

# STABILIZATION OF CONDITIONAL UNCERTAIN NEGATIVE-IMAGINARY SYSTEMS USING RICCATI EQUATION APPROACH\*

MOHAMED A. MABROK, ABHIJIT G. KALLAPUR, IAN R. PETERSEN<sup>†</sup> AND  
ALEXANDER LANZON<sup>‡</sup>

**Abstract.** In this paper, we derive a negative imaginary and a strict negative imaginary lemma based on Riccati equations. The resulting negative imaginary lemma is used to solve a controller synthesis problem. For a given uncertain system, a static controller is constructed to force the system to satisfy the negative imaginary property. As a result, the closed-loop system can be guaranteed to be robustly stable against any strict negative imaginary uncertainty, such as in the case of unmodeled spill-over dynamics in a lightly damped flexible structure. A numerical example is presented to illustrate the usefulness of the proposed results.

**Key words.** Negative imaginary systems, lightly damped systems, Riccati equations, uncertain system.

**1. Introduction.** Negative imaginary (NI) systems theory has many engineering applications. Such classes of systems include DC machines [2], electrical active filter circuits [9], lightly damped structures [4, 3, 7, 10, 5]. When force actuators and position sensors (such as piezoelectric sensors) are collocated on a flexible structure, the input/output map is NI. Stability results for interconnected systems with an NI frequency response have been applied to the decentralized control of large vehicle platoons in [1]. In [8] the authors show how the class of linear systems having negative imaginary transfer matrices is a direct extension of the class of linear Hamiltonian input-output systems. Also, an extension for negative imaginary systems to infinite-dimensional systems has been studied in [6].

NI systems theory was introduced by Lanzon and Petersen in [4]. In the SISO case, such systems are defined by considering the properties of the imaginary part of the frequency response  $G(j\omega) = D + C(j\omega I - A)^{-1}B$ , requiring the condition,  $j(G(j\omega) - G(j\omega)^*) \geq 0$  for all  $\omega \in (0, \infty)$ .

In general, NI systems are stable systems having a phase lag between 0 and  $-\pi$  for all  $\omega > 0$ . That is, their Nyquist plot lies below the real axis when the frequency varies in the open interval  $(0, \infty)$  (for strictly negative-imaginary systems, the Nyquist plot should not touch the real axis except at zero frequency and/or at infinity).

In this paper, we are concerned with the robust stabilization problem of uncertain systems when full state feedback is available, where the uncertainty is strict negative imaginary (SNI) and satisfies the condition  $\bar{\lambda}\Delta(0) < 1$ . Here,  $\bar{\lambda}(\cdot)$  denotes the maximum eigenvalue and  $\Delta(\cdot)$  is the uncertainty in the system model. We present a systematic method for designing a controller to force the closed-loop system to satisfy the NI property based on the Riccati equation approach. This controller has some advantages, since we guarantee its robust stability against SNI uncertainty. Also, by making the closed-loop system satisfy the NI property, we can choose a suitable SNI controller [7] to guarantee the stability and required closed-loop performance.

---

\*This work was supported by the Australian Research Council.

<sup>†</sup>School of Engineering and Information Technology, University of New South Wales at the Australian Defence Force Academy Canberra ACT 2600, Australia. [abdallamath@gmail.com](mailto:abdallamath@gmail.com), [abhijit.kallapur@gmail.com](mailto:abhijit.kallapur@gmail.com), [i.r.petersen@gmail.com](mailto:i.r.petersen@gmail.com).

<sup>‡</sup>Control Systems Centre, School of Electrical and Electronic Engineering, University of Manchester, Manchester M13 9PL, United Kingdom. [Alexander.Lanzon@manchester.ac.uk](mailto:Alexander.Lanzon@manchester.ac.uk).

This paper is further organized as follows: Section 2 introduces the concept of negative imaginary systems. and provides some technical results that will be used in deriving the main results in the paper. In section 3, a negative imaginary and a strict negative imaginary lemma based on Riccati equations have been introduced and a controller synthesis problem is addressed. Section 4 provides a numerical example to support the results.

**2. Preliminaries.** In this section, we introduce the definition for NI systems and present the NI Lemma. Also, we introduce some technical results which will be used in deriving the main results of this paper.

