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A Robust Output Feedback Consensus Protocol for Networked Negative-Imaginary Systems ⋆

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Abstract: The robust output feedback consensus problem for homogeneous networked Negative-Imaginary (*NI*) systems is investigated in this paper. By virtue of the *NI* systems stability theorem, a set of reasonable yet elegant conditions are derived for guaranteeing consensus under L_2 disturbances and *NI* model uncertainty. Furthermore, the eventual convergence sets of several special *NI* systems that are commonly studied in the literature are also presented. It is shown how the results in this work embed and generalise earlier results to these classes of systems. Numerical examples are given to illustrate the results of this paper.

Keywords: Consensus, Cooperative Control, Negative-Imaginary Systems, Robust Control

1. INTRODUCTION

Broadly speaking, Negative-Imaginary (*NI*) systems are systems with a negative imaginary frequency response. This class of systems has received extensive attention in recent years since it was introduced by Lanzon and Petersen (2008) and found its most successful application in the area of nano-positioning control where co-located force actuation and position measurement are typical (Petersen and Lanzon (2010) and Song et al. (2012)). Similarly, *NI* systems theory has been widely applied to the flexible structures with highly-resonant dynamics, which is typically challenging to tackle via classical methods, such as passivity (Khalil (1996)) or small gain analysis (Zhou et al. (1996)). Applications include flexible robot manipulators (Wilson et al. (2002)), ground and aerospace vehicles (Harigae et al. (2003)), atomic force microscopes (Mahmood et al. (2011)) and nano-positioning systems (Salapaka et al. (2002)), to name a few.

The topic of cooperative control has been very active over the past decade and it was immediately evident that decentralised control and communication networks play an important role in the system properties, including basic stability analysis. The output feedback consensus problem, or more precisely, the output synchronization problem was first studied in Chopra and Spong (2008), and a solution for weakly minimum phase nonlinear systems with relative degree of one was presented. Later on, a series of papers, such as Kim et al. (2010), Wang et al. (2010) and Su et al. (2013) extended the results to heterogeneous cases even with uncertainties. The output feedback consensus problem that we consider is to have all the outputs naturally converge to a common value (not necessarily constant) which is entirely determined by the subsystems themselves as well as the graph properties. This is different from cooperative control problems where the output of each agent is made to follow a given reference signal.

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This paper is motivated by applications in which the system goal cannot be accomplished by a single *NI* system due to limitations in its capability, such as coverage or precision. This in turn requires the coordination of multiple *NI* systems, which in this paper involves output feedback consensus under model uncertainty and disturbances. In this paper, a homogeneous network of *NI* systems and a fixed communication topology are assumed. The ith *NI* system is described in the s-domain by

$$\boldsymbol{y}_i^{m\times 1} = P(s)\boldsymbol{u}_i^{m\times 1},\ i = 1, \cdots, n, \qquad (1)$$

where $P(s)$ is the transfer function (generally MIMO), $n > 1$ is the number of agents and $m \geq 1$ is the dimension of both the output and input. Then, an elegant problem formulation, using the Laplacian matrix and the Kronecker product, is adopted such that the output feedback consensus problem turns out to be an internal stability problem, which can be solved by *NI* systems theory as detailed in Lanzon and Petersen (2008), Xiong et al. (2010) and Mabrok et al. (2013). The contributions of this paper can be summarized as: (a) it provides a novel viewpoint, (b) it only exploits output feedback information as opposed to the full state feedback which is common in the literature, (c) it gives a class of consensus protocols that can be tuned for performance and/or robustness, (d) it characterises the convergence set, and (e) it provides a robustness guarantee via *NI* systems theory.

The remainder of this paper is organised as follows: In Section 2, preliminary notation and definitions used in this paper are given. The problem formulation is described in Section 3 and then the first main result of this paper, a robust output feedback consensus protocol and convergence proof, are presented. Section 4 explicitly characterises the convergence set and specialises this to obtain easily interpretable results for several *NI* systems typically considered in the literature, such as single integrators, double integrators, and second order damped systems. Simulation results are presented in Section 5. Finally, concluding remarks are given in Section 6.

