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Analysis of robust performance for uncertain negative-imaginary systems using structured singular value

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Abstract— **Negative-Imaginary systems are important in engineering practise as this class of systems quite often appears in practical problems, for example, lightly damped flexible structures with collocated position sensors and force actuators. In this paper, the problem of assessing the robust $\mathcal{H}_\infty$ performance of uncertain negative-imaginary systems is investigated. It is shown that a structure singular value condition for a transformed input/output map equivalently gives a quantitative performance (measured via an $\mathcal{H}_\infty$-norm) test for systems with strictly negative-imaginary uncertainty.**

I. INTRODUCTION

The notion of negative-imaginary systems was first formalized in [1]. By definition, a square transfer function matrix $G(s)$ is said to be negative-imaginary if it is stable, and satisfies the following negative-imaginary condition: $j[G(j\omega) - G^*(j\omega)] \geq 0$ for all $\omega \in (0, \infty)$. Strictly negative-imaginary systems are defined in a similar way where the above non-strict negative-imaginary condition is replaced by a strict one. The definition of negative-imaginary systems implies that their Nyquist plot has phase lag between 0 to $-\pi$ for all $\omega > 0$, and hence is below real axis when the frequency varies in the open interval 0 to ∞. This is similar to positive real systems: the frequency response is constrained to lie in one half of the complex plane [2] [3]. However, negative-imaginary systems overcome the relative degree limitation of positive-real systems since negative-imaginary systems can have relative degree more than unity while positive-real systems can not.

Negative-imaginary systems have important applications in, for example, lightly damped flexible structures. When force actuators and position sensors (for instance piezoelectric sensors) are collocated on a flexible structure, the input/output map is negative-imaginary [1], [4]–[9]. These structures have long been of great interest to the engineering community (see [10]–[15] and references therein). They are in reality distributed parameter systems which are typically modeled with a very high (or infinite) order transfer functions. Quite often, for controller synthesis, a truncated plant model is used, the unmodeled dynamics give rise to spill-over dynamics that make it difficult to control [10] [11]. These unmodeled spill-over dynamics, quite often, belong to the class of strictly negative-imaginary systems [7]. In [1], it has been shown that the positive feedback interconnection of a negative-imaginary system and a strictly negative-imaginary system is internally stable if and only if the DC loop gain is less than unity. This mathematical stability result captures the graphical design methods related to positive position feedback control [10] [11] and integral control [15] in a systematic and rigorous framework. In this regard, it provides a natural tool for robust stability analysis and gives a direction for controller design of uncertain negative-imaginary systems. Based on this stability result, a static-state feedback robust stabilizing controller synthesis technique has been proposed in [7]. Preliminary results on dynamical stabilizing controller synthesis for uncertain negative-imaginary systems can be found in [5].

Robust stability is the minimum requirement of any practical control system. However, even if a closed-loop system is robustly stable, it is useless if it does not deliver the required performance. It is well known that the robust performance problem of uncertain linear time invariant feedback systems can be transformed into a robust stability problem by introducing a fictional bounded-real uncertainty, and then structure singular value theory is usually used to assess the resulting robust stability problem which involves a structured uncertainty. The standard definition of structure singular value assumes that the uncertainties are norm bounded [16]. However, the uncertainty considered in this paper is strictly negative-imaginary uncertainty (for example, the above mentioned highly resonant spill-over dynamics). When the robust performance problem is transformed into robust stability problem, the structured uncertainty involved is a mix of bounded-real uncertainty and strictly negative-imaginary uncertainty, which makes it a nontrivial problem. The main purpose of this paper is hence to extend the analysis framework for negative-imaginary systems to the robust performance problem that involves performance measured via an $\mathcal{H}_\infty$ norm and physically motivated uncertainty that satisfies a strictly negative-imaginary property.

Our derivation is based on algebraic operations on linear fractional transformations of feedback interconnected systems. The first lemma in the main result states that a structured singular value condition of a modified input-output map needs to be fulfilled for the closed-loop to remain

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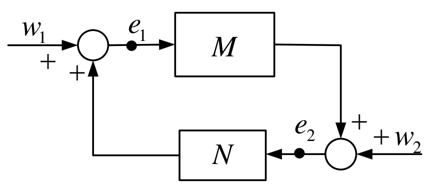

Fig. 1. Standard Feedback Interconnection

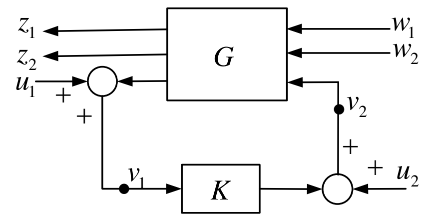

Fig. 2. LFT Interconnection

negative-imaginary when perturbations to the closed-loop are of a bounded-real nature. As a consequence of this result, important theorems and corollaries have been drawn to cast robust stability for a mixed bounded-real and strictly negative-imaginary uncertainty problem and robust performance problem in the presence of strictly negative-imaginary uncertainty. Due to space limitation, an illustrated example to demonstrate the usefulness of the proposed results and detailed proofs are omitted and will publish elsewhere. The rest of paper is organized as follows: Section II contains some mathematical preliminary work which is useful to streamline the main results of the paper. The main results are presented in Section III. Section IV concludes the paper.

