A Robust Kalman Conjecture
For First-Order Plants [★]

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Abstract: A robust Kalman conjecture is defined for the robust Lur'e problem. Specifically, it is conjectured that the nonlinearity's slope interval for which robust absolute stability is guaranteed corresponds to the robust interval of the uncertain plant. We verify this robust Kalman conjecture for first-order plants perturbed by various norm-bounded unstructured uncertainties. The analysis classifies the appropriate stability multipliers required for verification in these cases. Robust control of Lur'e-type nonlinear systems satisfying this novel conjecture can therefore be designed using linear robust control methods.

1. INTRODUCTION

The feedback interconnection of a plant $G(s)$ and a static nonlinearity φ (see Fig. 1) is known as the Lur'e structure.

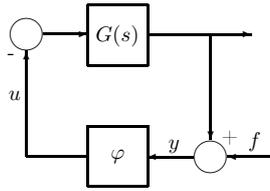Fig. 1. The Lur'e Structure.

Absolute stability theory studies the stability of the well-posed Lur'e structure for all nonlinearities φ from a given class of nonlinearities Φ. Determining the conditions for which the Lur'e structure loses its absolute stability has long attracted the interests of researchers since it was posed by Lur'e and Postnikov in 1944. The Lur'e problem represents a particular case of general nonlinear systems wherein the nonlinearity is separable and includes feedback linear systems with saturation constraints.

An operator $\varphi : \mathcal{L}_e \mapsto \mathcal{L}_e$ is static if $\exists \, \mathrm{N} : \mathbb{R} \to \mathbb{R}$ such that $(\varphi y)(t) = \mathrm{N}(y(t))$ and it is monotone non-decreasing if

$$[\mathrm{N}(y_2) - \mathrm{N}(y_1)][y_2 - y_1] \geq 0 \quad \forall \, y_1, \, y_2 \in \mathbb{R}.$$

Typically, conic conditions are used to further describe the families of nonlinearities. Given $\alpha, \, \beta \in \mathbb{R}$, a nonlinear operator is classified as follows:

- Sector-Bounded SB $[\alpha, \beta]$:

$$\alpha y_1^2 \leq y_1 \, \mathrm{N}(y_1) \leq \beta y_1^2.$$

- Slope-Restricted SR $[\alpha, \beta]$:

$$\alpha(y_2 - y_1)^2 \leq [\mathrm{N}(y_2) - \mathrm{N}(y_1)][y_2 - y_1] \leq \beta(y_2 - y_1)^2.$$

Without loss of generality any SB $[\alpha, \beta]$ or SR $[\alpha, \beta]$ can be mapped to SB $[0, \beta]$ or SR $[0, \beta]$ respectively by loop

transformations [1]. Two absolute stability conjectures have been proposed to answer the Lur'e problem. We can state these two conjectures as follows:

(1) **Aizerman Conjecture [2]:** The feedback interconnection between a linear plant $G(s)$ with any sector-bounded SB $[\alpha, \beta]$ nonlinearity φ_k is stable if the feedback interconnection between $G(s)$ and any constant gain $K \in [\alpha, \beta]$ is stable.

(2) **Kalman Conjecture [2]:** The feedback interconnection between a linear plant $G(s)$ with slope-restricted SR $[\alpha, \beta]$ nonlinearity φ_k is stable if the feedback interconnection between $G(s)$ and any constant gain $K \in [\alpha, \beta]$ is stable.
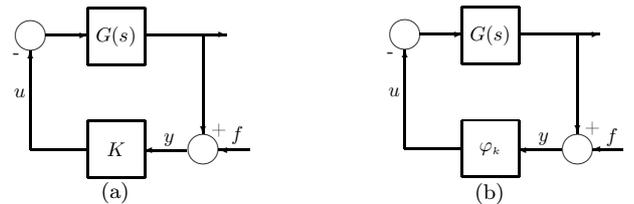
If the system in Fig. 2(a) is stable, then system in Fig. 2(b) is stable

Fig. 2. The Absolute Stability Conjecture.

Even though these conjectures have been shown to be false in general, they have played a vital role in rigorous development of modern absolute stability theory.

For first and second-order plants, the Aizerman conjecture has been shown to be true using the circle criterion and Popov criterion respectively [2](see the references therein). The Aizerman conjecture has been refuted for the generality of third-order plants by Pliss [2]. Thus, if stability is established via multipliers, the constant multipliers and popov multipliers provide the needed stability multipliers for first-order and second-order strictly-proper stable plants respectively. The Kalman conjecture has been proved to be valid for third-order plants [3], [4] by constructing an allowed multiplier that can be interpreted

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as a first-order Zames-Falb multiplier [5]. It also has been shown in [3] that for n^{th}-order plants with $n \geq 4$, there exists a Lur'e system with a nontrivial periodic solution and therefore not satisfying the Kalman conjecture.

