Authors' Camera Ready Manuscript. To appear in the
Proceedings of the 14th UKACC International Conference on Control, Winchester, UK, Apr 2024,
Please cite using bibliographic data of the associated published version.

Generalized Dissipativity and Nonlinear Negative Imaginary Systems

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Abstract— **Nonlinear negative imaginary systems find application in a range of engineering fields, including the control of flexible structures and air vehicles. Nevertheless, unlike their linear counterparts, the theory for nonlinear negative imaginary systems is not as well-established. In this paper, we propose a generalized *k*-th order dissipativity framework with respect to a supply rate which is a function of the *k*-th time-derivative of the system output. It is shown that positive realness and negative imaginaryness can be defined in this general framework in a unified manner. Then, necessary and sufficient conditions for first order dissipativity of nonlinear systems are obtained. These capture and are more general than the negative imaginary property. Moreover, the concept of exponentially negative imaginary systems for both linear and nonlinear systems is developed and the required conditions are obtained.**

Keywords: Negative Imaginary Systems, Dissipative Systems, Passive Systems, Nonlinear Systems.

I. Introduction

Negative Imaginary (NI) systems exhibit a frequency response characterized by a negative imaginary component [1], [2]. The theory of NI systems has been extensively employed in the analysis and design of linear time-invariant (LTI) control systems, particularly in applications involving flexible structures and air vehicles [1]–[3]. NI systems possess several desirable traits, including robust stability in the presence of positive feedback, dissipation relative to collocated inputs and outputs, and the availability of optimal controllers [4]. A pivotal finding in NI theory states that the positive feedback connection between an NI system and a strictly NI system leads to a robustly stable system [1], [2], [5]. This implies that when a system is identified as NI, it is highly advantageous to design a strictly NI controller to ensure robust stability.

The theory of negative imaginary systems offers a framework for assessing robustness and formulating robust controller designs. In essence, this theory enables the creation of controllers capable of preserving stability and performance, even when faced with uncertainties in the system, such as unaccounted-for spillover dynamics or fluctuations in resonant frequencies and damping coefficients [6]. Furthermore, [7] connects the frequency-domain NI conditions of LTI input-output NI systems with time-domain supply rates in a dissipativity framework for such LTI systems. However, numerous real-world systems exhibit nonlinear behavior, rendering them unsuitable for accurate representation by LTI models. For instance, systems like a mass-spring-damper setup with a nonlinearity in either the spring or damper fall under the category of nonlinear NI systems [8]. Analyzing and controlling nonlinear NI systems present considerable hurdles, as the established theory for NI systems does not directly apply in this context. Hence, there is a pressing need to establish a comprehensive framework tailored for nonlinear NI systems, one that can encapsulate their fundamental characteristics and facilitate their effective control.

Some preliminary work has been done for extending the NI property to nonlinear systems. In [8], [9], based on the time-domain interpretation of the NI property for LTI system and dissipativity property, a formal definition of the nonlinear negative imaginary property is presented and the NI robust stability results are extended to nonlinear systems. However, the notion of strictly negative imaginary (SNI) and steady-state gain is not well-defined for nonlinear system in these works. Moreover, the connection between a nonlinear system and its corresponding Jacobi linearized model in term of NI and SNI properties is not investigated. Additionally, [10] has extended LTI NI properties to linear time-varying systems, but the definitions and results therein are unable to handle full nonlinear dynamics.

In this paper, we introduce a k^{th} order dissipativity framework, centered around a supply rate function dependent on the k^{th} time-derivative of the system output. This framework demonstrates the ability to encompass both positive realness and negative imaginary characteristics within a unified context. Consequently, we derive the necessary and sufficient conditions, for achieving first-order dissipativity in nonlinear systems. Additionally, we introduce the notion of exponentially negative systems to encompass both linear and nonlinear systems, and establish the corresponding conditions for this characterization. Moreover, it is shown that a linearized model of the nonlinear negative imaginary system also is NI.

II. Preliminaries and Notation

A. Dissipative Nonlinear systems

Consider the nonlinear dynamical system $\mathcal{G}$ of form

$$\dot{x}(t) = F(x(t), u(t)), \quad (1)$$

$$y(t) = H(x(t), u(t)), \quad (2)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^l$ are the system state, control input, and output, respectively. It is assumed that $F(\cdot,\cdot)$ and $H(\cdot,\cdot)$ are smooth functions, and without loss of generality, $F(0,0) = 0$ and $H(0,0) = 0$.

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For the dynamical system $\mathcal{G}$, a function $r : \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R}$ such that $r(0, 0) = 0$ is called a *supply rate* if it is locally integrable.

Definition 1: A dynamical system $\mathcal{G}$ is dissipative with respect to the supply rate $r(u, y)$ if the dissipation inequality

$$0 \leq \int_0^t r(u(s), y(s))\mathrm{d}s, \tag{3}$$

is satisfied for all $t \geq 0$ and all $u(\cdot) \in \mathbb{R}^m$ with $x(0) = 0$ along the trajectory of $\mathcal{G}$.

