AN ECONOMIC PARAMETRISATION FOR PARAHERMITIAN MATRIX FUNCTIONS USED IN CONTROL SYSTEMS OPTIMISATION

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Abstract: Positive parahermitian matrix function descriptions occur frequently in optimisation problems that arise in control theory. Parahermitian matrix functions can however be parametrised in a number of different equivalent ways. This brief note discusses an economic parametrisation which leads to substantially less variables that are needed in optimisation.

Keywords: $\mathscr{H}_\infty$-control, optimisation, D-scales, μ-synthesis, parahermitian, positive functions

1. INTRODUCTION

Positive frequency response functions of the form $T(j\omega)^*T(j\omega)$, where T is a unit in $\mathscr{RH}_\infty$, occur frequently in $\mathscr{H}_\infty$-control. A popular example is D-scales in μ-synthesis based optimisations (Packard and Doyle, 1993; Young and Doyle, 1996). Such objects arise because of the simple property of an $\mathscr{H}_\infty$-norm which can be succinctly described by:

$$\begin{aligned} &\|TM\|_\infty < 1 \\ &\quad \Leftrightarrow \quad M(j\omega)^*[T(j\omega)^*T(j\omega)]M(j\omega) < I \quad \forall\omega \end{aligned}$$

for appropriate $T, M \in \mathscr{RL}_\infty$.

It is also frequently the case that these positive frequency response objects $T(j\omega)^*T(j\omega)$ are written as frequency responses of parahermitian rational matrix functions

$$\begin{bmatrix} B^T(-sI - A^T)^{-1} & I \end{bmatrix} \begin{bmatrix} P & S \\ S^T & R \end{bmatrix} \begin{bmatrix} (sI - A)^{-1}B \\ I \end{bmatrix}$$

for some real matrices A, B, P, S, R of compatible dimensions.

In state-space optimisation frameworks that invoke the Kalman-Yakubovich-Popov lemma (Rantzer, 1996) to transform frequency conditions (such as the frequency domain matrix inequality above) to state-space conditions, one typically optimises over P, S and R at different steps in the optimisation problem and then constructs the object of interest T via spectral factorisation at the end. However, parahermitian rational matrix functions do not have a unique parametrisation. It is consequently possible to economise on the number of variables by removing redundancy in the parametrisation. This is the aim of this brief note.

2. PARAHERMITIAN FUNCTIONS

The following well-known lemma shows that parahermitian rational matrix functions can be rewritten with arbitrary (1,1)-block. Variants of this result can be found in (Francis, 1987; Zhou *et al.*, 1996).

Lemma 1. Let A, B, P, S, R be real matrices of compatible dimensions such that $P = P^T$, $R = R^T$ and $\lambda_i(A) \neq -\lambda_j(A)\ \forall i, j$. Define the parahermitian rational matrix function

$$\Gamma(s) := \begin{bmatrix} B^T(-sI - A^T)^{-1} & I \end{bmatrix} \begin{bmatrix} P & S \\ S^T & R \end{bmatrix} \begin{bmatrix} (sI - A)^{-1}B \\ I \end{bmatrix}.$$

Then, given an arbitrary real matrix $\hat{P} = \hat{P}^T$ of the same dimensions as P, there exists a real matrix $\hat{S}$ of the same dimensions as S such that

$$\Gamma(s) = \begin{bmatrix} B^T(-sI - A^T)^{-1} & I \end{bmatrix} \begin{bmatrix} \hat{P} & \hat{S} \\ \hat{S}^T & R \end{bmatrix} \begin{bmatrix} (sI - A)^{-1}B \\ I \end{bmatrix}.$$

In fact, $\hat{S}$ is given by

$$\hat{S} = S + XB,$$

where the real matrix $X = X^T$ is the unique solution to the Lyapunov equation

$$XA + A^T X = (\hat{P} - P).$$

Proof Since $\lambda_i(A) + \lambda_j(A) \neq 0 \ \forall i, j$, the Lyapunov equation $XA + A^T X = (\hat{P} - P)$ has a unique solution (Zhou *et al.*, 1996, Lemma 2.7). Furthermore, since A is real and $(\hat{P} - P)$ is real and symmetric, such a solution is real and symmetric. Writing $\Gamma(s)$ as:

$$\Gamma(s) = \left[\begin{array}{cc|c} A & 0 & B \\ -P & -A^T & -S \\ \hline S^T & B^T & R \end{array}\right]$$

and applying the similarity transformation $\begin{bmatrix} I & 0 \\ X & I \end{bmatrix}$ yields the required result. □

The next lemma is the main result of this brief technical note and it gives a complete parametrisation of frequency functions of the form $T(j\omega)^* T(j\omega)$, where T is a unit in $\mathcal{RH}_\infty$. Using 0 as the arbitrary (1,1)-block in this parametrisation considerably reduces the number of potential decision variables in an eventual optimisation. Note that the dimension of the P matrix is the same as that of A and consequently, for high-order systems T, P would have a huge number of variables. The lemma below shows that there is no loss of generality in pinning all these variables at 0.