Table 1. Conjectures

Plant Order	Aizerman	Kalman
1st	True	True
2nd	True	True
3rd	False	True
$\geq 4^{th}$	False	False

The aim of this work is to study the Lur'e problem (see Fig. 1) when the stable linear plant is considered with uncertainty (see Fig. 3). We focus our attention on the family of *scalar* static (possibly nonlinear) nonlinearities slope-restricted in the interval $[0, k]$, henceforward, φ_k.
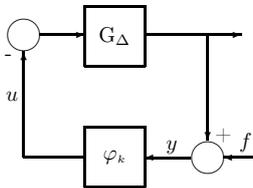

Fig. 3. The Robust Lur'e problem.

Most of the literature on this problem has been devoted to parametric uncertainty [6] using Kharitonov's Theorem [7] or equivalent results. For norm-bounded linear time-invariant (LTI) unstructured uncertainties, a robust circle criterion and robust Popov criterion for additive and multiplicative LTI uncertainties were presented in [8].

The integral quadratic constraint (IQC) theory [9] provides a unified framework where the norm-bounded general uncertainties can easily be embedded with the nonlinear block. With the IQC framework, the results in [8] can be extended to other robust stability criteria for different (possibly nonlinear) uncertainty descriptions.

We propose a conjecture for the robust Lur'e problem that may be true for low-order stable plants but will be false in general just as with the Aizerman and Kalman conjectures. In particular, this analysis investigates the appropriate stability multiplier needed to verify this conjecture for first-order stable plants with unstructured additive, input multiplicative, output multiplicative and feedback uncertainties.

2. NOTATION

Let $\mathbb{R}_+$ be the set of all non-negative real numbers and $\mathbf{RH}_\infty$ denote real rational stable transfer functions. RKC means robust Kalman conjecture. LFT denotes Linear Fractional Transformation while $\mathcal{L}_p^n[0, \infty)$ denotes the space of p-integrable functions $f : [0, \infty) \to \mathbb{R}^n(-\infty, \infty)$. The $\mathcal{L}_p$ norm is thus defined by $\|f\|_p^p = \int_0^\infty |f|^p \, \mathrm{d}t$.

A truncation of the function f at T is given by $f_T(t) = f(t)$, $\forall t \leq T$ and $f_T(t) = 0$, $\forall t > T$. $\mathcal{L}_{2e}^n[0, \infty)$ represents a space of extended functions whose truncations at any finite time are square integrable.

Let the operator S be a map from $\mathcal{L}_{2e}^n[0, \infty)$ to $\mathcal{L}_{2e}^n[0, \infty)$, with input u and output Su. This operator S is causal if $\mathrm{S}u(t) = \mathrm{S}(u_T)(t)$ for all $t < T$. Moreover, the operator S is $\mathcal{L}_2$-stable if for all $u \in \mathcal{L}_2^n[0, \infty)$, then S$u \in \mathcal{L}_2^n[0, \infty)$. Furthermore, the operator S is bounded and finite-gain $\mathcal{L}_2$-stable if there exists a constant γ such that $\|\mathrm{S}u\|_2 \leq \gamma \|u\|_2$. The supremum of such constants defines $\|\mathrm{S}\|_{\mathcal{L}_2}$.

Consider the feedback interconnection of a stable LTI SISO plant G and a bounded operator φ, shown in Fig. 1:

$$\begin{cases} y = f - Gu, \\ u = \varphi y. \end{cases} \tag{1}$$

Since G is a stable LTI SISO plant, any exogenous input in this part of the loop can be taken as the zero signal without loss of generality. It is well posed if the map $(y, u) \mapsto (0, f)$ has a causal inverse on $\mathcal{L}_{2e}^{2n}[0, \infty)$. Furthermore, this interconnection is $\mathcal{L}_2$-stable if for any $f \in \mathcal{L}_2^n[0, \infty)$, then $(Gu, \varphi y) \in \mathcal{L}_2^n[0, \infty)$. The transfer function of the linear plant G is denoted by $G(s)$. Subsequently, we assume that the interconnected system in Fig. 3 is well-posed by requiring that $G(s)$ be strictly-proper.

3. ROBUST INTERVAL

With the nonlinearity φ_k in Fig. 3 replaced by a static gain K, we then have an uncertain system as shown in Fig. 4(a). G_Δ is an uncertain stable plant that belongs to a family of stable plants $\mathcal{G}_\Delta$ defined in terms of norm-bound (possibly nonlinear) uncertainties Δ, where $\|\Delta\|_{\mathcal{L}_2} \leq 1$.
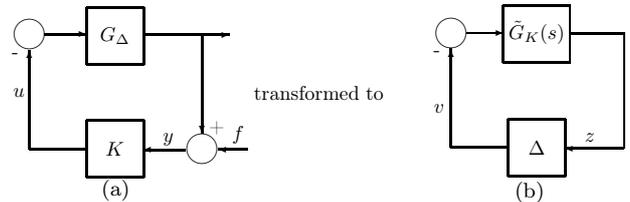
Using LFT , the system in Fig. 4(a) is transformed into Fig. 4(b)

Fig. 4. The Uncertain System.