Definition 2: ([11]) A dynamical system $\mathcal{G}$ is exponentially dissipative with respect to the supply rate $r(u, y)$ if there exists a constant $\varepsilon > 0$ such that

$$0 \leq \int_0^t e^{\varepsilon s} r(u(s), y(s))\mathrm{d}s, \tag{4}$$

is satisfied for all $t \geq 0$ and all $u(\cdot) \in \mathbb{R}^m$ with $x(0) = 0$ along the trajectory of $\mathcal{G}$.

Definition 3: For the dynamical system $\mathcal{G}$, a continuous, nonnegative-definite function $V_{\mathrm{s}} : \mathbb{R}^n \to \mathbb{R}$, $V_{\mathrm{s}}(0) = 0$ is called a *storage function* if

$$V_{\mathrm{s}}(x(t)) \leq V_{\mathrm{s}}(x(0)) + \int_0^t r(u(s), y(s))\mathrm{d}s, \tag{5}$$

for all $t \geq 0$ with $u(\cdot) \in \mathbb{R}^m$.

Definition 4: ([11]) For the dynamical system $\mathcal{G}$, a continuous, nonnegative-definite function $V_{\mathrm{s}} : \mathbb{R}^n \to \mathbb{R}$, $V_{\mathrm{s}}(0) = 0$ is called an *exponential storage function* if

$$e^{\varepsilon t} V_{\mathrm{s}}(x(t)) \leq V_{\mathrm{s}}(x(0)) + \int_0^t e^{\varepsilon s} r(u(s), y(s))\mathrm{d}s, \tag{6}$$

for all $t \geq 0$ with $u(\cdot) \in \mathbb{R}^m$.

It is shown that [11] if $\mathcal{G}$ is completely reachable and locally controllable, then $\mathcal{G}$ is dissipative (respectively exponentially dissipative) with respect to the supply rate $r(u, y)$ if and only if there exists a continuous function $V_{\mathrm{s}}(x)$ satisfying (5) (respectively, (6)). Moreover if $\mathcal{G}$ is also zero-state observable, then all the storage functions $V_{\mathrm{s}}(x)$ for $\mathcal{G}$ are positive definite. Finally, if V_{s} is continuously differentiable, then $\mathcal{G}$ is dissipative with respect to $r(u, y)$ if

$$\dot{V}_s(x(t)) \leq r(u(t), y(t)), \quad \forall t \geq 0. \tag{7}$$

Similarly, $\mathcal{G}$ is exponentially dissipative with respect to $r(u, y)$ if

$$\dot{V}_s(x(t)) + \varepsilon V_s(x(t)) \leq r(u(t), y(t)), \quad \forall t \geq 0. \tag{8}$$

Furthermore, a system $\mathcal{G}$ with storage function $V_s(\cdot)$ is *strictly dissipative* with respect to the supply rate $r(u, y)$ if

$$V_{\mathrm{s}}(x(t)) < V_{\mathrm{s}}(x(0)) + \int_0^t r(u(s), y(s))\mathrm{d}s, \quad t > 0. \tag{9}$$

Definition 5: A dynamical system $\mathcal{G}$ with $m = l$ is passive (respectively, exponentially passive) if it is dissipative (respectively, exponentially dissipative) with respect to the supply rate $r(u, y) = 2u^{\mathrm{T}} y$.

Now, consider the following LTI system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = 0, \tag{10}$$

$$y(t) = Cx(t) + Du(t), \tag{11}$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{m\times n}$, $D \in \mathbb{R}^{m\times m}$, and with the square transfer function matrix $G(s) = C(sI - A)^{-1}B + D$.

Definition 6: ([11]) A square transfer function matrix $G(s)$ is positive real if the following conditions are satisfied:

1) $G(s)$ has no pole in $\mathrm{Re}[s] > 0$.
2) For all ω such that $s = j\omega$ is not a pole of $G(s)$,

$$(G(j\omega) + G^*(j\omega)) \geq 0. \tag{12}$$

3) If $s = j\omega_0$ is a pole of $G(s)$, then it is a simple pole and the residue matrix $K = \lim_{s\longrightarrow j\omega_0} (s - j\omega_0)G(s)$ is Hermitian and positive semidefinite.

Moreover, a square transfer function $G(s)$ is strongly strictly positive real [12] if there exists $\varepsilon > 0$ such that $G(s - \varepsilon)$ is positive real and $G(s) + G^{\mathrm{T}}(-s)$ has full normal rank.

Theorem 1: ([11]) Consider the linear dynamical system (10)-(11) with the transfer function $G(s)$. Then, it follows that $G(s)$ is positive real if and only if $\int_0^T y^{\mathrm{T}}(t)u(t)\mathrm{d}t \geq 0$, $\forall T \geq 0$.

Theorem 2: ([11]) Consider the linear dynamical system (10)-(11) with the transfer function $G(s)$ such that $G(s) + G^{\mathrm{T}}(-s)$ has full normal rank. Then, it follows that $G(s)$ is strongly strictly positive real if and only if there exists $\varepsilon > 0$ such that $\int_0^T e^{\varepsilon t} y^{\mathrm{T}}(t)u(t)\mathrm{d}t \geq 0$, $\forall T \geq 0$.

