ C & I \end{bmatrix}^* \begin{bmatrix} \hat{U} \Lambda_{2\omega} \hat{U}^* & 0 \\ 0 & \hat{V}^{-*} \Lambda_{1\omega} \hat{V}^{-1} \end{bmatrix} \times \begin{bmatrix} I & P \\ C & I \end{bmatrix},$$

(b) $$\underline{s}(\omega)^2 (\hat{V}^{-*} \Lambda_{1\omega} \hat{V}^{-1}) < P^* (\hat{U} \Lambda_{2\omega} \hat{U}^*) P,$$

$$\bar{s}(\omega)^2 (\hat{V}^{-*} \Lambda_{1\omega} \hat{V}^{-1}) > P^* (\hat{U} \Lambda_{2\omega} \hat{U}^*) P,$$

(c) $$\exists \underline{\xi}_{1\omega}, \bar{\xi}_{1\omega} : \underline{\xi}_{1\omega} I < (\hat{V}^{-*} \Lambda_{1\omega} \hat{V}^{-1}) < \bar{\xi}_{1\omega} I,$$

$$\overline{w}_1(\omega)^{-2} < \underline{\xi}_{1\omega}, \quad \bar{\xi}_{1\omega} < \underline{w}_1(\omega)^{-2},$$

$$\bar{\xi}_{1\omega} < k_1(\omega)^2 \underline{\xi}_{1\omega},$$

(d) $$\exists \underline{\xi}_{2\omega}, \bar{\xi}_{2\omega} : \underline{\xi}_{2\omega} I < (\hat{U} \Lambda_{2\omega} \hat{U}^*) < \bar{\xi}_{2\omega} I,$$

$$\underline{w}_2(\omega)^2 < \underline{\xi}_{2\omega}, \quad \bar{\xi}_{2\omega} < \overline{w}_2(\omega)^2,$$

$$\bar{\xi}_{2\omega} < k_2(\omega)^2 \underline{\xi}_{2\omega}.$$

Choose $\hat{U}(s) = I_m$ and $\hat{V}(s) = I_n$ when diagonal weights are required whereas $\hat{U}(s)$ and $\hat{V}(s)$ as units in $\mathscr{RL}_\infty$ that approximately interpolate the frequency-by-frequency unitary matrices containing the left and right singular vectors, respectively, of the scaled nominal plant P when non-diagonal weights are required. Note also that the above problem is a quasi-convex optimisation problem if the controller $C \in \mathscr{C}(P)$ is held fixed.

4. Solution algorithm

Here, a sub-optimal solution is proposed to optimisation problem (1). This solution is based on the last reformulation given in Section 3.1. Analogous to DK iterations and other solution methods for these types of optimisations, an iterative algorithm must be used since the posed problem is not simultaneously convex in all variables.

The following quantities need to be specified as inputs to the algorithm: a scaled nominal plant P that satisfies the assumption given at the beginning of Section 3, frequency functions $\underline{s}(\omega)$ and $\bar{s}(\omega)$ that are boundaries for an allowable loop-shape, frequency functions $\underline{w}_i(\omega)$ and $\overline{w}_i(\omega)$ that delimit the allowable region for the singular values of loop-shaping weight $W_i(j\omega)$ $(i = 1, 2)$, and frequency function $k_i(\omega)$ that bounds the condition number of loop-shaping weight $W_i(j\omega)$ $(i = 1, 2)$.

Now, the solution algorithm is presented:

1. Find a controller $C_0^\star$ such that the interconnection $[P, C_0^\star]$ is internally stable. This controller can be found by a variety of standard techniques and provides a feasible initial starting point for the algorithm.

   Furthermore, let $\hat{U}(s) = I_m$ (resp. $\hat{V}(s) = I_n$) if a diagonal loop-shaping weight W_2 (resp. W_1) is required, or let $\hat{U}(s)$ (resp. $\hat{V}(s)$) be constructed as described in (Papageorgiou & Glover, 1997; Lanzon, 2000) if a non-diagonal weight W_2 (resp. W_1) is required.