Notation

$\mathcal{R}$ and $\mathcal{RH}_\infty$ denote the set of all real-rational and proper real-rational stable transfer function matrices, respectively. Let $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers, respectively. The superscript $(\cdot)^{n\times m}$ denotes a matrix with m columns and n rows. Let $\mathbb{C}_+$ and $\bar{\mathbb{C}}_+$ be, respectively, the open and closed right-half complex planes. Let A^*, $\bar{\lambda}(A)$ and $\bar{\sigma}(A)$ be the complex conjugate, the largest eigenvalue and singular value of matrix A. Furthermore, $\|P\|_\infty$ denote the $\mathcal{H}_\infty$-norm of P and $\mu_{\mathbf{\Delta}}(M)$ denote the structured singular value of M with respect to a given set $\mathbf{\Delta}$ [16]. Let $F_\ell(\cdot,\cdot)$ and $F_u(\cdot,\cdot)$ denote, respectively, the lower and upper Linear Fractional Transformation (LFT) and $(\cdot\star\cdot)$ denotes the Redheffer Star-Product of two transfer function matrices [16]. Let $[M,N]$ denote the positive feedback interconnection as depicted in Fig. 1 of transfer function matrices $M(s)$ and $N(s)$, and $[M,N]$ is said to be internally stable if the transfer function matrix from $\begin{pmatrix} w_1^T & w_2^T \end{pmatrix}^T$ to $\begin{pmatrix} e_1^T & e_2^T \end{pmatrix}^T$ exists and belongs to $\mathcal{RH}_\infty$. Also, let $\langle G,K\rangle$ be the LFT feedback interconnection shown in Fig. 2 of transfer function matrices $G(s)$ and $K(s)$, and correspondingly, let $T(G,K)$ be the transfer function matrix from $\begin{pmatrix} \omega_1^T & \omega_2^T & \begin{bmatrix} u_1^T & u_2^T \end{bmatrix} \end{pmatrix}^T$ to $\begin{pmatrix} z_1^T & z_2^T & \begin{bmatrix} v_1^T & v_2^T \end{bmatrix} \end{pmatrix}^T$. We say $\langle G,K\rangle$ is internally stable when $T(G,K)\in\mathcal{RH}_\infty$.

II. Mathematical Preliminaries

In this section, some background material is presented which is required to establish the main results of this paper. First, two sets of negative-imaginary systems are defined as follows:

Definition 1: [1] Let the set of negative-imaginary transfer function matrices be defined as

$$\mathcal{I} := \{X\in\mathcal{RH}_\infty^{n\times n}: \quad j[X(j\omega)-X(j\omega)^*]\geq 0\ \forall\omega\in(0,\infty)\}. \quad (1)$$

and the set of strictly negative-imaginary transfer function matrices be defined as

$$\mathcal{I}_s := \{X\in\mathcal{RH}_\infty^{n\times n}: \quad j[X(j\omega)-X(j\omega)^*]> 0\ \forall\omega\in(0,\infty)\}\subset\mathcal{I}. \quad (2)$$

It is easy to see that $X\in\mathcal{I}$ implies that $s[X(s)-X(\infty)]$ is positive-real [1].

The following theorem establishes the internal stability for a positive feedback interconnection of a negative-imaginary system and a strictly negative-imaginary system. It is as follows:

Theorem 1: [1] Given $M(s)\in\mathcal{I}$ and $N(s)\in\mathcal{I}_s$, and suppose $M(\infty)N(\infty)=0$ and $N(\infty)\geq 0$. Then, the positive-feedback interconnection of these two systems as shown in Fig. 1 is internally stable, if and only if,

$$\bar{\lambda}(M(0)N(0))<1. \quad (3)$$

Now, we present some technical lemmas which will streamline the proofs in the subsequent sections. The following lemma gives an equivalent condition for stability of a transfer function matrix that has a blocking zero at $s=0$.

Lemma 1: Given $X(s)\in\mathcal{R}$ satisfying $X(\infty)=0$. Then $Y(s)=sX(s)\in\mathcal{RH}_\infty$ and $Y(0)=0$, if and only if, $X(s)\in\mathcal{RH}_\infty$.

Proof: This is trivial by noting that $Y(s)$ has a blocking zero at $s=0$, hence $X(s)=\frac{1}{s}Y(s)$ has the same poles as $Y(s)$. ■

The following lemma gives an equivalent simpler condition for the input-output stability of a particular Redheffer Star-Product. This will be used in Section III to establish the stability equivalence result of two different star-product interconnections.