Lemma 2. Given $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ with A Hurwitz.

I. For every $C \in \mathbb{R}^{m\times n}$ and $D \in \mathbb{R}^{m\times m}$ such that $T(s) := \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in \mathcal{RH}_\infty$ satisfies $T^{-1} \in \mathcal{RH}_\infty$, there exist $Q_{12} \in \mathbb{R}^{n\times m}$ and $Q_{22} = Q_{22}^T \in \mathbb{R}^{m\times m}$ such that

$$T(j\omega)^* T(j\omega) = \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^* \begin{bmatrix} 0 & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} > 0$$

for all $\omega \in \mathbb{R} \cup \{\infty\}$.

II. For every $Q_{12} \in \mathbb{R}^{n\times m}$ and $Q_{22} = Q_{22}^T \in \mathbb{R}^{m\times m}$ such that

$$\begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^* \begin{bmatrix} 0 & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} > 0$$

for all $\omega \in \mathbb{R} \cup \{\infty\}$, there exist $C \in \mathbb{R}^{m\times n}$ and $D \in \mathbb{R}^{m\times m}$ such that $T(s) := \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in \mathcal{RH}_\infty$ satisfies $T^{-1} \in \mathcal{RH}_\infty$ and

$$T(j\omega)^* T(j\omega) = \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^* \begin{bmatrix} 0 & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}$$

for all $\omega \in \mathbb{R} \cup \{\infty\}$.

Proof *I.* For any $C \in \mathbb{R}^{m\times n}$ and $D \in \mathbb{R}^{m\times m}$ such that $T(s) := \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in \mathcal{RH}_\infty$ satisfies $T^{-1} \in \mathcal{RH}_\infty$, $T(j\omega)^* T(j\omega) > 0 \ \forall \omega \in \mathbb{R} \cup \{\infty\}$ and

$$T(j\omega)^* T(j\omega) = \begin{bmatrix} B^T(-j\omega I - A^T)^{-1} & I \end{bmatrix} \times \begin{bmatrix} C^T C & C^T D \\ D^T C & D^T D \end{bmatrix} \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}$$

for all $\omega \in \mathbb{R} \cup \{\infty\}$. Now using Lemma 1 with $\hat{P} = 0$, there exist $Q_{12} \in \mathbb{R}^{n\times m}$ and $Q_{22} = Q_{22}^T \in \mathbb{R}^{m\times m}$ (in fact $Q_{22} = D^T D$) such that the required result holds.

II. For any $Q_{12} \in \mathbb{R}^{n\times m}$ and $Q_{22} = Q_{22}^T \in \mathbb{R}^{m\times m}$ such that

$$\begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^* \begin{bmatrix} 0 & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} > 0$$

for all $\omega \in \mathbb{R} \cup \{\infty\}$, it follows by spectral factorisation results (Zhou *et al.*, 1996, Theorem 13.19 I(a)) and (Zhou *et al.*, 1996, Corollary 13.20) that there exist a $C \in \mathbb{R}^{m\times n}$ and a $D \in \mathbb{R}^{m\times m}$ such that $T(s) := \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in \mathcal{RH}_\infty$ satisfies $T^{-1} \in \mathcal{RH}_\infty$ and

$$T(j\omega)^* T(j\omega) = \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^* \begin{bmatrix} 0 & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}$$

for all $\omega \in \mathbb{R} \cup \{\infty\}$. □

3. CONCLUSION

Lemma 2 shows that there is no loss of generality in pinning the (1,1)-block of parahermitian frequency response matrix function parametrisations for positive frequency response objects of the form $T(j\omega)^* T(j\omega)$ to zero. In so doing, the number of variables that need to be found in an optimisation is considerably reduced as the size of this (1,1)-block is the same as the size of the A-matrix (i.e. the order of $T(s)$). Huge savings can consequently be envisaged for high order $T(s)$.

REFERENCES

Francis, B. A. (1987). *A course in $\mathcal{H}_\infty$ control theory*. Vol. 88 of *Lecture notes in Control and Information Sciences*. Springer-Verlag.

Packard, A. and J. Doyle (1993). The complex structured singular value. *Automatica* **29**(1), 71–109.

Rantzer, A. (1996). On the Kalman-Yakubovich-Popov lemma. *Systems and Control Letters* **28**(1), 7–10.

Young, P. M. and J. C. Doyle (1996). Properties of the mixed μ problem and its bounds. *IEEE Transactions on Automatic Control* **41**(1), 155–159.

Zhou, K., J. C. Doyle and K. Glover (1996). *Robust and Optimal Control*. Prentice-Hall, Inc.