Now, we state a well-known theorem for which sufficiency and necessity follows from small-gain argument [10] and contradiction argument respetively.

Theorem 1. *Assume the feedback system in Fig. 4(b) is well-posed. Suppose that both $\tilde{G}_K(s) \in \mathbf{RH}_\infty$ and Δ are causal and finite-gain $\mathcal{L}_2$-stable. Under these conditions, the feedback system in Fig. 4(b) is $\mathcal{L}_2$-stable for all Δ with $\|\Delta\|_{\mathcal{L}_2} \leq 1$ if and only if $\|\tilde{G}_K\|_\infty < 1$.*

The robust interval $\mathcal{I}_r = [0, K_r)$ is the largest interval such that the feedback interconnection of any plant $G_\Delta \in \mathcal{G}_\Delta$ with a constant gain $K \in \mathcal{I}_r$ is stable. Thus, K_r is the supremum of $\mathcal{I}_r$ for which the feedback system shown in Fig. 4(a) is stable.

3.1 Graphical Interpretation

From the proof of theorem 1, we find that there exists an LTI uncertainty Δ_{LTI} that renders the feedback system shown in Fig. 4(a) unstable. Thus in this section we develop a graphical interpretation of the robust interval

for such LTI uncertainties Δ_{LTI}.

When there is no uncertainty (i.e. $\Delta = 0$, $G_\Delta = G$) in Fig. 4(a), then Theorem 2 holds:

Theorem 2 (Nyquist Criterion). *The feedback intercon-nection of a rational stable transfer function $G(s)$, i.e. $G(s) \in \mathbf{RH}_\infty$, and a constant gain K is stable if and only if*

- $\inf_{\omega \in \mathbb{R}} |1 + KG(j\omega)| \neq 0$,
- *$KG(j\omega)$ does not encircle the $-1 + 0j$ point.*

Now consider the case for which the uncertainty in Fig. 4(a) is linear (i.e. Δ_{LTI}). The uncertain plant $G_{\Delta_{LTI}}$ now belongs to a family of stable LTI plants $\mathcal{G}_{\Delta_{LTI}}$. Thereom 2 can then be extended to the (LTI) robust case as follows:

Corollary 1. *Let $\mathcal{G}_{\Delta_{LTI}}$ be a family of stable ratio-nal transfer functions. The feedback interconnection of a $G_{\Delta_{LTI}}(s) \in \mathcal{G}_{\Delta_{LTI}}$ and a constant gain K is stable if and only if*

$$\inf_{\omega \in \mathbb{R}} |1 + KG_{\Delta_{LTI}}(j\omega)| > 0 \qquad (2)$$

Graphically, $\mathcal{G}_{\Delta_{LTI}}$ is represented by a "loose" region about the Nyquist plot of the nominal plant model (see Fig. 5).
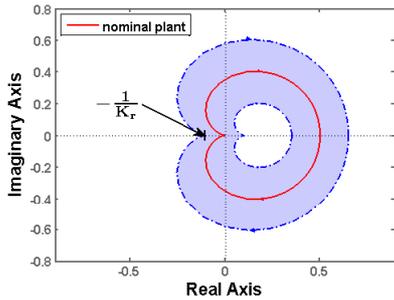

Fig. 5. Robust Nyquist Plot

Corollary 1 provides the interval of gains for which elements of $\mathcal{G}_{\Delta_{LTI}}$ in feedback interconnection are stable. The supremum of such gains is denoted by bound K_r and parameterizes the robust interval. Thus the "loose" region cannot include the real interval $(-\infty, -1/K_r]$.

4. ROBUST ABSOLUTE STABILITY

In this section, the robust absolute stability of the Lur'e problem (Fig. 3) is analysed. The uncertainty Δ is com-bined with nonlinearity φ_k to form $\tilde{\Delta}$ defined in (5) and an augmented Lur'e problem is obtained, see Fig. 6.

(a) Robust Lur'e problem   (b) Augmented Lur'e problem
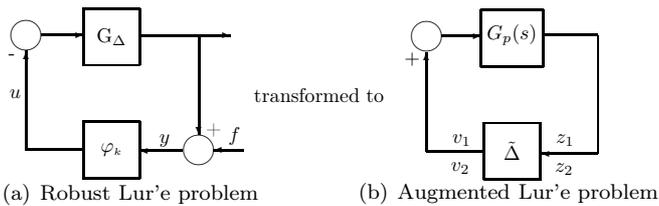
Using LFT , the system in Fig. 6(a) is transformed into Fig. 6(b)

Fig. 6. Augmented Lur'e problem

The unifying framework of integral quadratic constraints (IQC) gives useful input-output characterizations of the structure of an operator on a Hilbert space. IQCs are defined by quadratic forms which are in turn defined in terms of bounded self-adjoint operators. With the IQC framework, the norm-bounded general uncertainties can easily be embedded with the nonlinear block.