Lemma 2: Given $T=\left(\begin{array}{cc|c} T_{11} & T_{12} & T_{13}\\ \hline T_{21} & T_{22} & T_{23}\\ T_{31} & T_{32} & T_{33}\end{array}\right)\in\mathcal{RH}_\infty$. Then,

$$\left(\left[\begin{array}{cc|cc} 0 & 0 & I & 0\\ \hline 0 & I & 0 & -\sqrt{2}I\\ I & 0 & 0 & 0\\ 0 & \sqrt{2}I & 0 & -I\end{array}\right]\star T\right)\in\mathcal{RH}_\infty$$

$$\iff \quad (I+T_{22})^{-1}\in\mathcal{RH}_\infty. \quad (4)$$

Proof: This equivalence can directly be seen by expanding the Redheffer Star-Product of $\left[\begin{array}{cc:cc} 0 & 0 & I & 0 \\ 0 & I & 0 & -\sqrt{2}I \\ \hdashline I & 0 & 0 & 0 \\ 0 & \sqrt{2}I & 0 & -I \end{array}\right] \star T$ (see for example Section 10.4 of [16]). ■

For compactness of notation, an uncertainty set and two sets of block structures are introduced as follows:

Definition 2: Let the stable strictly bounded-real uncertainty set be defined as:

$$\mathcal{B}^{\circ}\mathbf{\Delta} = \{\Delta \in \mathcal{RH}_{\infty} : \|\Delta\|_{\infty} < 1\}, \tag{5}$$

let a complex block-structure $\mathbf{\Delta_{TOT}}$ and a real block-structure $\mathbf{\Delta_{REAL}}$ be defined respectively as:

$$\mathbf{\Delta_{TOT}} = \left\{ \begin{bmatrix} \bar{\Delta}_1 & 0 \\ 0 & \bar{\Delta}_2 \end{bmatrix} : \ \bar{\Delta}_1 \in \mathbb{C}^{q\times p},\ \bar{\Delta}_2 \in \mathbb{C}^{m\times m} \right\},$$

$$\mathbf{\Delta_{REAL}} = \left\{ \begin{bmatrix} \bar{\Delta}_1 & 0 \\ 0 & \bar{\Delta}_2 \end{bmatrix} : \ \bar{\Delta}_1 \in \mathbb{R}^{q\times p},\ \bar{\Delta}_2 \in \mathbb{R}^{m\times m} \right\}.$$

The following lemma is a simple restatement of the Main-Loop Theorem in μ analysis. It is given here for ease of reference in subsequent proofs.

Lemma 3: (Main-Loop Theorem) [16] Let $M \in \mathcal{RH}_{\infty}^{(p+m)\times(q+m)}$ be partitioned as $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$. Then, the followings are equivalent:

$$\sup_{\omega\in\mathbb{R}} \mu_{\mathbf{\Delta_{TOT}}}(M(j\omega)) \leq 1$$

$$\Longleftrightarrow \begin{cases} \|M_{11}\|_{\infty} \leq 1 \quad \text{and} \\ \sup\limits_{\Delta_1\in\mathcal{B}^{\circ}\mathbf{\Delta}} \|F_u(M,\Delta_1)\|_{\infty} \leq 1 \end{cases}$$

Proof: This is precisely the Main-Loop Theorem with $\sup\limits_{\omega\in\mathbb{R}}$ included and specialised to a two full block in the statement (see Theorem 11.7 of [16]). ■

The following lemma gives a necessary and sufficient condition for robust stability of a perturbed system against a two full-block structured uncertainty. This lemma will be used in subsequent section to derive the robust performance analysis results for systems with strictly negative-imaginary uncertainty by equivalently formulating the robust stability analysis results for systems with a mixed perturbations of bounded-real and strictly negative-imaginary uncertainties.

Lemma 4: Given $N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \in \mathcal{RH}_{\infty}^{(p+m)\times(q+m)}$ and $\Delta_2 \in \mathcal{RH}_{\infty}^{m\times m}$. Then,

$[\Delta_2, F_u(N,0)]$ is internally stable and $\|F_\ell(N,\Delta_2)\|_{\infty} \leq 1$,

if and only if

$$\left[\begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{pmatrix}, N\right] \quad \text{is internally stable } \forall \Delta_1 \in \mathcal{B}^{\circ}\mathbf{\Delta}.$$

III. Main results

In this section, the robust performance analysis problem for uncertain negative-imaginary systems is equivalently cast into a specific μ analysis framework. For a given controller that internally stabilizes an LFT closed-loop system

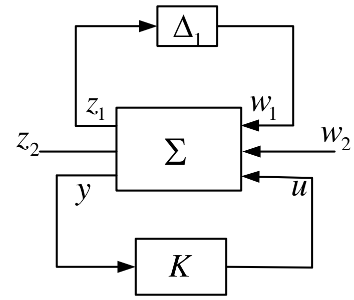

Fig. 3. LFT Interconnection

in the presence of strictly negative-imaginary uncertainty, the achieved performance can be quantified by solving the proposed analytical problem. Hence, the following lemma gives an equivalent condition for the LFT interconnection shown in Fig. 3 to be negative-imaginary, from the signal vector w_2 to the output signal vector z_2, when other two loops are closed with a given controller K and with a fictional strictly bounded-real uncertainty Δ_1.