   Set $i = 0$, where i denotes the iteration number, and let $\varepsilon^\star_{\max,0} = -1$.
2. Increment i by 1.
3. Let $\hat{P} := \hat{U}^\sim P \hat{V}$ and $\hat{C} := \hat{V}^{-1} C^\star_{i-1} \hat{U}^{-\sim}$ (where $C^\star_{i-1}$ is the controller synthesised in the previous iteration) and solve the following quasi-convex optimisation problem *at each frequency* ω:[2]

   Minimise γ_ω^2

   such that $\exists \Lambda_{1\omega} \in \mathbf{\Lambda}_n,\ \Lambda_{2\omega} \in \mathbf{\Lambda}_m$ satisfying

   (a) $$\begin{bmatrix} 0 & \hat{P} \\ 0 & I \end{bmatrix}^* \begin{bmatrix} \Lambda_{2\omega} & 0 \\ 0 & \Lambda_{1\omega} \end{bmatrix} \begin{bmatrix} 0 & \hat{P} \\ 0 & I \end{bmatrix} \leqslant \gamma_\omega^2 \begin{bmatrix} I & \hat{P} \\ \hat{C} & I \end{bmatrix}^* \begin{bmatrix} \Lambda_{2\omega} & 0 \\ 0 & \Lambda_{1\omega} \end{bmatrix} \begin{bmatrix} I & \hat{P} \\ \hat{C} & I \end{bmatrix},$$

   (b) $\underline{s}(\omega)^2 \Lambda_{1\omega} < \hat{P}^* \Lambda_{2\omega} \hat{P} < \bar{s}(\omega)^2 \Lambda_{1\omega}$,

   (c) $\exists \underline{\xi}_{1\omega}, \bar{\xi}_{1\omega} : \underline{\xi}_{1\omega}(\hat{V}^*\hat{V}) < \Lambda_{1\omega} < \bar{\xi}_{1\omega}(\hat{V}^*\hat{V})$,

   $\overline{w}_1(\omega)^{-2} < \underline{\xi}_{1\omega}, \quad \bar{\xi}_{1\omega} < \underline{w}_1(\omega)^{-2}$,

   $\bar{\xi}_{1\omega} < k_1(\omega)^2 \underline{\xi}_{1\omega}$,

   (d) $\exists \underline{\xi}_{2\omega}, \bar{\xi}_{2\omega} : \underline{\xi}_{2\omega}(\hat{U}^*\hat{U})^{-1} < \Lambda_{2\omega} < \bar{\xi}_{2\omega}(\hat{U}^*\hat{U})^{-1}$,

   $\underline{w}_2(\omega)^2 < \underline{\xi}_{2\omega}, \quad \bar{\xi}_{2\omega} < \overline{w}_2(\omega)^2$,

   $\bar{\xi}_{2\omega} < k_2(\omega)^2 \underline{\xi}_{2\omega}$.

   The above minimisation problem can thus be easily solved using LMI routines.

   Denote by $\Lambda^\star_{1\omega}$ and $\Lambda^\star_{2\omega}$ the values of $\Lambda_{1\omega}$ and $\Lambda_{2\omega}$ that achieve the minimum of the above optimisation problem at each frequency ω.
4. a. Construct *diagonal* transfer function matrices $D_1^\star(s)$ and $D_2^\star(s)$ in $\mathscr{GH}_\infty$ by fitting stable minimum phase transfer functions to each magnitude function on the main diagonal of $(\Lambda^\star_{1\omega})^{-1/2}$ and $(\Lambda^\star_{2\omega})^{1/2}$, respectively.