B. Negative Imaginary Systems

A negative imaginary (NI) system is defined as follows:

Definition 7: ([2]) A square transfer function matrix $G(s)$ is NI if the following conditions are satisfied:

1) $G(s)$ has no pole in $\mathrm{Re}[s] > 0$.
2) For all $\omega > 0$ such that $s = j\omega$ is not a pole of $G(s)$,

$$j\,(G(j\omega) - G^*(j\omega)) \geq 0. \tag{13}$$

3) If $s = j\omega_0$ with $\omega_0 > 0$ is a pole of $G(s)$, then it is a simple pole and the residue matrix $K = \lim_{s\longrightarrow j\omega_0} (s - j\omega_0)jG(s)$ is Hermitian and positive semidefinite.
4) If $s = 0$ is a pole of $G(s)$, then $\lim_{s\longrightarrow 0} s^k G(s) = 0$ for all $k \geq 3$ and $\lim_{s\longrightarrow 0} s^2 G(s)$ is Hermitian and positive semidefinite.

The negative imaginary lemma presents necessary and sufficient conditions that describes NI systems, similar to the positive-real lemma [13], [14]. This result was presented in [1], [15] and then extended in [16], [17].

Lemma 1: ([17]) Let $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be a minimal state space realization of a transfer function matrix $G(s)$ for the system in (10)-(11). Then, $G(s)$ is NI if and only if there exist matrices $P = P^{\mathrm{T}} > 0$, $W \in \mathbb{R}^{m\times m}$, and $L \in \mathbb{R}^{m\times n}$

such that

$$PA + A^{\mathrm{T}}P + L^{\mathrm{T}}L = 0, \tag{14}$$

$$PB - A^{\mathrm{T}}C^{\mathrm{T}} + L^{\mathrm{T}}W = 0, \tag{15}$$

$$CB + B^{\mathrm{T}}C^{\mathrm{T}} - W^{\mathrm{T}}W = 0. \tag{16}$$

Next, the notation of strictly negative imaginary (SNI) and strongly strictly negative imaginary (SSNI) are defined and similar necessaty and sufficient conditions are presented.

Definition 8: ([1], [16]) A square transfer function matrix $G(s)$ is SNI if the following conditions are satisfied:

1) $G(s)$ has no pole in $\mathrm{Re}[s] \geq 0$.
2) For all $\omega > 0$, $j\,(G(j\omega) - G^*(j\omega)) > 0$.

Definition 9: ([12]) A square transfer function matrix $G(s)$ is SSNI if

1) For some $\varepsilon > 0$, the transfer function $G(s - \varepsilon)$ is NI.
2) $j[G(s) - G^{\mathrm{T}}(-s)]$ has full normal rank.

The theory of nonlinear NI systems is not yet well investigated. Preliminary work has been carried out in [8], [9], [18].

Definition 10: The nonlinear system $\mathcal{G}$ is said to be NI if there exists a positive definite continuously differentiable storage function $V : \mathbb{R}^n \to \mathbb{R}$ such that

$$V(x(t)) \leq V(x(0)) + \int_0^t \dot{y}^{\mathrm{T}}(\tau)u(\tau)\mathrm{d}\tau, \quad \forall t > 0, \tag{17}$$

for all $t \geq 0$ with $u(\cdot) \in \mathbb{R}^m$.

III. Generalized dissipativity

In this section, we try to unify disspativeness and negative imaginaryness of nonlinear system in one framework.

Definition 11: A dynamical system $\mathcal{G}$ with a relative degree of k is said to be k^{th} order dissipative with respect to the supply rate $r(u, y)$, $k = 0, 1, \ldots$ if the dissipation inequality

$$0 \leq \int_0^t r(u(s), y^{(k)}(s))\mathrm{d}s, \tag{18}$$

is satisfied for all $t \geq 0$ and all $u(\cdot) \in \mathbb{R}^m$ with $x(0) = 0$ along the trajectory of $\mathcal{G}$ where $y^{(k)}(t)$ denotes the k^{th} time derivative of $y(t)$ with $y^{(0)}(t) = y(t)$.

Similarly, the k^{th} order storage function can be defined as follows.

Definition 12: For the dynamical system $\mathcal{G}$ with a relative degree of k, a continuous, nonnegative-definite function $V_{\mathrm{s}}^k : \mathbb{R}^n \to \mathbb{R}$, $V_{\mathrm{s}}^k(0) = 0$ is called a *k^{th} order storage function* if

$$V_{\mathrm{s}}^k(x(t)) \leq V_{\mathrm{s}}^k(x(0)) + \int_0^t r(u(s), y^{(k)}(s))\mathrm{d}s, \tag{19}$$

for all $t \geq 0$ with $u(\cdot) \in \mathbb{R}^m$.

Definition 13: ([11]) For the dynamical system $\mathcal{G}$ with a relative degree of k, a continuous, nonnegative-definite function $V_{\mathrm{s}}^k : \mathbb{R}^n \to \mathbb{R}$, $V_{\mathrm{s}}^k(0) = 0$ is called an *exponentially k^{th} order storage function* if

$$e^{\varepsilon t}V_{\mathrm{s}}^k(x(t)) \leq V_{\mathrm{s}}^k(x(0)) + \int_0^t e^{\varepsilon s} r(u(s), y^{(k)}(s))\mathrm{d}s, \tag{20}$$

for all $t \geq 0$ with $u(\cdot) \in \mathbb{R}^m$.

Definition 14: A dynamical system $\mathcal{G}$ with a relative degree of k and $m = l$ is k^{th} order passive if it is k^{th} order dissipative with respect to the supply rate $r(u, y^{(k)}) = 2u^{\mathrm{T}}y^{(k)}$.