To establish the main results of this paper, we consider a generalized definition for NI systems which allows for poles at the origin as follows:

**DEFINITION 2.1.** *A square transfer function matrix  $G(s)$  is NI if all the following conditions are satisfied:*

1.  $G(s)$  has no pole in  $\text{Re}[s] > 0$ .
2. For all  $\omega \geq 0$  such that  $j\omega$  is not a pole of  $G(s)$ ,  $j(G(j\omega) - G(j\omega)^*) \geq 0$ .
3. If  $s = j\omega_0$ ,  $\omega_0 > 0$  is a pole of  $G(s)$  then it is a simple pole. Furthermore, if  $s = j\omega_0$ ,  $\omega_0 > 0$  is a pole of  $G(s)$ , the residual matrix  $K = \lim_{s \rightarrow j\omega_0} (s - j\omega_0)jG(s)$  is positive semidefinite Hermitian. If  $s = 0$  is a pole of  $G(s)$ , then it is either a simple pole or a double pole. If it is double pole, then,  $\lim_{s \rightarrow 0} s^2G(s) \geq 0$ .

**DEFINITION 2.2.** *A square transfer function matrix  $G(s)$  is SNI if the following conditions are satisfied:*

1.  $G(s)$  has no pole in  $\text{Re}[s] \geq 0$ .
2. For all  $\omega > 0$ ,  $j(G(j\omega) - G(j\omega)^*) > 0$ .

Consider the following LTI system,

$$\dot{x}(t) = Ax(t) + Bu(t), \quad (2.1)$$

$$y(t) = Cx(t) + Du(t), \quad (2.2)$$

where,  $A \in R^{n \times n}$ ,  $B \in R^{n \times m}$ ,  $C \in R^{m \times n}$ , and  $D \in R^{m \times m}$ .

**LEMMA 2.3.** *Let  $\left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$  be a minimal realization of the transfer function matrix  $G(s)$  for the system in (2.1)-(2.2). Then,  $G(s)$  is NI if and only if  $D = D^T$  and there exist matrices  $P = P^T \geq 0$ ,  $W \in R^{m \times m}$ , and  $L \in R^{m \times n}$  such that the following LMI is satisfied:*

$$\begin{bmatrix} PA + A^T P & PB - A^T C^T \\ B^T P - CA & -(CB + B^T C^T) \end{bmatrix} = \begin{bmatrix} -L^T L & -L^T W \\ -W^T L & -W^T W \end{bmatrix} \leq 0. \quad (2.3)$$

The following lemma gives spectral conditions for a transfer functions which will be used in deriving the SNI lemma.

**LEMMA 2.4.** *Let  $G(s) = \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$  be a minimal realization. Given  $A$  have no pure imaginary eigenvalues,  $\omega_0 > 0$  and  $\lambda \in C$  is not an eigenvalue of  $\frac{CB + B^T C^T}{2} > 0$ . Then,  $\lambda$  is an eigenvalue of  $H(j\omega_0) = \frac{1}{2}j\omega_0(G(j\omega_0) - G(j\omega_0)^*)$  if and only if  $j\omega_0$  is an eigenvalue of the matrix*

$$N_\lambda = \begin{bmatrix} A + BR_\lambda^{-1}CA & BR_\lambda^{-1}B^T \\ -A^T C^T R_\lambda^{-1}CA & -A^T - A^T C^T R_\lambda^{-1}B^T \end{bmatrix},$$

where  $R_\lambda = 2\lambda I - CB - B^T C^T$ .

Now, consider the following theorem that defines an SNI system based on the spectrum of its corresponding Hamiltonian matrix.

**THEOREM 2.5.** Let  $G(s) = \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$  be a minimal realization and  $CB + B^T C^T > 0$ . Then,  $G(s)$  is SNI if and only if

1.  $A$  is a Hurwitz matrix, and  $D = D^T$ ,
2. the Hamiltonian matrix

$$N_0 = \begin{bmatrix} A + BQ^{-1}CA & BQ^{-1}B^T \\ -A^T C^T Q^{-1}CA & -A^T - A^T C^T Q^{-1}B^T \end{bmatrix}$$

has no positive pure imaginary eigenvalues. Here,  $Q = -(CB + B^T C^T)$ .

**3. Main results.** In this section, we use algebraic Riccati equations to give a new representation for the NI and SNI lemmas. Then we will use the NI lemma to derive a static controller such that the closed-loop system satisfies the NI property.

**THEOREM 3.1.** Let  $G(s) = \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$  be a minimal realization with  $CB + B^T C^T > 0$ . Then  $G(s)$  is NI if and only if  $D = D^T$  and there exists a matrix  $P \geq 0$  such that  $P$  is a solution to the following algebraic Riccati equation

$$PA_0 + A_0^T P + PBR^{-1}B^T P + Q = 0, \quad (3.1)$$

where

$$\begin{aligned} A_0 &= A - BR^{-1}CA, \\ R &= CB + B^T C^T, \text{ and} \\ Q &= A^T C^T R^{-1}CA. \end{aligned}$$

**THEOREM 3.2.** Let  $G(s) = \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$  be a minimal realization and  $CB + B^T C^T > 0$ . Then  $G(s)$  is SNI if and only if

1.  $A$  has no  $j\omega$ -axis eigenvalues and  $D = D^T$ ,
2. there exists a matrix  $P > 0$  such that  $P$  is a solution to the following algebraic Riccati equation

$$PA_0 + A_0^T P + PBR^{-1}B^T P + Q = 0, \quad (3.2)$$

where all the eigenvalues of the matrix  $A_0 + BR^{-1}B^T P$  lie in the open left half of the complex plane or at the origin.