Definition 1 ([9]). *A bounded operator $\tilde{\Delta} : \mathcal{L}_{2e}^n \mapsto \mathcal{L}_{2e}^n$, with input z and output v, is said to satisfy the IQC defined by a measurable bounded Hermitian-valued $\Pi : j\mathbb{R} \to \mathbb{C}^{(2n) \times (2n)}$, if for all $z \in \mathcal{L}_2^n$,*

$$\int_{-\infty}^{\infty} \begin{bmatrix} \widehat{z}(j\omega) \\ \widehat{v}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \widehat{z}(j\omega) \\ \widehat{v}(j\omega) \end{bmatrix} d\omega \geq 0, \qquad (3)$$

where $\widehat{z}$ and $\widehat{v}$ are the Fourier transform of the signals z and w respectively.

Theorem 3 (IQC Theorem [9]). *Consider the feedback interconnection in Fig. 6(b). Let $G_p(s) \in \mathbf{RH}_\infty$ and let $\tilde{\Delta}$ be a bounded causal operator. Assume that:*

(i) The feedback interconnection between $G_p(s)$ and $\tau\tilde{\Delta}$ is well posed for all $\tau \in [0, 1]$.

(ii) There exists a measurable Hermitian–valued Π such that the operator $\tau\tilde{\Delta}$ satisfies the IQC defined by Π for all $\tau \in [0, 1]$.

(iii) There exists $\epsilon > 0$ such that

$$\begin{bmatrix} G_p(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G_p(j\omega) \\ I \end{bmatrix} \leq -\epsilon I \qquad \forall \omega \in \mathbb{R}. \quad (4)$$

Then, the feedback system in Fig. 6(b) is $\mathcal{L}_2$-stable.

For different uncertainty descriptions, the generalized plant $G_p(s)$ will have different forms whereas the aug-mented nonlinearity will preserve the diagonal structure

$$\tilde{\Delta} = \begin{bmatrix} \varphi_k & 0 \\ 0 & \Delta \end{bmatrix}, \qquad (5)$$

where φ_k is a SR $[0, k]$ nonlinearity. Δ is a causal and boun-ded operator. Thus, $\Pi(j\omega)$ will have the same structure

$$\Pi(j\omega) = \begin{bmatrix} 0 & 0 & kM(j\omega)^* & 0 \\ 0 & 1 & 0 & 0 \\ kM(j\omega) & 0 & -M(j\omega) - M(j\omega)^* & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad (6)$$

throughout this work, where the positive operator $M(j\omega)$ is a stability multiplier for φ_k [5]. Depending on the stability criteria, $M(j\omega)$ has different parametrization:

- Constant Multipliers: $M(j\omega) = \eta : \eta > 0$.
- Popov Multipliers: $M(j\omega) = \eta + j\omega\lambda : \eta > 0, \ \lambda \in \mathbb{R}$.
- Zames-Falb Multipliers: $M(j\omega) = \eta + H(j\omega) : \eta > 0$ and $\|h\|_1 < \eta$ where h is the impulse response of $H(j\omega)$.

Remark 1. *Due to the structure of $\tilde{\Delta}$, assumption (i) is guaranteed if it is well-posed for $\tau = 1$.*

5. ROBUST KALMAN CONJECTURE

Conjecture 1 (Robust Kalman Conjecture).

Suppose G_Δ is an uncertain stable plant that belongs to a family of stable plants $\mathcal{G}_\Delta$ defined in terms of norm-bound (possibly nonlinear) uncertainties Δ, where $\|\Delta\|_{\mathcal{L}_2} \leq 1$. If the feedback interconnection between G_Δ and any constant gain $K \in [0, K_r)$ is stable, then the

feedback interconnection of G_Δ and any slope-restricted nonlinearity φ_k : SR $[0, K_r)$ is stable.
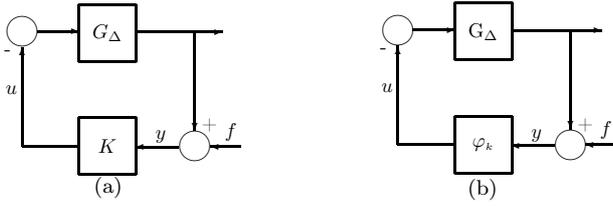
If the system in Fig. 7(a) is stable, then system in Fig. 7(b) is stable

Fig. 7. Robust Kalman Conjecture (RKC).

The RKC provides the condition for which the robust absolute stability of the robust Lur'e problem is exact.

The key idea of verifying this Robust Kalman Conjecture is *determining if the robust interval of the uncertain plant coincides with the slope interval of φ_k for which the robust Lur'e structure is absolutely stable*. In subsequent sections, we investigate this conjecture for first-order stable plants with various uncertainty descriptions.