Moreover, if $V_{\mathrm{s}}^k(\cdot)$ is continuously differentiable, then $\mathcal{G}$ is k^{th} order dissipative with respect to $r(u, y)$ if

$$\dot{V}_s^k(x(t)) \leq r(u(t), y^{(k)}(t)), \quad \forall t \geq 0. \tag{21}$$

Similarly, if $V_{\mathrm{s}}^k(\cdot)$ is continuously differentiable, $\mathcal{G}$ is exponentially k^{th} order dissipative with respect to $r(u, y)$ if

$$\dot{V}_s^k(x(t)) + \varepsilon V_s^k(x(t)) \leq r(u(t), y^{(k)}(t)), \quad \forall t \geq 0. \tag{22}$$

It is clear that for $k = 0$, the above definitions are identical to the normal dissipativeness. However, for $k = 1$, the first order storage function $V_{\mathrm{s}}^1(\cdot)$ is a more generalized version and captures negative imaginaryness in (17) while first order passivity is equal to the negative imaginary property for nonlinear systems. Hence, the above unified framework will allow us to consider more general supply rate for the class of nonlinear negative imaginary systems. One important option for the supply rate $r(u, y)$ is the quadratic function

$$r(u, y) = y^{\mathrm{T}}Qy + 2y^{\mathrm{T}}Su + u^{\mathrm{T}}Ru, \tag{23}$$

with $Q \in \mathbb{R}^{l\times l}$, $Q^{\mathrm{T}} = Q$, $S \in \mathbb{R}^{l\times m}$, and $R \in \mathbb{R}^{m\times m}$, $R^{\mathrm{T}} = R$.

The next two lemma provide the stability results in the proposed generalized dissipativity.

Lemma 2: If $\mathcal{G}$ is k^{th} order dissipative with respect to the supply rate $r(u, y)$ with a continuously differentiable and positive definite k^{th} order storage function $V_s^k(\cdot)$, then the equilibrium point $x = 0$ of the unforced ($u(t) = 0$) system $\mathcal{G}_{\mathrm{a}}$ is Lyapunov stable.

Proof: The result follows from (21) by imposing $u(t) = 0$. ■

Lemma 3: If $\mathcal{G}_{\mathrm{a}}$ is exponentially k^{th} order dissipative with respect to the supply rate $r(u, y)$ with a continuously differentiable and positive definite exponentially k^{th} order storage function $V_s^k(\cdot)$, then the equilibrium point $x = 0$ of the unforced ($u(t) = 0$) system $\mathcal{G}_{\mathrm{a}}$ is asymptotically stable.

Proof: The result follows from (22) by imposing $u(t) = 0$. ■

A. First-order dissipativity, first-order passivity and nonlinear negative imaginary systems

The next theorem provides necessary and sufficient conditions for first order dissipativeness with respect to a quadratic function for a nonlinear affine dynamical system $\mathcal{G}_{\mathrm{a}}$ of the form

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t), \tag{24}$$

$$y(t) = h(x(t)), \tag{25}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^l$ are the system state, control input, and output, respectively. It is assumed that $f(\cdot)$, $g(\cdot)$, and $h(\cdot)$ are smooth functions, and without loss of generality, $f(0) = 0$ and $h(0) = 0$.

Theorem 3: Let $\mathcal{G}_\mathrm{a}$ be zero-state observable and completely reachable. Then, $\mathcal{G}_\mathrm{a}$ is first order dissipative with respect to the quadratic supply rate (23) if and only if there exist functions $V_\mathrm{s}^1 : \mathbb{R}^n \to \mathbb{R}$, $l : \mathbb{R}^n \to \mathbb{R}^p$, $\mathcal{W} : \mathbb{R}^n \to \mathbb{R}^{p\times m}$ such that $V_s^1(\cdot)$ is continuously differentiable and positive definite, and for all $x \in \mathbb{R}^n$

$$\nabla^\mathrm{T} V_\mathrm{s}^1(x)f(x) - f^\mathrm{T}(x)\nabla^\mathrm{T} h(x)Q\nabla h(x)f(x) + l^\mathrm{T}(x)l(x) = 0 \tag{26}$$

$$R + g^\mathrm{T}(x)\nabla^\mathrm{T} h(x)S + S^\mathrm{T}\nabla h(x)g(x) + g^\mathrm{T}(x)\nabla^\mathrm{T} h(x)Q\nabla h(x)g(x) - \mathcal{W}^\mathrm{T}(x)\mathcal{W}(x) = 0 \tag{27}$$

$$\frac{1}{2}\nabla^\mathrm{T} V_\mathrm{s}^1(x)g(x) - f^\mathrm{T}(x)\nabla^\mathrm{T} h(x)Q\nabla h(x)g(x) - f^\mathrm{T}(x)\nabla^\mathrm{T} h(x)S + l^\mathrm{T}(x)\mathcal{W}(x) = 0 \tag{28}$$