   b. Solve the following co-spectral and spectral factorisations (Francis, 1987, Section 7.3)

   $$(W^\star_{1,i})(W^\star_{1,i})^\sim = (\hat{V})(D_1^\star)(D_1^\star)^\sim(\hat{V})^\sim,$$

   $$(W^\star_{2,i})^\sim(W^\star_{2,i}) = (\hat{U})(D_2^\star)^\sim(D_2^\star)(\hat{U})^\sim,$$

   to obtain loop-shaping weights $W^\star_{1,i}(s)$ and $W^\star_{2,i}(s)$ in $\mathscr{GH}_\infty$ that have the required structure. Note that the above factorisations reduce to simply setting $W^\star_{1,i}(s) = D_1^\star(s)$ (resp. $W^\star_{2,i}(s) = D_2^\star(s)$) when $\hat{V}(s) = I_n$ (resp. $\hat{U}(s) = I_m$).
5. Compute $b_{\text{opt}}(W^\star_{2,i} P W^\star_{1,i})$ as detailed in Glover and McFarlane (1989) and let this value be denoted by $\varepsilon^\star_{\max,i}$. Furthermore, synthesise a controller $C^\star_{\infty,i}$ that achieves a robust stability margin $b(W^\star_{2,i} P W^\star_{1,i}, C^\star_{\infty,i}) = \varepsilon^\star_{\max,i}$ using the state-space formula given in Glover (1984, Theorem 6.3). Finally, let $C_i^\star = W^\star_{1,i} C^\star_{\infty,i} W^\star_{2,i}$.

   Note that all these calculations are easily computed using well-known formulae which are coded in commercially available software.
6. Evaluate $(\varepsilon^\star_{\max,i} - \varepsilon^\star_{\max,i-1})$. If this difference (which is always positive) is very small (say less than 0.01) and has remained this small for the last few iterations, then EXIT. Otherwise return to Step 2.

The resulting outputs from the algorithm are: the maximised value of $b_{\text{opt}}(P_s)$ obtained by the algorithm in the variable $\varepsilon^\star_{\max,i}$, loop-shaping weights $W^\star_{1,i}(s)$ and $W^\star_{2,i}(s)$ that achieve this maximised robust stability margin, and a controller $C^\star_{\infty,i}(s)$ that achieves $b(W^\star_{2,i} P W^\star_{1,i}, C^\star_{\infty,i}) = \varepsilon^\star_{\max,i}$.

This algorithm is an ascent one, meaning that the value $\varepsilon^\star_{\max,i}$ is monotonically non-decreasing as i increases and that at each iteration i, the reciprocal of the square-root of the minimum cost γ_ω^2 obtained in Step 3 of the algorithm is greater than or equal to $\varepsilon^\star_{\max,i-1}$ for all frequencies ω. Note however that iterative algorithms as the one presented above cannot be guaranteed to converge to the *global maximum*. Only monotonicity properties can be guaranteed. Use of this algorithm on various practical design examples has however shown that the algorithm is quite insensitive to the choice of the initial stabilising controller $C_0^\star$. This is almost certainly due to the fact that the synthesised controller $C^\star_{\infty,i}$ is allowed to change at both Steps 3 (since changing weights imply changing $C^\star_{\infty,i}$) and 5 of the algorithm—i.e., the algorithm has enough freedom to rectify a poor choice of initial stabilising controller.

[2] Since the controller C will be held fixed during Step 3 of the solution algorithm, the *single* optimisation problem obtained at the end of Section 3.1 is decoupled into an *infinite number* of optimisation problems, each solved at a different frequency ω. In doing so, the optimisation problems do not remain identical (or equivalent) because in Step 3 we minimise γ_ω^2 at each frequency ω whereas in the optimisation problem at the end of Section 3.1 only $\gamma^2 = \sup_\omega \gamma_\omega^2$ is minimised. This is not a difficulty for the following three reasons: (i) minimising γ_ω^2 at each frequency ω in Step 3 will guarantee that γ^2 in the optimisation problem at the end of Section 3.1 is also minimised, even though this is not true vice versa, (ii) there is nothing wrong in achieving a higher robust stability margin $1/\gamma_\omega$ at each frequency ω than what is possible over all frequencies $1/\gamma$, and (iii) as the iterations proceed in this algorithm, γ_ω will typically become flat across frequency and approximately equal to γ due to the controller synthesis Step 5.