Lemma 5: Given a controller $K \in \mathcal{R}^{r\times l}$ and a generalized plant

$$\Sigma = \left[\begin{array}{c|cc:c} A & B_1 & B_2 & B_3 \\ \hline C_1 & D_{11} & D_{12} & D_{13} \\ C_2 & D_{21} & D_{22} & D_{23} \\ \hdashline C_3 & D_{31} & D_{32} & D_{33} \end{array}\right] \tag{6}$$

with $A \in \mathbb{R}^{n\times n}$ and $\det(A) \neq 0$, $D_{11} \in \mathbb{R}^{p\times q}$, $D_{22} = 0 \in \mathbb{R}^{m\times m}$, $D_{33} = 0 \in \mathbb{R}^{l\times r}$, $D_{21} = 0$, $D_{23} = 0$, (A, B_3) stabilizable and (C_3, A) detectable. Then, $\langle\Sigma, K\rangle$ is internally stable, $F_\ell(F_u(\Sigma,\Delta_1),K) \in \mathcal{I}$ for all $\Delta_1 \in \mathcal{B}^{\circ}\mathbf{\Delta}$ and $\|F_\ell(F_\ell(\Sigma,K),-sI)\|_{\infty} \leq 1$, if and only if, $\langle G, K\rangle$ is internally stable, $\sup\limits_{\omega\in\mathbb{R}} \mu_{\mathbf{\Delta_{TOT}}}[F_\ell(G,K)] \leq 1$ and $\det(I + F_u(F_\ell(G,K),\Delta_1)(j\omega)) \neq 0$ for all $\omega \in \mathbb{R}\cup\{\infty\}$ and $\Delta_1 \in \mathcal{B}^{\circ}\mathbf{\Delta}$, where G is given in (7) (at the bottom on the next page).

Proof: First, we proof the following sequence of equivalent formulations extended from that in [5]:

(a) $\langle\Sigma, K\rangle$ is internally stable, $F_\ell(F_u(\Sigma,\Delta_1),K) \in \mathcal{RH}_{\infty}$ for all $\Delta_1 \in \mathcal{B}^{\circ}\mathbf{\Delta}$, and $j\big[F_\ell(F_u(\Sigma,\Delta_1),K)(j\omega) - F_\ell(F_u(\Sigma,\Delta_1),K)(j\omega)^*\big] \geq 0$ for all $\omega \in (0,\infty)$ and $\Delta_1 \in \mathcal{B}^{\circ}\mathbf{\Delta}$.

(b) $\langle\hat{\Sigma}, K\rangle$ is internally stable, $F_\ell(F_u(\hat{\Sigma},\Delta_1),K) \in \mathcal{RH}_{\infty}$ for all $\Delta_1 \in \mathcal{B}^{\circ}\mathbf{\Delta}$, and $\big[F_\ell(F_u(\hat{\Sigma},\Delta_1),K)(j\omega) + F_\ell(F_u(\hat{\Sigma},\Delta_1),K)(j\omega)^*\big] \geq 0$ for all $\omega \in \mathbb{R}$ and $\Delta_1 \in \mathcal{B}^{\circ}\mathbf{\Delta}$, where

$$\hat{\Sigma} = \left[\begin{array}{c|cc:c} A & B_1 & B_2 & B_3 \\ \hline C_1 & D_{11} & D_{12} & D_{13} \\ C_2A & C_2B_1 & C_2B_2 & C_2B_3 \\ \hdashline C_3 & D_{31} & D_{32} & 0 \end{array}\right]. \tag{8}$$

[The equivalence between $F_\ell(F_u(\hat{\Sigma},\Delta_1),K) \in \mathcal{RH}_{\infty}$ and $F_\ell(F_u(\Sigma,\Delta_1),K) \in \mathcal{RH}_{\infty}$ can be proved via invoking Lemma 1. The proof for the rest part of the statement in (a) and (b) follows the lines of the step from (b) to (c) in the proof of Theorem 4 in [5].]

(c) $\langle G, K\rangle$ is internally stable, $\sup_{\Delta_1 \in \mathcal{B}^\circ \mathbf{\Delta}} \|F_u(F_\ell(G,K),\Delta_1)\|_\infty \leq 1$, and $\det(I + F_u(F_\ell(G,K),\Delta_1)(j\omega)) \neq 0$ for all $\omega \in \mathbb{R} \cup \{\infty\}$ and $\Delta_1 \in \mathcal{B}^\circ \mathbf{\Delta}$, where G is given in (7).

[First, note that

$$G = \left[\begin{array}{cc|cc} 0 & 0 & I & 0 \\ 0 & I & 0 & -\sqrt{2}I \\ \hline I & 0 & 0 & 0 \\ 0 & \sqrt{2}I & 0 & -I \end{array}\right] \star \hat{\Sigma},$$

it follows from properties of LFT that

$$T(G,K) = \left[\begin{array}{cc|cc} 0 & 0 & I & 0 \\ 0 & I & 0 & -\sqrt{2}I \\ \hline I & 0 & 0 & 0 \\ 0 & \sqrt{2}I & 0 & -I \end{array}\right] \star T(\Sigma,K), \text{ and}$$

$$T(\hat{\Sigma},K) = \left[\begin{array}{cc|cc} 0 & 0 & I & 0 \\ 0 & I & 0 & -\sqrt{2}I \\ \hline I & 0 & 0 & 0 \\ 0 & \sqrt{2}I & 0 & -I \end{array}\right] \star T(G,K).$$

The rest of the proof to show the equivalence from (b) to (c) is similar to the lines of the proof to show the equivalence from the step (c) to (d) given in the proof of Theorem 4 in [5] by appropriately invoking Lemma 2.]