We consider the case where $G(s)$ is a first-order plant

$$G(s) = \frac{a}{s+b}, \qquad a > 0 \text{ and } b > 0. \qquad (7)$$

5.1 Additive Uncertainty

Consider the well-posed uncertain nonlinear system with $\bar{w} \in \mathbb{R}_+$ and Δ satisfying $\|\Delta\|_{\mathcal{L}_2} \leq 1$ as shown in Fig. 8.
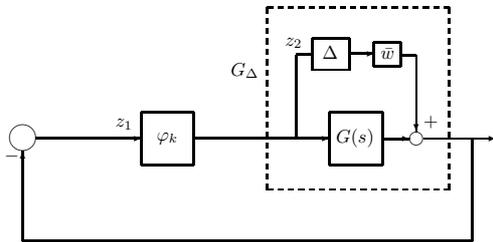

Fig. 8. Robust Lur'e for Additive Uncertainty

Lemma 1. *Let $G_\Delta \in \mathcal{G}_\Delta$ be a first-order plant (7) with an additive uncertainty i.e. $G_\Delta = G + \bar{w}\Delta$ as shown in Fig. 8, then the robust interval $\mathcal{I}_r [0, K_r)$ of G_Δ is $[0, 1/\bar{w})$.*

Result 1. *Let $G_\Delta \in \mathcal{G}_\Delta$ be a first-order plant (7) with an additive uncertainty as shown in Fig. 8 .By virtue of lemma 1, the robust interval $(\mathcal{I}_r)$ of G_Δ is defined as $[0, 1/\bar{w})$. Thus the system in Fig. 8 is $\mathcal{L}_2$-stable for any $\varphi_k \in$ SR $[0, \frac{1}{\bar{w}})$ nonlinearity.*

Proof. Choose $M(j\omega) = \frac{1}{k^2\bar{w}^2}$. Substituting $M(j\omega)$ in (6), then (4) gives

$$\frac{1}{k\bar{w}^2}\frac{2ab}{\omega^2+b^2} + 2(\frac{1}{k^2\bar{w}^2} - 1) > 0 \qquad \forall \omega \in \mathbb{R}. \qquad (8)$$

The first term of (8) is always positive and the second term is positive provided $k < \frac{1}{\bar{w}}$.

Thus, $M(j\omega) = \frac{1}{k^2\bar{w}^2}$ satisfies the conditions in Theorem 3. As a result, the system in Fig. 8 is $\mathcal{L}_2$-stable for any $\varphi_k \in$ SR $[0, \frac{1}{\bar{w}})$ nonlinearity. $\qquad \square$

Remark 2. *Result 1 demonstrates that the robust Kalman conjecture is true for first-order plant with additive uncertainty. As indicated in Table 2, the robust circle criterion provides an appropriate constant multiplier for this purpose.*

Graphical Interpretation of Result 1

Given the linear uncertain plant as $G_{\Delta_{LTI}}(s) = G(s) + \bar{w}\Delta_{LTI}$, the "loose" region is defined by discs of radius $\bar{w} = 0.01$ centered at each frequency as shown in Fig. 9. The bound K_r is then obtained by invoking Corollary 1:

$$|1 + K\frac{a}{j\omega+b}| > |K\bar{w}\Delta_{LTI}| \qquad \forall \omega \in \mathbb{R}. \qquad (9)$$

Remark 3. *An equivalent interpretation of (9) is that $G(j\omega)$ avoids and does not encircle the circle of radius $|\bar{w}\Delta_{LTI}|$ centered at $-1/K$ [11].*
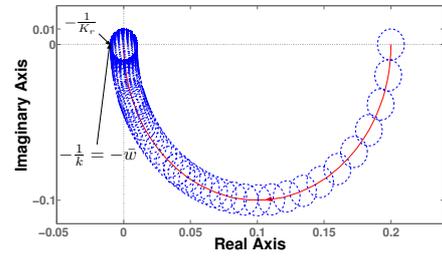

Fig. 9. Robust Nyquist Plot

The analytical result of Result 1 corresponds with the graphical interpretation of circle criterion for the "loose" region.

Table 2. RKC for Additive Uncertainty

	Robust Interval $(\mathcal{I}_r)$	Multipliers :	φ_k Slope Interval Constant	
$\bar{w} > 0$	$[0, 1/\bar{w})$		SR $[0, 1/\bar{w})$	-

5.2 Input Multiplicative Uncertainty

Consider the well-posed uncertain nonlinear system with $\bar{w} \in \mathbb{R}_+$ and Δ satisfying $\|\Delta\|_{\mathcal{L}_2} \leq 1$ as shown in Fig. 10.
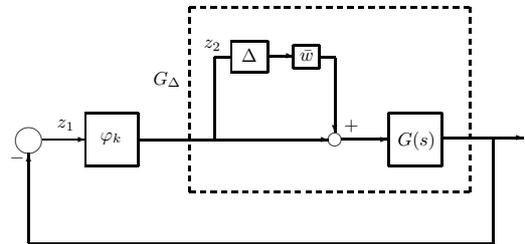