5. Numerical example

The following numerical example will illustrate the algorithm proposed in Section 4. The plant used to demonstrate the applicability of the proposed algorithm is a scaled-down version of the high incidence research model developed by the Defence Evaluation and Research Agency in Bedford, UK. A physical model of this was constructed at the University of Cambridge in order to investigate problems associated with the control of air-vehicles at high angles of attack. Details of the identification experiments carried out on this plant may be found in Papageorgiou and Glover (1999a).

The scaled nominal open-loop plant P is given by the following identified state-space model

$$\left[\begin{array}{cccc|cc} 0 & 1 & 0 & 0 & 0 & 0 \\ -23.8 & -3.36 & 4.60 & -0.239 & 1.67 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -16.8 & -0.0248 & 22.8 & -0.916 & 0 & 1.39 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right]$$

in cascade with a low-pass filter with corner frequency of 50 rad/s and a Padé approximation of a 0.05 s time delay to model the actuators on each input channel (i.e. P has 8 states). The loop-shape boundaries $\underline{s}(\omega)$ and $\bar{s}(\omega)$ selected so that the performance specifications are satisfied are given by

$$\underline{s}(\omega) = \frac{1000\left|\frac{j\omega}{0.3}+1\right|}{\left|\frac{j\omega}{0.03}+1\right|^2\left|\frac{j\omega}{30}+1\right|^7} \quad \text{and}$$

$$\bar{s}(\omega) = \frac{10^6\left|\frac{j\omega}{2}+1\right|^4}{\left|\frac{j\omega}{0.2}+1\right|^5\left|\frac{j\omega}{200}+1\right|^3}.$$

For instance, the bandwidth determines the rise time and the low-frequency gain determines the sensitivity reduction and hence the reference tracking capabilities. The frequency functions $\underline{w}_1(\omega)$, $\overline{w}_1(\omega)$ and $k_1(\omega)$ that confine the singular values/condition number of pre-compensator W_1 were chosen to be 10^{-10}, 10^{10} and 20, respectively. These bounds never became active in this particular design example. In other design problems (say, for plants with very large condition number and/or very stringent requirements on gains from plant output disturbances to control signal or from plant input disturbances to output signal), it may be necessary to specify more complicated, perhaps frequency dependent, bounds to satisfy the problem specifications.

For the purpose of illustration, two design cases will be considered, one requiring a diagonal pre-compensator W_1 and the other requiring a non-diagonal pre-compensator W_1. The post-compensator will be held fixed in both design cases for simplicity of illustration. This will also usually be the case in practice. W_2 was in fact chosen to be a first-order low-pass filter with a corner frequency of 300 rad/s on each output channel for sensor noise rejection. The algorithm was coded up in Matlab 5.3 and run on a 400 MHz Pentium II PC. One hundred equally spaced frequency points on a logarithmic scale between $\omega=10^{-4}$ and 10^4 rad/s were chosen for Step 3 of the algorithm (i.e. 100 optimisation problems were solved at Step 3 of the algorithm at each iteration). Four iterations were sufficient for practical convergence of the algorithm in both design cases and it took less than 6 min in both designs. The singular values of the designed pre-compensator W_1, the correspondingly achieved loop-shape and the singular values of the simultaneously synthesised robustly stabilising controller C_∞ for both design cases are shown in Fig. 2. In both cases, it can be easily seen that the loop-shape lies in the pre-specified region and that it rolls-off at a very small rate around cross-over. Furthermore, both the loop-shaping weight W_1 and the controller C_∞ introduce some phase lead around cross-over to improve the robust stability margin. The synthesised precompensator W_1 had 8 states when it was required to have a diagonal structure but it had 38 states, which could be easily model reduced to 12 states, when it was required to have a non-diagonal structure. The condition number of W_1 was less than 4 at all frequencies in both design scenarios, which is typically considered good. The order of the synthesised controller C_∞ was 17 states in the diagonal design case and 46 states (which could easily be model reduced to 11 states) in the non-diagonal design case. Interestingly, the final implementable controller $C = W_1C_\infty W_2$ after model reduction had 17 states in both design situations. Finally, and perhaps of greatest importance, the robust stability margin obtained in the diagonal design was 0.368 whereas that obtained in the non-diagonal design was slightly higher at 0.382, as one would expect.