Also, note that $F_\ell(F_\ell(\Sigma,K),-sI) = F_\ell(F_\ell(G,K),0)$, it is then followed by Lemma 3, the statement in (a) together with $\|F_\ell(F_\ell(\Sigma,K),-sI)\|_\infty \leq 1$, is equivalent to the statement in (c) with $\sup_{\Delta_1 \in \mathcal{B}^\circ \mathbf{\Delta}} \|F_u(F_\ell(G,K),\Delta_1)\|_\infty \leq 1$ replaced with $\sup_{\omega \in \mathbb{R}} \mu_{\mathbf{\Delta_{TOT}}}[F_\ell(G,K)] \leq 1$, which completes the proof. ■

Four remarks are appropriate to be given at this stage.

Remark 1: The assumption that $D_{33} = 0$ is made without loss of generality as if it were non-zero, it could always be loopshifted to the controller K. Also, $D_{22} = 0$ could easily be replaced by $D_{22} = D_{22}^* \neq 0$ with appropriate minor modifications in the lemma statement. Finally, $D_{21} = 0$ and $D_{23} = 0$ could be replaced by $D_{12} = 0$ and $D_{32} = 0$ as this would be a dual generalized plant.

Remark 2: The assumption $\det(A) \neq 0$ is imposed for mathematical convenience to proof the stability equivalence between $F_\ell(F_u(\Sigma,\Delta_1),K)$ and $F_\ell(F_u(\hat{\Sigma},\Delta_1),K)$. This assumption can be replaced by $\left\| F_\ell\left(\left[\begin{array}{c|cc} A & B_1 & B_3 \\ \hline C_1 & D_{11} & D_{13} \\ C_3 & D_{31} & D_{33} \end{array}\right], K\right)\right\|_\infty = \|F_\ell(F_\ell(\Sigma,K),0)\|_\infty \leq 1$, which can be interpreted as nominal performance and the lemma statement still holds. This latter assumption is used instead of $\det(A) \neq 0$ in the subsequent theorems and corollaries.

Remark 3: The zero D-term assumptions of the generalized plant in the suppositions of the lemma statement guarantee properness of $F_\ell(F_\ell(\Sigma,K),-sI)$. It is easy to see that the assumption $\|F_\ell(F_\ell(\Sigma,K),-sI)\|_\infty \leq 1$ can be interpreted as a nominal performance property of the transformed interconnection $\langle G, K\rangle$.

Remark 4: For all $\omega \in \mathbb{R}$ such that $\mu_{\mathbf{\Delta_{TOT}}}[F_\ell(G,K)] < 1$, the condition $\det(I + F_u(F_\ell(G,K),\Delta_1)(j\omega)) \neq 0$ is automatically fulfilled for $\Delta_1 \in \mathcal{B}^\circ \mathbf{\Delta}$. Consequently, this determinant condition needs to be checked only at the frequencies where $\mu_{\mathbf{\Delta_{TOT}}}[F_\ell(G,K)] = 1$.

The following lemma gives an equivalent μ condition to estimate the least upper bound of the upper LFT of a constant real matrix with a contractive real matrix. This lemma will be used in subsequent corollaries to quantify the largest family of possible strictly negative-imaginary uncertainties for which the robust performance of the closed-loop system is guaranteed.

Lemma 6: Given $Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \in \mathbb{R}^{(p+m)\times(q+m)}$ such that $\bar{\sigma}(Q_{11}) \leq 1$. Suppose two real block sets $\mathbf{\Omega_1}$ and $\mathbf{\Omega_2}$ are defined as

$$\mathbf{\Omega_1} = \left\{\bar{\Delta}_1 \in \mathbb{R}^{q\times p} : \ \bar{\sigma}(\bar{\Delta}_1) < 1\right\}, \tag{9}$$

$$\mathbf{\Omega_2} = \left\{\bar{\Delta}_2 \in \mathbb{R}^{m\times m} : \ \bar{\sigma}(\bar{\Delta}_2) < 1\right\}. \tag{10}$$

Then,

$$\begin{aligned} &\inf\left\{\beta > 0 : \ \mu_{\mathbf{\Delta_{REAL}}}\left(\begin{pmatrix} I & 0 \\ 0 & \frac{1}{\beta}I \end{pmatrix} Q\right) \leq 1\right\} \\ &= \sup_{\bar{\Delta}_1 \in \mathbf{\Omega_1}} \bar{\sigma}(F_u(Q,\bar{\Delta}_1)). \end{aligned} \tag{11}$$

Proof: This is a readily consequence of applying Lemma 3 at zero frequency i.e to a real matrix with two real full-block structured uncertainty. ■

The least upper bound of $\bar{\sigma}(F_u(Q,\bar{\Delta}_1))$ $\forall \bar{\Delta}_1 \in \mathbb{R}^{q\times p}$ satisfying $\bar{\sigma}(\bar{\Delta}_1) < 1$ can be estimated via numerical methods. However, the computation complexity increases as the dimension of $\bar{\Delta}_1$, i.e., $q \times p$ increases. The above lemma gives a analytical method so that real μ can be computed as a reasonably tight upper bound using real structured singular value techniques [17] [18].