Proof: Suppose that $\mathcal{G}_\mathrm{a}$ is first order dissipative with respect to the quadratic supply rate (23), it follows from (21) that

$$\dot{V}_s^1(x(t)) \leq r(u(t), \dot{y}(t)), \forall t \geq 0. \tag{29}$$

Define the function $d : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ as

$$d(x,u) = -\dot{V}_\mathrm{s}^1(x) + r(u,\dot{y}) = -\nabla V_\mathrm{s}^1(x)(f(x)+g(x)u) + r\big(u, \nabla h(x)(f(x)+g(x)u)\big).$$

Then, it follows from (29) that $d(x,u) \geq 0$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and it is quadratic in u. Hence, there exist functions $l : \mathbb{R}^n \to \mathbb{R}^p$, $\mathcal{W} : \mathbb{R}^n \to \mathbb{R}^{p\times m}$ such that

$$\begin{aligned} d(x,u) &= \big(l(x) + \mathcal{W}(x)u\big)^\mathrm{T}\big(l(x) + \mathcal{W}(x)u\big) \\ &= -\nabla^\mathrm{T} V_\mathrm{s}^1(x)(f(x)+g(x)u) + r\big(u, \nabla h(x)(f(x)+g(x)u)\big) \\ &= -\nabla^\mathrm{T} V_\mathrm{s}^1(x)(f(x)+g(x)u) + 2(f(x)+g(x)u)^\mathrm{T}\nabla^\mathrm{T} h(x)Su \\ &+ (f(x)+g(x)u)^\mathrm{T}\nabla^\mathrm{T} h(x)Q\nabla h(x)(f(x)+g(x)u) + u^\mathrm{T} Ru \end{aligned}$$

By equating coefficients of equal power, conditions (26)-(28) are obtained.

Conversely, suppose that there exist $V_\mathrm{s}^1 : \mathbb{R}^n \to \mathbb{R}$, $l : \mathbb{R}^n \to \mathbb{R}^p$, $\mathcal{W} : \mathbb{R}^n \to \mathbb{R}^{p\times m}$ such that $V_s^1(\cdot)$ is continuously differentiable and positive definite, and (26)-(28) are satisfied. Then, for any $t \in \mathbb{R}$, $t \geq 0$, it follows that

$$\begin{aligned} \int_0^t r(u,\dot{y})\mathrm{d}t &= \int_0^t \big(\dot{y}^\mathrm{T} Q\dot{y} + 2\dot{y}^\mathrm{T} Su + u^\mathrm{T} Ru\big)\mathrm{d}t \\ &= \int_0^t \big(\nabla^\mathrm{T} V_s^1(x)(f(x)+g(x)u) + l^\mathrm{T}(x)l(x) \\ &\quad + 2l^\mathrm{T}(x)\mathcal{W}(x)u + u^\mathrm{T}\mathcal{W}^\mathrm{T}(x)\mathcal{W}(x)u\big)\mathrm{d}t \\ &= \int_0^t \big(\dot{V}_s^1(x) + \big(l(x)+\mathcal{W}(x)u\big)^\mathrm{T}\big(l(x)+\mathcal{W}(x)u\big)\big)\mathrm{d}t \\ &\geq V_s^1(x(t)) - V_s^1(x(0)), \end{aligned}$$

and hence $\mathcal{G}_\mathrm{a}$ is first order dissipative with respect to the quadratic supply rate (23). This completes the proof. ■

The next result presents necessary and sufficient conditions for negative imaginary (equivalently, first-order passive) nonlinear systems.

Corollary 1: Let $\mathcal{G}_\mathrm{a}$ be zero-state observable and completely reachable. Then, $\mathcal{G}_\mathrm{a}$ is negative imaginary if and only if there exist functions $V_\mathrm{s}^1 : \mathbb{R}^n \to \mathbb{R}$, $l : \mathbb{R}^n \to \mathbb{R}^p$, $\mathcal{W} : \mathbb{R}^n \to \mathbb{R}^{p\times m}$ such that $V_s^1(\cdot)$ is continuously differentiable and positive definite, and for all $x \in \mathbb{R}^n$

$$\nabla^\mathrm{T} V_\mathrm{s}^1(x)f(x) + l^\mathrm{T}(x)l(x) = 0, \tag{30}$$

$$g^\mathrm{T}(x)\nabla^\mathrm{T} h(x) + \nabla h(x)g(x) - \mathcal{W}^\mathrm{T}(x)\mathcal{W}(x) = 0, \tag{31}$$

$$\frac{1}{2}\nabla^\mathrm{T} V_\mathrm{s}^1(x)g(x) - f^\mathrm{T}(x)\nabla^\mathrm{T} h(x) + l^\mathrm{T}(x)\mathcal{W}(x) = 0. \tag{32}$$

Proof: The result follows from Theorem 3 with $l = m$, $Q = 0$, $R = 0$, and $S = I_m$. ■

Remark 1: It should be noted that the result in Corollary 1 is similar to the one presented in [8] with a minor constant coefficient difference. However, as it will be shown, the proposed framework can provide a more general definition for SSNI of nonlinear systems which is not considered in [8]. Moreover, the above results are consistent with the ones provided in Lemma 1 for linear systems.