**3.1. Synthesis Result.** In order to present the synthesis result, consider the following state space representation for a linear uncertain system

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u, \\ z &= C_1 x, \\ w &= \Delta(s)z, \end{aligned} \quad (3.3)$$

where,  $A \in R^{n \times n}$ ,  $B_1 \in R^{n \times m}$ ,  $B_2 \in R^{n \times r}$ ,  $C_1 \in R^{m \times n}$ , and  $\Delta(s)$  represents the uncertainty matrix. Also, suppose that  $K$  is a static controller such that  $u = Kx$ ,

then the closed-loop interconnection of the system (3.3) with the static controller  $K$  is given by;

$$\begin{aligned}\dot{x} &= (A + B_2K)x + B_1w, \\ z &= C_1x, \\ w &= \Delta(s)z.\end{aligned}\tag{3.4}$$

Our aim is to construct the controller  $K$  such that the corresponding closed-loop system (3.4) satisfies the NI property.

Consider the following transformation (Schur transformation)

$$A_f = U^T(A - B_2(C_1B_2)^{-1}C_1A)U = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},\tag{3.5}$$

$$B_f = U^T(B_2(C_1B_2)^{-1} - B_1R^{-1}) = \begin{bmatrix} B_{f1} \\ B_{f2} \end{bmatrix},\tag{3.6}$$

$$\tilde{B}_1 = U^T B_1 = \begin{bmatrix} B_{11} \\ B_{22} \end{bmatrix}.\tag{3.7}$$

The transformation (3.5) can be completed such that  $A_{11}$  is a stable matrix (with a zero eigenvalue) and  $A_{22}$  is an anti-stable matrix.

**THEOREM 3.3.** *Consider an uncertain system model as in (3.3) with  $C_1B_2$  invertible and  $R = C_1B_1 + B_1^T C_1^T > 0$ . Then there exists a controller  $K$  such that the closed-loop system in (3.4) is NI if there exist matrices  $T \geq 0$  and  $S \geq 0$  such that*

$$-A_{22}T - TA_{22}^T + B_{f2}RB_{f2}^T = 0,\tag{3.8}$$

$$-A_{22}S - SA_{22}^T + B_{22}R^{-1}B_{22}^T = 0\tag{3.9}$$

and  $S - T < 0$ . Here,  $A_{22}$  is the anti-stable block of the matrix  $A_f$  defined in (3.5). Furthermore, the controller gain matrix is

$$K = (C_1B_2)^{-1}(B_1^T P - C_1A - R(B_2^T C_1^T)^{-1}B_2^T P),\tag{3.10}$$

where  $P = UP_f U^T$  and  $P_f$  is a solution to the algebraic Riccati equation

$$P_f A_f + A_f^T P_f - P_f B_f R B_f^T P_f + P_f \tilde{B}_1 R^{-1} \tilde{B}_1^T P_f = 0.\tag{3.11}$$

**REMARK 1.** *In the case of no anti-stable modes in the matrix  $A - B_2(C_1B_2)^{-1}C_1A$ , the controller matrix is*

$$K = -(C_1B_2)^{-1}C_1A.$$

**NOTE 1.** *One possible step-wise approach to choosing a control strategy would be:*

- *Given an uncertain system of the form (3.3), make sure it satisfies the requirement of Theorem 3.3.*
- *Find the corresponding static controller such that the closed-loop system is NI as given in Theorem 3.3 or Remark 1.*
- *Find a suitable SNI controller, such as an integral resonant controller to guarantee the robustness, stability and performance requirements of the closed-loop system [7].*

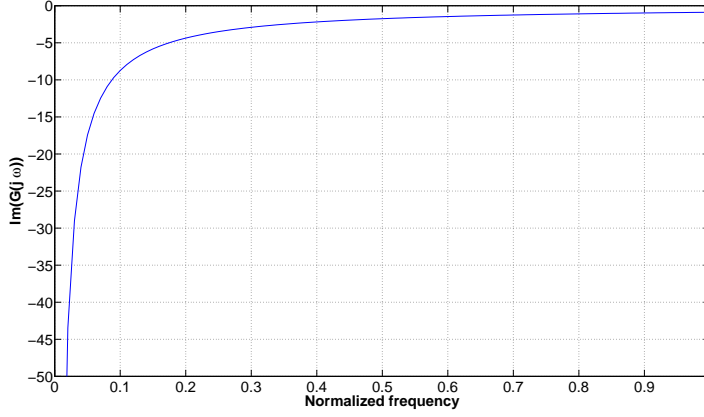


FIG. 4.1. Plot of the imaginary part of the closed-loop system.