More meaningful engineering significant robust stability and robust performance results can be obtained as an immediate useful consequence of the above lemmas as follows.

Theorem 2: (Robust Stability) Given the suppositions of Lemma 5 except $\det(A) \neq 0$, $\gamma > 0$, and G is as given

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$$G = \left[\begin{array}{c|ccc} V^{-1}A & V^{-1}B_1 & \sqrt{2}B_2U^{-1} & V^{-1}B_3 \\ \hline C_1 - D_{12}U^{-1}C_2A & D_{11} - D_{12}U^{-1}C_2B_1 & \sqrt{2}D_{12}U^{-1} & D_{13} - D_{12}U^{-1}C_2B_3 \\ -\sqrt{2}U^{-1}C_2A & -\sqrt{2}U^{-1}C_2B_1 & (I - C_2B_2)U^{-1} & -\sqrt{2}U^{-1}C_2B_3 \\ C_3 - D_{32}U^{-1}C_2A & D_{31} - D_{32}U^{-1}C_2B_1 & \sqrt{2}D_{32}U^{-1} & -D_{32}U^{-1}C_2B_3 \end{array}\right],$$

$$U = I + C_2B_2 \ \text{ and } \ V = I + B_2C_2. \tag{7}$$

in (7) such that $\left\|F_\ell(\left[\begin{array}{c|cc} A & B_1 & B_3 \\ \hline C_1 & D_{11} & D_{13} \\ C_3 & D_{31} & D_{33}\end{array}\right], K)\right\|_\infty \leq 1$, $\langle G, K\rangle$ is internally stable, $\sup_{\omega\in\mathbb{R}} \mu_{\mathbf{\Delta_{TOT}}}[F_\ell(G,K)] \leq 1$ and $\det(I + F_u(F_\ell(G,K),\Delta_1)(j\omega)) \neq 0$ for all $\omega \in \mathbb{R}\cup\{\infty\}$ and $\Delta_1 \in \mathcal{B}^\circ\mathbf{\Delta}$. Then $\left[\begin{pmatrix}\Delta_1 & 0\\ 0 & \Delta_2\end{pmatrix}, F_\ell(\Sigma,K)\right]$ is internally stable for all $\Delta_1 \in \mathcal{B}^\circ\mathbf{\Delta}$ and $\Delta_2 \in \mathcal{I}_s$ satisfying $\Delta_2(\infty) \geq 0$ and $\bar\lambda(\Delta_2(0)) < \gamma (\leq \gamma)$, if and only if

$$\inf\left\{\beta > 0:\ \mu_{\mathbf{\Delta_{REAL}}}\left(\begin{pmatrix} I & 0\\ 0 & \frac{1}{\beta}I\end{pmatrix} F_\ell(\Sigma,K)(0)\right) \leq 1\right\} \leq \frac{1}{\gamma} (< \frac{1}{\gamma}). \tag{12}$$

Proof: ($\Longleftarrow$) Since $\langle G, K\rangle$ is internally stable, $\sup_{\omega\in\mathbb{R}} \mu_{\mathbf{\Delta_{TOT}}}[F_\ell(G,K)] \leq 1$ and $\det(I + F_u(F_\ell(G,K),\Delta_1)(j\omega)) \neq 0$ for all $\omega \in \mathbb{R}\cup\{\infty\}$ and $\Delta_1 \in \mathcal{B}^\circ\mathbf{\Delta}$, it follows from Lemma 5 and Remark 2 that $\langle\Sigma, K\rangle$ is internally stable and $F_u(F_\ell(\Sigma,K),\Delta_1) \in \mathcal{I}$ for all $\Delta_1 \in \mathcal{B}^\circ\mathbf{\Delta}$. Hence, we have $F_\ell(\Sigma,K) \in \mathcal{RH}_\infty$. Also note that $\begin{pmatrix}\Delta_1 & 0\\ 0 & \Delta_2\end{pmatrix} \in \mathcal{RH}_\infty$, thus, by Theorem 5.7 of [16], the closed-loop system $\left[\begin{pmatrix}\Delta_1 & 0\\ 0 & \Delta_2\end{pmatrix}, F_\ell(\Sigma,K)\right]$ is internally stable if and only if

$$\det\left[I - F_\ell(\Sigma,K)(s_0)\begin{pmatrix}\Delta_1(s_0) & 0\\ 0 & \Delta_2(s_0)\end{pmatrix}\right] \neq 0\ \forall s_0 \in \bar{\mathbb{C}}_+. \tag{13}$$

Let $F_\ell(\Sigma,K) = \begin{bmatrix}\bar N_{11} & \bar N_{12}\\ \bar N_{21} & \bar N_{22}\end{bmatrix}$, and note that $\|\bar N_{11}\|_\infty = \left\|F_\ell(\left[\begin{array}{c|cc} A & B_1 & B_3 \\ \hline C_1 & D_{11} & D_{13} \\ C_3 & D_{31} & D_{33}\end{array}\right], K)\right\|_\infty \leq 1$. Then from the small-gain theorem [16] we have $(I - \bar N_{11}\Delta_1)^{-1} \in \mathcal{RH}_\infty$ for all $\Delta_1 \in \mathcal{B}^\circ\mathbf{\Delta}$. Hence, $\det(I - \bar N_{11}(s_0)\Delta_1(s_0)) \neq 0$ for all $s_0 \in \bar{\mathbb{C}}_+$ and $\Delta_1 \in \mathcal{B}^\circ\mathbf{\Delta}$.