In the non-diagonal pre-compensator design case, even though the synthesised weight and controller were of much higher order, the achieved robust stability margin was slightly higher too. It was also noted that the controller $C = W_1C_\infty W_2$ could be model reduced to 17 states in *both* design cases with negligible deterioration of the designed properties. Thus, one should not be scared 'a priori' of a non-diagonal design simply on the grounds that it may give higher order weights, as these can often be model reduced afterwards. However, if diagonal weights achieve approximately the same robust stability margin as non-diagonal weights, then it is futile to use more complicated weights to achieve very minor improvements.

6. Conclusions

An optimisation problem has been introduced which facilitates the design of loop-shaping weights in the $\mathscr{H}_\infty$ loop-shaping design procedure. The weights can be chosen to

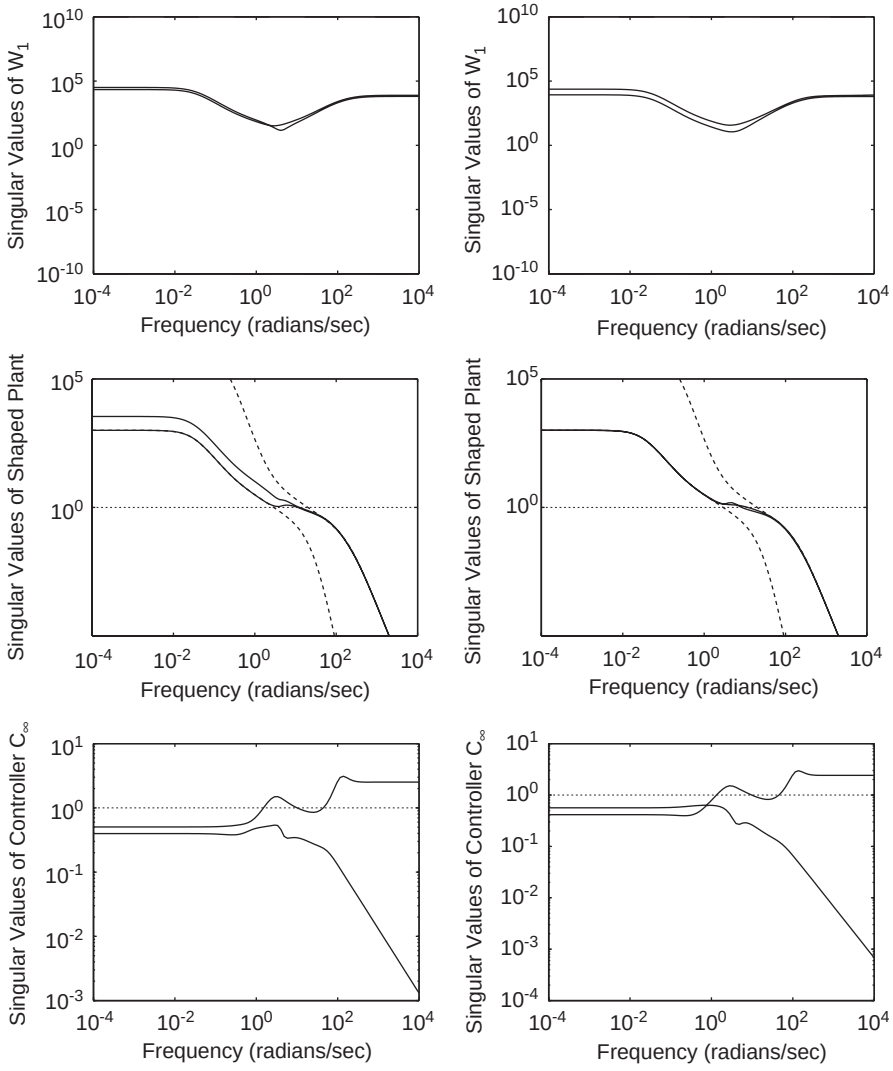

Fig. 2. Results for diagonal W_1 are shown on the left whereas results for non-diagonal W_1 are shown on the right.

have a diagonal or full block structure. Since the algorithm is not time-consuming, it enables the designer to determine whether non-diagonal weights are beneficial. The proposed procedure also eliminates much of the trial and error that typically goes on in designing "good" loop-shaping weights. The proposed optimisation problem thus allows the designer to concentrate on more fundamental design issues than simply finding weights that achieve the desired loop-shape.