Fig. 10. Robust Lur'e for Input Multiplicative Uncertainty

Lemma 2. *Let $G_\Delta \in \mathcal{G}_\Delta$ be a first-order plant (7) with input multiplicative uncertainty i.e. $G_\Delta = G[1 + \bar{w}\Delta]$ as shown in Fig. 10. The robust interval $\mathcal{I}_r [0, K_r)$ of G_Δ is:*

$$\mathcal{I}_r = \begin{cases} [0, \infty) & \text{if } \bar{w} \leq 1, \\ [0, \frac{b}{a(\bar{w}-1)}) & \text{if } \bar{w} > 1. \end{cases} \qquad (10)$$

Result 2. *Let $G_\Delta \in \mathcal{G}_\Delta$ be a first-order plant (7) with input multiplicative uncertainty as shown in Fig. 10. By virtue of lemma 2, the robust interval ($\mathcal{I}_r$) of G_Δ is defined as (10). Thus the system in Fig. 10 is $\mathcal{L}_2$-stable for any*

$$\varphi_k \in \begin{cases} \text{SR}\,[0,\infty) & \text{if } \bar{w} \leq 1, \\ \text{SR}\,[0,\dfrac{b}{a(\bar{w}-1)}) & \text{if } \bar{w} > 1. \end{cases} \quad (11)$$

Proof. Proof omitted. □

Remark 4. *Result 2 demonstrates that the robust Kalman conjecture is true for first-order plant with input multiplicative uncertainty $\forall \bar{w} > 0$. As indicated in Table 3, the robust Popov criterion provides an appropriate Popov multiplier for this purpose. However, it can be shown that the robust circle criterion is only adequate when $\bar{w} \geq 2$.*

Graphical Interpretation of Result 2

In this case, the "loose" region is defined by discs of radius $\bar{w}|G(j\omega)|$ centered at each frequency. The robust Nyquist and the corresponding Popov plots having ellipses with semi-axes of $\bar{w}|G(j\omega)|$ and $\omega\bar{w}|G(j\omega)|$ centered at each frequency are illustrated in Fig. 11 for $\bar{w}$=1.3.
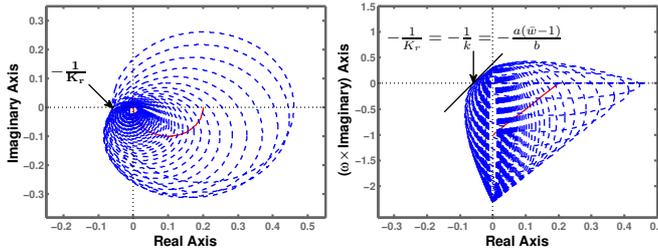

Fig. 11. Robust Nyquist and Robust Popov Plots

The analytical result of Result 2 corresponds with the graphical interpretation of the Popov criterion for the "loose region".

Table 3. RKC for Input Multiplicative Uncertainty

	Robust Interval ($\mathcal{I}_r$)	φ_k Slope Interval Constant Multi.	Popov Multi.
$\bar{w} \leq 1$	$[0,\infty)$	$\text{SR}\,[0,\frac{4b}{a\bar{w}^2})$	$\text{SR}\,[0,\infty)$
$1 < \bar{w} < 2$	$[0,\frac{b}{a(\bar{w}-1)})$	$\text{SR}\,[0,\frac{4b}{a\bar{w}^2})$	$\text{SR}\,[0,\frac{b}{a(\bar{w}-1)})$
$\bar{w} \geq 2$	$[0,\frac{b}{a(\bar{w}-1)})$	$\text{SR}\,[0,\frac{b}{a(\bar{w}-1)})$	$\text{SR}\,[0,\frac{b}{a(\bar{w}-1)})$

5.3 OUTPUT MULTIPLICATIVE UNCERTAINTY

Consider the well-posed uncertain nonlinear system with $\bar{w} \in \mathbb{R}_+$ and Δ satisfying $\|\Delta\|_{\mathcal{L}_2} \leq 1$ as shown in Fig. 12.
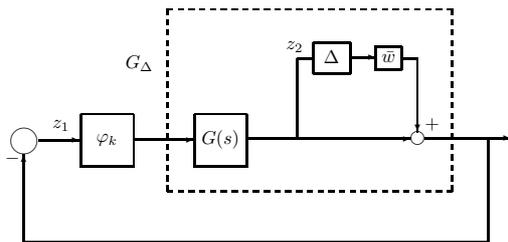

Fig. 12. Robust Lur'e for Output Multiplicative Uncertainty

Lemma 3. *Let $G_\Delta \in \mathcal{G}_\Delta$ be a first-order plant (7) with a output multiplicative uncertainty i.e. $G_\Delta = [1 + \bar{w}\Delta]G$ as shown in Fig. 12. The robust interval $\mathcal{I}_r\,[0,K_r)$ of G_Δ is:*

$$\mathcal{I}_r = \begin{cases} [0,\infty) & \text{if } \bar{w} \leq 1, \\ [0,\dfrac{b}{a(\bar{w}-1)}) & \text{if } \bar{w} > 1. \end{cases} \quad (12)$$