Note that $\bar N_{21}(\infty) = 0$ and $\bar N_{22}(\infty) = 0$, it follows that $F_u(F_\ell(\Sigma,K),\Delta_1)(\infty) = 0$ for all $\Delta_1 \in \mathcal{B}^\circ\mathbf{\Delta}$. Since $F_u(F_\ell(\Sigma,K),\Delta_1) \in \mathcal{I}$ for all $\Delta_1 \in \mathcal{B}^\circ\mathbf{\Delta}$, then $F_u(F_\ell(\Sigma,K)(0),\Delta_1(0)) \geq F_u(F_\ell(\Sigma,K)(\infty),\Delta_1(\infty)) = 0$ via Lemma 2 of [1]. Hence, $\forall \Delta_1(0) \in \mathbb{R}^{q\times p}$ satisfying $\bar\sigma(\Delta_1(0)) < 1$, $\bar\lambda(F_u(F_\ell(\Sigma,K)(0),\Delta_1(0)) = \bar\sigma(F_u(F_\ell(\Sigma,K)(0),\Delta_1(0)))$. Furthermore, since $\inf\left\{\beta > 0:\ \mu_{\mathbf{\Delta_{REAL}}}\left(\begin{pmatrix} I & 0\\ 0 & \frac{1}{\beta}I\end{pmatrix} F_\ell(\Sigma,K)(0)\right) \leq 1\right\} \leq \frac{1}{\gamma}(< \frac{1}{\gamma})$, it follows from Lemma 6 that
$\bar\lambda(F_u(F_\ell(\Sigma,K)(0),\Delta_1(0)) = \bar\sigma(F_u(F_\ell(\Sigma,K)(0),\Delta_1(0)))$
$\leq \frac{1}{\gamma}(< \frac{1}{\gamma})$ for all $\Delta_1(0) \in \mathbb{R}^{q\times p}$ satisfying $\bar\sigma(\Delta_1(0)) < 1$. Also, since $\Delta_2 \in \mathcal{I}_s$ satisfies $\Delta_2(\infty) \geq 0$ and $\bar\lambda(\Delta_2(0)) < \gamma(\leq\gamma)$, it follows that $\bar\lambda(F_u(F_\ell(\Sigma,K),\Delta_1)(0)\Delta_2(0)) < 1$. Consequently, it follows from Theorem 1 that $\det(I - F_u(F_\ell(\Sigma,K),\Delta_1)(s_0)\Delta_2(s_0)) \neq 0$ for all $s_0 \in \bar{\mathbb{C}}_+$, $\Delta_1 \in \mathcal{B}^\circ\mathbf{\Delta}$ and $\Delta_2 \in \mathcal{I}_s$ satisfying $\Delta_2(\infty) \geq 0$ and $\bar\lambda(\Delta_2(0)) < \gamma(\leq\gamma)$. Hence,

$$\begin{aligned}&\det\left[I - F_\ell(\Sigma,K)(s_0)\begin{pmatrix}\Delta_1(s_0) & 0\\ 0 & \Delta_2(s_0)\end{pmatrix}\right]\\ &= \det(I - \bar N_{11}(s_0)\Delta_1(s_0))\\ &\quad\times\det(I - F_u(F_\ell(\Sigma,K),\Delta_1)(s_0)\Delta_2(s_0))\\ &\neq 0\ \forall s_0 \in \bar{\mathbb{C}}_+\ \forall\Delta_1 \in \mathcal{B}^\circ\mathbf{\Delta}\ \forall\Delta_2 \in \mathcal{I}_s\\ &\quad\text{satisfying } \Delta_2(\infty) \geq 0 \text{ and } \bar\lambda(\Delta_2(0)) < \gamma(\leq\gamma).\end{aligned}$$

($\Longrightarrow$) This can be proved via a contra-positive argument on choosing $\Delta_1 = 0$ and $\Delta_2 = \frac{1/\bar\lambda(\bar N_{22}(0))}{s+1} I$ as the destabilizing $\begin{pmatrix}\Delta_1 & 0\\ 0 & \Delta_2\end{pmatrix}$. ■

The following corollary is an immediate consequence of Theorem 2. It is not a restatement of Theorem 2, but a different version of the robust stability result, where the real μ condition is used to quantify the largest family of perturbations that are a mixture of bounded-real and strictly negative-imaginary uncertainties for which robust stability of the perturbed closed-loop system is guaranteed.