Software code has been developed in Matlab and can be requested via email from the author.

References

Francis, B. A. (1987). *A course in $\mathscr{H}_\infty$ control theory*, *Lecture notes in control and information sciences* (Vol. 88). Berlin: Springer.

Glover, K. (1984). All optimal Hankel-norm approximations of linear multivariable systems and their $\mathscr{L}_\infty$-error bounds. *International Journal of Control*, *39*(6), 1115–1193.

Glover, K., & McFarlane, D. (1989). Robust stabilization of normalized coprime factor plant descriptions with $\mathscr{H}_\infty$-bounded uncertainty. *IEEE Transactions on Automatic Control*, *34*(8), 821–830.

Hyde, R. A. (1995). $\mathscr{H}_\infty$ aerospace control design—a VSTOL flight application. In *Advances in industrial control series*. Berlin: Springer.

Lanzon, A. (2000). *Weight selection in robust control: An optimisation approach*. Ph.D. thesis, University of Cambridge, UK. Download from `http://rsise.anu.edu.au/~alanzon`.

Lanzon, A. (2001). Simultaneous synthesis of weights and controllers in $\mathscr{H}_\infty$ loop-shaping. In *Proceedings of the 40th IEEE conference on decision and control* (Vol. 1) Orlando, FL, USA (pp. 670–675).

Lanzon, A., Bombois, X., & Anderson, B. D. O. (2003). On weight adjustments in $\mathscr{H}_\infty$ control design. In *Proceedings of the European control conference*, Cambridge, UK.

Lanzon, A., & Cantoni, M. W. (2000). A state-space algorithm for the simultaneous optimisation of performance weights and controllers in μ-synthesis. In *Proceedings of the 39th IEEE conference on decision and control* (Vol. 1) Sydney, Australia (pp. 611–616).

Lanzon, A., & Cantoni, M. W. (2003). On the formulation and solution of robust performance problems. *Automatica*, *39*(10), 1707–1720.

Lanzon, A., & Richards, R. J. (1999). A frequency domain optimisation algorithm for simultaneous design of performance weights and

controllers in μ-synthesis. In *Proceedings of the 38th IEEE conference on decision and control* (Vol. 5) Phoenix, AZ, USA (pp. 4523–4528).

McFarlane, D., & Glover, K. (1990). *Robust controller design using normalized coprime factor plant descriptions*. *Lecture notes in control and information sciences* (Vol. 138, 1st ed.). Berlin: Springer.

McFarlane, D., & Glover, K. (1992). A loop shaping design procedure using $\mathscr{H}_\infty$ synthesis. *IEEE Transactions on Automatic Control*, *37*(6), 759–769.

Papageorgiou, G., & Glover, K. (1997). A systematic procedure for designing non-diagonal weights to facilitate $\mathscr{H}_\infty$ loop-shaping. In *Proceedings of the 36th IEEE conference on decision and control*, San Diego, CA, USA (pp. 2127–2132).

Papageorgiou, G., & Glover, K. (1999a). Design, development and control of the HIRM wind tunnel model. In *Proceedings of the 38th IEEE conference on decision and control*, Phoenix, AZ, USA (pp. 1529–1537).

Papageorgiou, G., & Glover, K. (1999b). $\mathscr{H}_\infty$ loop-shaping: Why is it a sensible procedure for designing robust flight controllers? In *Proceedings of the AIAA conference on guidance, navigation and control*.

Vinnicombe, G. (2000). *Uncertainty and feedback: $\mathscr{H}_\infty$ loop-shaping and the ν-gap metric*. Imperial College Press.

Zhou, K., Doyle, J. C., & Glover, K. (1996). *Robust and optimal control*. Englewood Cliffs, NJ: Prentice-Hall.

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