**4. Illustrative Example.** In this section, we provide a numerical example in order to validate our results.

Consider the following uncertain system of the general form (3.3) where,

$$A = \begin{bmatrix} -1 & 0 & -1 \\ 1 & 1 & -1 \\ -5 & 1 & 1 \end{bmatrix}; B_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}; B_2 = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}; C_1 = [0 \quad 2 \quad -3]. \quad (4.1)$$

This system satisfies Theorem 3.3. Applying Schur decomposition to the matrix  $(A - B_2(C_1 B_2)^{-1} C_1 A)$  in (3.5) gives  $A_f = \begin{bmatrix} -2.8557 & -3.5104 & -39.4820 \\ 0 & 0 & 2.8020 \\ 0 & 0 & 10.8557 \end{bmatrix}$ .

The solution to Lyapunov equations (3.8) and (3.9) gives  $T = 0.0156$  and  $S = 0.0120$  which implies that  $X = -0.0036$ . It follows from Theorem 3.3 that  $P_f = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 275.5235 \end{bmatrix} \geq 0$ . This implies that the controller gain matrix (3.10) is given by  $K = [140.2792 \quad -14.2713 \quad -55.5191]$ , where,  $P = U P_f U^T \geq 0$  and  $U = \begin{bmatrix} -0.2864 & -0.2395 & -0.9277 \\ -0.7972 & 0.5967 & 0.0921 \\ -0.5315 & -0.7659 & 0.3618 \end{bmatrix}$  is the Schur transformation matrix. According to Theorem 3.3, the closed-loop feedback system (3.4) from  $w$  to  $z$  is NI. To illustrate this we plot the imaginary part of the transfer function matrix of the closed-loop system from  $w$  to  $z$  in Fig. 4.1.

**5. Conclusion.** In this paper, the algebraic Riccati equation approach was used to derive a negative imaginary (NI) lemma and a strict negative imaginary (SNI) lemma. The NI lemma was employed to solve a negative imaginary controller synthesis problem for an uncertain system. A static controller was chosen to force the plant to satisfy the negative imaginary property under certain assumptions. This controller can be used to guarantee the robustness stability of the closed-loop system with strict negative imaginary uncertainty.

## REFERENCES

- [1] CHAOHONG CAI AND GREGORY HAGEN, *Stability analysis for a string of coupled stable subsystems with negative imaginary frequency response*, IEEE Transactions on Automatic Control, 55 (Aug. 2010), pp. 1958–1963.
- [2] S. ENGELKEN, S. PATRA, A. LANZON, AND I. R. PETERSEN, *Stability analysis of negative imaginary systems with real parametric uncertainty - the SISO case*, IET Control Theory and Applications, In press (2010).
- [3] J. L. FANSON AND T. K. CAUGHLEY, *Positive position feedback control for large space structures*, AIAA Journal, 28 (1990), pp. 717–724.
- [4] A. LANZON AND I. R. PETERSEN, *Stability robustness of a feedback interconnection of systems with negative imaginary frequency response*, IEEE Transactions on Automatic Control, 53 (2008), pp. 1042–1046.
- [5] S. O. R. MOHEIMANI, B. J. G. VAUTIER, AND B. BHIKKAJI, *Experimental implementation of extended multivariable ppf control on an active structure*, IEEE Transactions on Control Systems Technology, 14 (2006), pp. 443–455.
- [6] M. R. OPMEER, *Infinite-dimensional negative imaginary systems*, IEEE Transactions on Automatic Control, 56 (2011), pp. 2973–2976.
- [7] I. R. PETERSEN AND A. LANZON, *Feedback control of negative imaginary systems*, IEEE Control Systems Magazine, 30 (2010), pp. 54–72.
- [8] A.J. VAN DER SCHAFT, *Positive feedback interconnection of hamiltonian systems*, in Proceedings of the 50th IEEE Conference on Decision and Control and European Control Conference (CDC-ECC) Orlando, FL, USA, 2011.
- [9] A. WATERS, *Active filter design*, London : Macmillan, 1991.
- [10] Y. K. YONG, B. AHMED, AND S. O. REZA MOHEIMANI, *Atomic force microscopy with a 12-electrode piezoelectric tube scanner*, Review of Scientific Instruments, 81 (March 2010).