Result 3. *Let $G_\Delta \in \mathcal{G}_\Delta$ be a first-order plant (7) with output multiplicative uncertainty as shown in Fig. 12. By virtue of lemma 3, the robust interval ($\mathcal{I}_r$) of G_Δ is defined as (12). Thus the system in Fig. 12 is $\mathcal{L}_2$-stable for any*

$$\varphi_k \in \begin{cases} \text{SR}\,[0,\infty) & \text{if } \bar{w} \leq 1, \\ \text{SR}\,[0,\dfrac{b}{a(\bar{w}-1)}) & \text{if } \bar{w} > 1. \end{cases} \quad (13)$$

Proof. Proof omitted. □

Remark 5. *Result 3 demonstrates that the robust Kalman conjecture is true for first-order plant with output multiplicative uncertainty for all $\bar{w} > 0$. As indicated in Table 4, the robust Zames-Falb criterion provides an appropriate first-order Zames-Falb multiplier for this purpose. It can be shown that the robust circle criterion is only adequate when $\bar{w} \geq 2$ just as in the input multiplicative case. Also the robust Popov criterion does not yield additional information over and above the robust circle criterion.*

Graphical Interpretation of Result 3

The "loose" region and bound K_r is same as defined for the input-multiplicative case. The corresponding robust Zames-Falb plots $\frac{1}{k^2\bar{w}^2}[\frac{j\omega+b}{j\omega+(b+ka)}](\mathcal{G}_{\Delta_{LTI}} + \frac{1}{k})$: $k \in \mathcal{I}_r$ indeed lies in the RHS plane as illustrated in Fig. 13(a) and Fig. 13(b) for $\bar{w}$=1 and $\bar{w}$=1.3 respectively.

(a) $\bar{w} = 1$, $k = 20$ (b) $\bar{w} = 1.3$, $k = K_r$
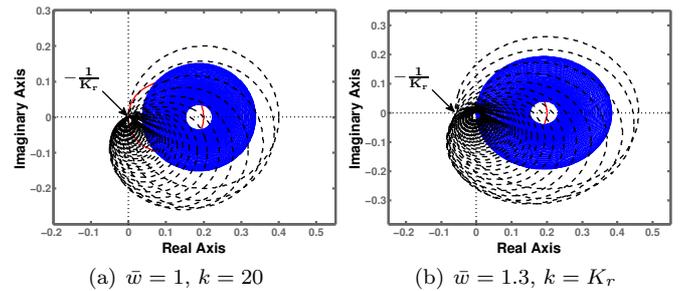
The robust Zames-Falb plots are scaled by 1500.

Fig. 13. Robust Nyquist (--) and Robust Zames-Falb Plots (-)

The analytical result of Result 3 corresponds with the graphical interpretation of the Zames-Falb criterion for the "loose region".

Table 4. RKC for Output Multiplicative Uncertainty

	Robust Interval ($\mathcal{I}_r$)	φ_k Slope Interval Constant M.	Popov M.	Zames-Falb M.
$\bar{w} \leq 1$	$[0,\infty)$	$\text{SR}\,[0,\frac{4b}{a\bar{w}^2})$	-	$\text{SR}\,[0,\infty)$
$1 < \bar{w} < 2$	$[0,\frac{b}{a(\bar{w}-1)})$	$\text{SR}\,[0,\frac{4b}{a\bar{w}^2})$	-	$\text{SR}\,[0,\frac{b}{a(\bar{w}-1)})$
$\bar{w} \geq 2$	$[0,\frac{b}{a(\bar{w}-1)})$	$\text{SR}\,[0,\frac{b}{a(\bar{w}-1)})$	-	$\text{SR}\,[0,\frac{b}{a(\bar{w}-1)})$

5.4 FEEDBACK UNCERTAINTY

Consider the well-posed uncertain nonlinear system with $\bar{w} \in \mathbb{R}_+$ and Δ satisfying $\|\Delta\|_{\mathcal{L}_2} \leq 1$ as shown in Fig. 14.
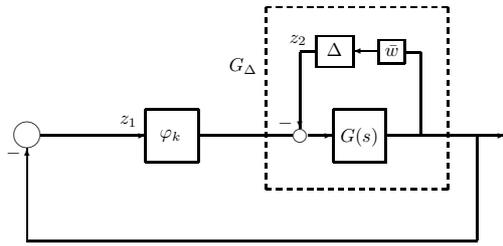

Fig. 14. Robust Lur'e for Feedback Uncertainty

Remark 6. *The uncertain plant G_Δ is stable if $\bar{w} < \frac{b}{a}$.*

Lemma 4. *Let $G_\Delta \in \mathcal{G}_\Delta$ be a first-order plant (7) with an feedback uncertainty i.e. $G_\Delta = G[1 + \Delta G \bar{w}]^{-1}$ as shown in Fig. 14. For $\bar{w} < \frac{b}{a}$ the robust interval $\mathcal{I}_r [0, K_r)$ of G_Δ is $[0, \infty)$.*