Example 1: Consider the nonlinear controlled Lienard system

$$\ddot{z}(t) + \alpha(z(t))\dot{z}(t) + \beta(z(t)) = u(t), \tag{33}$$

with output $y(t) = \frac{1}{2}z(t)$ where $\alpha(z)$ is an even function and $\beta(z)$ is an odd function. By defining the states as $x_1(t) = z(t)$ and $x_2(t) = \dot{z}(t)$, the system (33) can be written as follows:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -\alpha(x_1(t))x_2(t) - \beta(x_1(t)) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \tag{34}$$

$$y(t) = \begin{bmatrix} \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}. \tag{35}$$

One can use the result of Corollary 1 to determine the conditions of $\alpha(x)$ and $\beta(x)$ that guarantee the Lienard system is nonlinear negative imaginary. Consider the energy function

$$V(x_1(t), x_2(t)) = \frac{1}{2}\dot{x}_2^2(t) + \int_0^{x_1(t)} \beta(s)\mathrm{d}s, \tag{36}$$

where $x_1\beta(x_1) \geq 0$ and hence $V(x_1(t), x_2(t)) \geq 0$ is a candidate first order storage function. Now, it follows from (30) that

$$\begin{bmatrix} \beta(x_1) & x_2 \end{bmatrix} \begin{bmatrix} x_2(t) \\ -\alpha(x_1)x_2 - \beta(x_1) \end{bmatrix} + l^\mathrm{T}(x)l(x) = 0$$

which leads to $\alpha(x_1)x_2^2 = l^\mathrm{T}(x)l(x)$ and consequently, we should have $\alpha(x_1) \geq 0$, $\forall x_1 \in \mathbb{R}$. The conditions (31) and (32) hold automatically by selecting $\mathcal{W}(x) = 0$.

The next result shows the relation between first order dissipativity of a nonlinear system and its linearized model.

Theorem 4: Let $\mathcal{G}_\mathrm{a}$ be zero-state observable and completely reachable and suppose it is first order dissipative with respect to the quadratic supply rate (23). Then, there exist matrices $P = P^\mathrm{T} > 0$, $W \in \mathbb{R}^{m\times m}$, and $L \in \mathbb{R}^{m\times n}$ such that

$$A^\mathrm{T} P + PA + L^\mathrm{T} L - A^\mathrm{T} C^\mathrm{T} QCA = 0, \tag{37}$$

$$PB - A^\mathrm{T} C^\mathrm{T} QCB - A^\mathrm{T} C^\mathrm{T} S + L^\mathrm{T} W = 0, \tag{38}$$

$$R + B^\mathrm{T} C^\mathrm{T} S + S^\mathrm{T} CB + B^\mathrm{T} C^\mathrm{T} QCB - W^\mathrm{T} W = 0, \tag{39}$$

where

$$A = \frac{\partial f}{\partial x}\Big|_{x=0}, B = g(0), C = \frac{\partial h}{\partial x}\Big|_{x=0}. \quad (40)$$

Proof: As $\mathcal{G}_\mathrm{a}$ is zero-state observable and completely reachable and it is first order dissipative with respect to the quadratic supply rate (23), there exist functions $V_\mathrm{s}^1 : \mathbb{R}^n \to \mathbb{R}$, $l : \mathbb{R}^n \to \mathbb{R}^p$, $\mathcal{W} : \mathbb{R}^n \to \mathbb{R}^{p\times m}$ such that $V_s^1(\cdot)$ is continuously differentiable and positive definite, and (26)-(28) are satisfied. Considering the Taylor series expansion of $V_s^1(\cdot)$ about $x = 0$, it follows that there exists a nonnegative-definite matrix P such that

$$V_s^1(x) = x^\mathrm{T} P x + V_\mathrm{sr}^1(x), \quad (41)$$

where $V_\mathrm{sr}^1(x)$ contains the higher-order terms of $V_s^1(x)$. Let $f(x) = Ax + f_\mathrm{r}(x)$, $l(x) = Lx + L_\mathrm{r}(x)$, and $h(x) = Cx + h_\mathrm{r}(x)$ with $f_\mathrm{r}(x)$, $L_\mathrm{r}(x)$, and $h_\mathrm{r}(x)$ represent the nonlinear terms of $f(x)$, $l(x)$, and $h(x)$, respectively. Moreover, let $g(x) = B + G_\mathrm{r}(x)$ and $\mathcal{W}(x) = W + \mathcal{W}_\mathrm{r}(x)$ with $G_\mathrm{r}(x)$, and $h_\mathrm{r}(x)$ contains the nonconstant term of $g(x)$ and $\mathcal{W}(x)$, respectively. It follows from (26) that

$$x^\mathrm{T}(A^\mathrm{T}P + PA + L^\mathrm{T}L - A^\mathrm{T}C^\mathrm{T}QCA)x + \gamma(x) = 0, \quad (42)$$

where $\gamma(x)$ includes the higher order terms in (26) with $\lim_{\|x\|\to 0} \frac{|\gamma(x)|}{\|x\|^2} = 0$. Hence, it follows that (37) is satisfied. Next, it follows from (28) that

$$x^\mathrm{T}(PB - A^\mathrm{T}C^\mathrm{T}QCB - A^\mathrm{T}C^\mathrm{T}S + L^\mathrm{T}W) + \Gamma(x) = 0, \quad (43)$$

where $\Gamma(x)$ includes the higher order terms in (28) with $\lim_{\|x\|\to 0} \frac{|\Gamma(x)|}{\|x\|} = 0$. Hence, it follows that (38) is satisfied. Finally, setting $x = 0$ in (27) yields to (39). ■

The next result presents the relation between a nonlinear negative imaginary system and its linearized one.