Corollary 1: (Robust Stability) Given the suppositions of Theorem 2 except $\gamma > 0$. Then $\left[\begin{pmatrix}\Delta_1 & 0\\ 0 & \Delta_2\end{pmatrix}, F_\ell(\Sigma,K)\right]$ is internally stable for all $\Delta_1 \in \mathcal{B}^\circ\mathbf{\Delta}$ and $\Delta_2 \in \mathcal{I}_s$ satisfying $\Delta_2(\infty) \geq 0$ if and only if

$$\frac{1}{\inf\left\{\beta > 0:\ \mu_{\mathbf{\Delta_{REAL}}}\left(\begin{pmatrix} I & 0\\ 0 & \frac{1}{\beta}I\end{pmatrix} F_\ell(\Sigma,K)(0)\right) \leq 1\right\}} > \bar\lambda(\Delta_2(0)). \tag{14}$$

Proof: This result is a straightforward consequence of Theorem 2 obtained by setting

$$\frac{1}{\gamma} = \inf\left\{\beta > 0:\ \mu_{\mathbf{\Delta_{REAL}}}\left(\begin{pmatrix} I & 0\\ 0 & \frac{1}{\beta}I\end{pmatrix} F_\ell(\Sigma,K)(0)\right) \leq 1\right\}$$

for sufficiency and $\gamma = \bar\lambda(\Delta_2(0))$ for necessity. ■

Theorem 3: (Robust Performance) Given the suppositions of Lemma 5 except $\det(A) \neq 0$, $\gamma > 0$, and G is as given in (7) such that $\left\|F_\ell(\left[\begin{array}{c|cc} A & B_1 & B_3 \\ \hline C_1 & D_{11} & D_{13} \\ C_3 & D_{31} & D_{33}\end{array}\right], K)\right\|_\infty \leq 1$, $\langle G, K\rangle$ is internally stable, $\sup_{\omega\in\mathbb{R}} \mu_{\mathbf{\Delta_{TOT}}}[F_\ell(G,K)] \leq 1$ and $\det(I + F_u(F_\ell(G,K),\Delta_1)(j\omega)) \neq 0$ for all $\omega \in \mathbb{R}\cup\{\infty\}$ and $\Delta_1 \in \mathcal{B}^\circ\mathbf{\Delta}$. Then $[\Delta_2, F_u(F_\ell(\Sigma,K),0)]$ is internally stable and $\|F_\ell(F_\ell(\Sigma,K),\Delta_2)\|_\infty \leq 1$ for all $\Delta_2 \in \mathcal{I}_s$ satisfying $\Delta_2(\infty) \geq 0$ and $\bar\lambda(\Delta_2(0)) < \gamma(\leq\gamma)$, if and only if the condition in (11) is satisfied.

Proof: This result is straightforward to obtain by combing Theorem 2 and Lemma 4. ■

This theorem broadly states that the internal stability of a transformed feedback interconnection ($\langle G, K\rangle$) and a structure singular value condition of the transformed input/output map ($F_\ell(G,K)$) guarantees that the worst-case performance from w_1 to z_1 in Fig. 4 remains smaller than unity (with the pre-specified weighting functions absorbed into Σ) for all possible strictly negative-imaginary uncertainties Δ_2 satisfying an extra condition at zero frequency and infinity. For

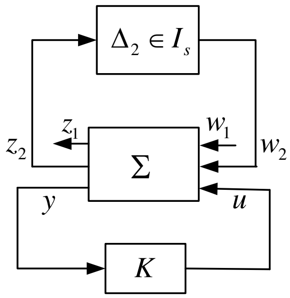

Fig. 4. LFT Interconnection for performance analyis

instance, w_1 represents the exogenous signals such as commands, disturbances, etc. whereas z_1 represents performance signals such as the error signals, control inputs, etc. in the feedback interconnection.

The following corollary is an immediate consequence of Theorem 3. It can be used to quantify the largest family of strictly negative-imaginary perturbations in terms of a DC loop gain condition for which robust performance of the perturbed closed-loop system is guaranteed.

Corollary 2: (Robust Performance) Given the suppositions of Theorem 3 except $\gamma > 0$. Then, $[\Delta_2, F_u(F_\ell(\Sigma, K), 0)]$ is internally stable and $\|F_\ell(F_\ell(\Sigma, K), \Delta_2)\|_\infty \leq 1$ for all $\Delta_2 \in \mathcal{I}_s$ satisfying $\Delta_2(\infty) \geq 0$, if and only if the condition in (14) is satisfied.

Proof: Similar to the proof of Corollary 1. ■

Remark 5: Note that, $\left\| F_\ell\left(\left[\begin{array}{c|cc} A & B_1 & B_3 \\ \hline C_1 & D_{11} & D_{13} \\ C_3 & D_{31} & D_{33} \end{array} \right], K\right) \right\|_\infty \leq 1$ implies the nominal performance of the system as structured in Fig. 4 is satisfied, i.e., the infinity norm of the transfer function matrix from w_1 to z_1 is less than one when the physical negative imaginary uncertainty $\Delta_2 = 0$.

IV. CONCLUSIONS

This paper considerably extends the robust stability analysis reformulation technique of [5] to a generalised framework to analyse the robust performance problems for uncertain negative-imaginary systems. To characterise the robust performance, conditions are derived in the μ framework. This work will underpin future developments for controller synthesis to achieve a robust performance level for uncertain negative-imaginary systems.

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