Result 4. *Let $G_\Delta \in \mathcal{G}_\Delta$ be a first-order plant (7) with feedback uncertainty as shown in Fig. 14. By virtue of lemma 4, for $\bar{w} < \frac{b}{a}$ the robust interval $(\mathcal{I}_r)$ of G_Δ is defined as $[0, \infty)$. Thus, for $\bar{w} < \frac{b}{a}$ the system in Fig. 14 is $\mathcal{L}_2$-stable for any $\varphi_k \in \mathrm{SR}[0, \infty)$.*

Proof. Proof omitted. $\square$

Remark 7. *Result 4 demonstrates that the robust Kalman conjecture is true for first-order plant with feedback uncertainty. As indicated in Table 5, the robust circle criterion provides an appropriate constant multiplier for this purpose.*

Graphical Interpretation of Result 4

In this case, the "loose" region is defined by discs of $G(j\omega)[1 + \bar{w}\Delta_{LTI}G(j\omega)]^{-1}$ as shown in Fig. 15 for $\bar{w} = 3$.
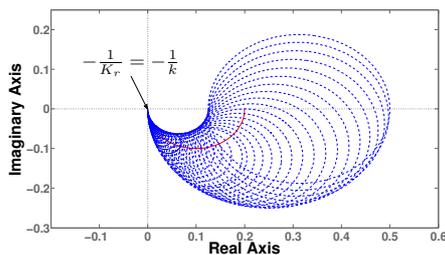

Fig. 15. Robust Nyquist Plot

The analytical result of Result 4 corresponds with the graphical interpretation of the circle criterion for the "loose" region.

Table 5. RKC for Feedback Uncertainty

	Robust Interval $(\mathcal{I}_r)$	Multipliers :	φ_k Slope Interval Constant	-
$\bar{w} < \frac{b}{a}$	$[0, \infty)$		$\mathrm{SR}[0, \infty)$	-

6. CONCLUSION

We have presented a new robust absolute stability conjecture called the robust Kalman conjecture. As indicated in Table 6, we have shown with different stability multipliers, that this robust Kalman conjecture is true for first-order stable plants with additive, (input and output) multiplicative and feedback uncertainties. A graphical interpretation of the result is given for each case when the norm-bound perturbations includes LTI uncertainties. The practical significance of this conjecture albiet for higher-order systems is highlighted by recent considerations of robust preservation in antiwindup control systems [12, 13].

Table 6. Robust Kalman Conjecture For First-Order Plants

	Uncertainty Descriptions			
	Additive	Input Multiplicative	Output Multiplicative	Feedback
Sufficient Multiplier	Constant	Popov	First-order Zames-Falb	Constant

REFERENCES

[1] F. D'Amato, M. Rotea, A. Megretski and U. Jönsson. "New results for analysis of systems with repeated nonlinearities", *Automatica*, vol. 37, pp. 739–747, 2001.

[2] V. Bragin, V. Vagaitsev, N. Kuznetsov and G. Leonov. "Algorithms for finding hidden oscillations in nonlinear systems: The Aizerman and Kalman conjectures and Chua's circuits", *Journal of Computer and Systems Sciences International*, vol. 50(4), pp. 511–543, 2011.

[3] N. E. Barabanov. "On the Kalman problem", *Siberian Mathematical Journal*, vol. 29(3), pp. 3–11, 1988.

[4] G. Leonov, I. Burkin and A. Shepeljavyi. *Frequency methods in oscillation theory.* Kluwer Academic Publishers, 1996.

[5] G. Zames and P. L. Falb. "Stability conditions for systems with monotone and slope-restricted nonlinearities", *SIAM Journal on Control and Optimization*, vol. 6, pp. 89–108, 1968.

[6] S. Impram and N. Munro. "A note on absolute stability of uncertain systems", *Automatica*, vol. 37, pp. 605–610, 2001.

[7] V. L. Kharitnov. "Asymptotic stability of a family of systems of linear differential equations", *In: Translation in Differential Equations*, vol. 14, pp. 1483–1485, 1979.

[8] Y. Z. Tsypkin and B. T. Polyak. *Robustness of dynamic systems with parameter uncertainties.* Basel, Switzerland: BirkhaK user Verlag., pp. 113-121, 1992.

[9] A. Megretski and A. Rantzer. "System analysis via integral quadratic constraints", *IEEE Journal of Automatic Control*, vol. 42(6), pp. 819–830, 1997.

[10] M. Vidyasagar. *Nonlinear Systems Analysis.* Prentice-Hall International, 1978.

[11] B. T. Polyak and Y. Z. Tsypkin. "Robust Nyquist Criterion", *Automation and Remote Control*, vol. 53, pp. 972–977, 1992.

[12] W. P. Heath, R. Morales and G. Li. "Robustness preserving antiwindup: Examples and counterexamples", *In: Proceedings of IEEE International Computer-Aided Control System Design Symposium*, Denver, USA, September 2011.

[13] R. Morales, W. P. Heath and G. Li. "Robustness preserving anti-windup for SISO systems", *In: Proceedings of 50th IEEE Conf. on Decision and Control*, Orlando, USA, December 2011.