Corollary 2: Let $\mathcal{G}_\mathrm{a}$ be zero-state observable, completely reachable negative imaginary system. Then, its linearized model at $x = 0$ is a negative imaginary system.

Proof: By setting $l = m$, $Q = 0$, $R = 0$, and $S = I_m$, equations (37)-(39) yields to the conditions (14)-(16) and hence as per Lemma 1, the linear model given by the matrices in (40) is also NI. ■

Corollary 3: A linear system $G(s)$ in (10)-(11) is NI if and only if $\int_0^t u^\mathrm{T}(s)\dot{y}(s)\mathrm{d}s \geq 0$.

Proof: Applying the result of Corollary 1 for $\mathcal{G}_\mathrm{a}$ with $f(x) = Ax$, $g(x) = B$, and $h(x) = Cx$ yields to conditions (14)-(16) in Lemma 1 and hence $\mathcal{G}_\mathrm{a}$ is NI. ■

B. Exponentially Negative Imaginary Systems

Next, the above results are generalized to exponentially dissipativeness properties.

Theorem 5: Let $\mathcal{G}_\mathrm{a}$ be zero-state observable and completely reachable. Then, $\mathcal{G}_\mathrm{a}$ is first order exponentially dissipative with respect to the quadratic supply rate (23) if and only if there exist functions $V_\mathrm{s}^1 : \mathbb{R}^n \to \mathbb{R}$, $l : \mathbb{R}^n \to \mathbb{R}^p$, $\mathcal{W} : \mathbb{R}^n \to \mathbb{R}^{p\times m}$ such that $V_s^1(\cdot)$ is continuously differentiable and positive definite, and for all $x \in \mathbb{R}^n$

$$\nabla^\mathrm{T} V_\mathrm{s}^1(x)f(x) + \varepsilon V_\mathrm{s}^1(x) - f^\mathrm{T}(x)\nabla^\mathrm{T}h(x)Q\nabla h(x)f(x) + l^\mathrm{T}(x)l(x) = 0, \quad (44)$$

and the conditions (27)-(28) in Theorem 3 hold.

Proof: The proof is similar to the proof of Theorem 3. ■

Based on the above framework, we can define the notation of exponentially negative imaginary (ENI) systems as follows:

Definition 15: The nonlinear system $\mathcal{G}_\mathrm{a}$ or (in general case $\mathcal{G}$) with $x(0) = 0$ is called to be exponentially NI if it is exponentially dissipative with respect to the supply rate $r(u, \dot{y})$, i.e. $\int_0^t e^{\varepsilon s} r(u(s), \dot{y}(s))\mathrm{d}s \geq 0$.

Next, necessary and sufficient conditions for ENI are presented.

Corollary 4: Let $\mathcal{G}_\mathrm{a}$ be zero-state observable and completely reachable. Then, $\mathcal{G}_\mathrm{a}$ is exponentially negative imaginary (equivalently, first-order exponentially passive) if and only if there exist function $V_\mathrm{s}^1 : \mathbb{R}^n \to \mathbb{R}$, $l : \mathbb{R}^n \to \mathbb{R}^p$, $\mathcal{W} : \mathbb{R}^n \to \mathbb{R}^{p\times m}$ such that $V_s^1(\cdot)$ is continuously differentiable and positive definite, and for all $x \in \mathbb{R}^n$

$$\nabla^\mathrm{T} V_\mathrm{s}^1(x)f(x) + \varepsilon V_\mathrm{s}^1(x) + l^\mathrm{T}(x)l(x) = 0, \quad (45)$$

and conditions (31)-(32) in Corollary (1) hold.

Proof: The result follows from Theorem 5 with $l = m$, $Q = 0$, $R = 0$, and $S = I_m$. ■

The next theorem shows the relation between ENI of a nonlinear system and its linearized model.

Theorem 6: Let $\mathcal{G}_\mathrm{a}$ be zero-state observable and completely reachable and suppose it is first order exponentially dissipative with respect to the quadratic supply rate (23). Then, there exist matrices $P = P^\mathrm{T} > 0$, $W \in \mathbb{R}^{m\times m}$, and $L \in \mathbb{R}^{m\times n}$ such that

$$A^\mathrm{T}P + PA + L^\mathrm{T}L + \varepsilon P - A^\mathrm{T}C^\mathrm{T}QCA = 0, \quad (46)$$

and conditions (38)-(39) hold with matrices A, B, and C defined in (40).

Proof: The proof is similar to the proof of Theorem 3. ■

Corollary 5: A linear system $G(s)$ in (10)-(11) is ENI if and only if for some $\varepsilon > 0$, there exist matrices $P = P^\mathrm{T} > 0$, $W \in \mathbb{R}^{m\times m}$, and $L \in \mathbb{R}^{m\times n}$ such that

$$PA + A^\mathrm{T}P + \varepsilon P + L^\mathrm{T}L = 0, \quad (47)$$

$$PB - A^\mathrm{T}C^\mathrm{T} + L^\mathrm{T}W = 0, \quad (48)$$

$$CB + B^\mathrm{T}C^\mathrm{T} - W^\mathrm{T}W = 0. \quad (49)$$

Proof: Applying the result of Corollary 4 for $\mathcal{G}_\mathrm{a}$ with $f(x) = Ax$, $g(x) = B$, and $h(x) = Cx$ yields to conditions (47)-(49). ■

Remark 2: It is important to note that as per above results, the notation of ENI can be defined for both linear and nonlinear systems while the notation of SSNI in Definition 9 has been only defined for linear systems. Hence, based on the proposed generalized dissipativity framework defined in this paper, the notation of exponentially disspativeness is extended to exponentially negative imaginary property for both linear and nonlinear systems. The remaining open research problem is to investigate the relation between SSNI and ENI properties for linear systems.

IV. Conclusion

This paper introduces a generalized notation of dissipativity for nonlinear systems where both positive realness and negative imaginaryness can be defined in a unified framework. The general necessary and sufficient conditions for first order dissipativity and exponentially dissipativity are provided and as the special cases, the corresponding conditions for negative imaginaryness and exponential negative imaginaryness are obtained. The relation between these proprieties for linear and nonlinear systems are also developed. Future work is to extend the robust stability result of LTI negative imaginary systems to nonlinear systems based on the proposed framework.

References

[1] A. Lanzon and I. R. Petersen, "Stability robustness of a feedback interconnection of systems with negative imaginary frequency response," *IEEE Tran. Auto. Cont.*, vol. 53, no. 4, pp. 1042–1046, 2008.

[2] M. A. Mabrok, A. G. Kallapur, I. R. Petersen, and A. Lanzon, "Generalizing negative imaginary systems theory to include free body dynamics: Control of highly resonant structures with free body motion," *IEEE Tran. Auto. Cont.*, vol. 59, no. 10, pp. 2692–2707, 2014.

[3] A. Preumont, *Vibration Control of Active Structures: An Introduction*, N. Jacobs, Ed. Springer, 2011.

[4] I. R. Petersen, "Negative imaginary systems theory and applications," *Annual Reviews in Control*, vol. 42, pp. 309–318, 2016.

[5] A. Lanzon and H. J. Chen, "Feedback stability of negative imaginary systems," *IEEE Transactions on Automatic Control*, vol. 62, no. 11, pp. 5620–5633, Nov. 2017.

[6] Z. Song, A. Lanzon, S. Patra, and I. R. Petersen, "A negative-imaginary lemma without minimality assumptions and robust state-feedback synthesis for uncertain negative-imaginary systems," *Systems and Control Letters*, vol. 61, no. 12, pp. 1269–1276, Dec. 2012.

[7] A. Lanzon and P. Bhowmick, "Characterization of input-output negative imaginary systems in a dissipative framework," *IEEE Transactions on Automatic Control*, vol. 68, no. 2, pp. 959–974, Feb. 2023.

[8] A. G. Ghallab and I. R. Petersen, "Negative imaginary systems theory for nonlinear systems: A dissipativity approach," *arXiv preprint arXiv:2201.00144*, 2022.

[9] A. G. Ghallab, M. A. Mabrok, and I. R. Petersen, "Extending negative imaginary systems theory to nonlinear systems," in *2018 IEEE Conference on Decision and Control (CDC)*. IEEE, 2018, pp. 2348–2353.

[10] S. Kurawa, P. Bhowmick, and A. Lanzon, "Negative imaginary theory for a class of linear time-varying systems," *IEEE Control Systems Letters*, vol. 5, no. 3, pp. 1001–1006, Jul. 2021.

[11] W. M. Haddad and V. Chellaboina, *Nonlinear Dynamical Systems and Control: A Lyapunov-Based Approach*. Princeton University Press, 2008.

[12] A. Ferrante, A. Lanzon, and L. Ntogramatzidis, "Foundations of not necessarily rational negative imaginary systems theory: Relations between classes of negative imaginary and positive real systems," *IEEE Transactions on Automatic Control*, vol. 61, pp. 3052–3057, 2016. [Online]. Available: https://api.semanticscholar.org/CorpusID:22665363

[13] B. D. O. Anderson and S. Vongpanitlerd, *Network Analysis and Synthesis: A Modern Systems Approach*. Englewood Cliffs, N.J., USA: Prentice-Hall, 1973.

[14] B. Brogliato, R. Lozano, B. Maschke, and O. Egeland, *Dissipative Systems Analysis and Control*, 2nd ed., ser. Communications and Control Engineering. London, UK: Springer, 2007.

[15] A. Lanzon and I. R. Petersen, "A modified positive-real type stability condition," in *Proceedings of the European Control Conference*, 2007, pp. 3912–3918.

[16] J. Xiong, I. R. Petersen, and A. Lanzon, "A negative imaginary lemma and the stability of interconnections of linear negative imaginary systems," *IEEE Tran. Auto. Cont.*, vol. 55, no. 10, pp. 2342–2347, 2010.

[17] M. A. Mabrok, A. G. Kallapur, I. R. Petersen, and A. Lanzon, "Stability analysis for a class of negative imaginary feedback systems including an integrator," in *2011 8th Asian Control Conference (ASCC)*, 2011, pp. 1481–1486.

[18] M. A. Mabrok, M. A. Alyami, and E. E. Mahmoud, "On the dissipativity property of negative imaginary systems," *Alexandria Engineering Journal*, vol. 60, no. 1, pp. 1403–1410, 2021.