2. PRELIMINARIES

2.1 Notation

$\mathbb{R}^{m\times n}$ and $\mathbb{C}^{m\times n}$ denote the family of $m \times n$ real and complex matrices, respectively. I_n is the $n \times n$ identity matrix. $\mathbf{1}_n$ and $1_{n\times n}$ are the $n\times 1$ vector and $n\times n$ matrix with all elements being 1, respectively. Given $M \in \mathbb{R}^{n\times n}$, $M > (<)0$ means M is positive (negative) definite and $M \geq (\leq)0$ means M is positive (negative) semi-definite. $\bar{\lambda}(M)$ denotes the largest eigenvalue of M and $\mathcal{N}(M)$ denotes the null space of M. M^T and M^* are the transpose and the complex conjugate transpose of M. In addition, given $s \in \mathbb{C}$, Re$[s]$ is the real part of s. Given $a_1,\ a_2 \in \mathbb{C}$, $\text{diag}(a_1, a_2) = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$. Finally, given $\boldsymbol{z} \in \mathbb{R}^{n\times 1}$, ave$(\boldsymbol{z})$ is the average operation of all elements of $\boldsymbol{z}$. OLHP is short for open left half plane.

2.2 Graph Theory

A graph can be mathematically expressed by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ is a nonempty finite set of n nodes and an edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is used to model the communications links among agents. The adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$, where $a_{ii} = 0$ and $\forall i,\ i \neq j$, $a_{ij} = 1$ if $(v_i, v_j) \in \mathcal{E}$ and 0 otherwise. In an undirected graph, $a_{ij} = a_{ji}$. The in-degree of node i is defined as $d_i = \sum_j a_{ij}$ and $\mathcal{D} = \text{diag}\{d_1, d_2, \cdots, d_n\} \in \mathbb{R}^{n\times n}$ is the in-degree matrix. Then, the Laplacian matrix of graph $\mathcal{G}$ is given by

$$\mathcal{L}_n = \mathcal{D} - \mathcal{A}. \tag{2}$$

A sequence of successive edges of $\mathcal{E}$ in the form of $\{(v_i, v_k), (v_k, v_l), \ldots, (v_m, v_j)\}$ is defined as a directed path from node i to node j. An undirected path in an undirected graph is defined analogously. An undirected graph is said to be connected if there is a path from node i to node j for all the distinct nodes $v_i, v_j \in \mathcal{V}$. It is well-known Ren and Beard (2008) that $\mathcal{L}_n$ has one unique zero eigenvalue associated with the eigenvector $\mathbf{1}_n$ and all the other eigenvalues are positive and real, when the graph is undirected and connected, or in other words,

$$\mathcal{L}_n \geq 0, \det(\mathcal{L}_n) = 0, \mathcal{N}(\mathcal{L}_n) = \text{span}\{\mathbf{1}_n\}. \tag{3}$$

2.3 Negative-Imaginary Systems

Before proceeding to the main result, let us first recall the definitions of *NI* and *SNI* systems:

Definition 1. (Mabrok et al. (2013)) A square transfer function matrix $P(s)$ is *NI* if the following conditions are satisfied:

(1) $P(s)$ has no pole in Re$[s] > 0$;
(2) $\forall\omega > 0$ such that $j\omega$ is not a pole of $P(s)$, $j(P(j\omega) - P(j\omega)^*) \geq 0$;
(3) If $s = j\omega_0$ where $\omega_0 > 0$ is a pole of $P(s)$, then it is a simple pole and the residue matrix $K = \lim_{s\to j\omega_0}(s - j\omega_0)jP(s)$ is Hermitian and positive semi-definite;
(4) If $s = 0$ is a pole of $P(s)$, then $\lim_{s\to 0} s^k P(s) = 0\ \forall k \geq 3$ and $P_2 = \lim_{s\to 0} s^2 P(s)$ is Hermitian and positive semi-definite.

It can be observed that Definition 1 for *NI* systems captures the definitions in Lanzon and Petersen (2008) and Xiong et al. (2010). Examples of *NI* systems can be found in Mabrok et al. (2013), and these include single-integrator system, double-integrator system, undamped and damped flexible structure, to name a few typically considered in consensus literature.

Definition 2. A square transfer function matrix $P_s(s)$ is *SNI* if the following conditions are satisfied:

(1) $P_s(s)$ has no pole in Re$[s] \geq 0$;
(2) $\forall\omega > 0, j(P_s(j\omega) - P_s(j\omega)^*) > 0$.

3. MAIN RESULT

In this section, the problem of output feedback consensus over networked *NI* systems is considered. Networked homogeneous *NI* agents are defined in the s-domain to have the form (1). Since $\boldsymbol{u}_i$ and $\boldsymbol{y}_i$ are vectors and $P(s)$ is a MIMO transfer function, the Laplacian matrix describing the network interconnection is modified via a Kronecker product to $\mathcal{L}_n \otimes I_m$ and the total networked plant under consideration is depicted in Fig. 1:

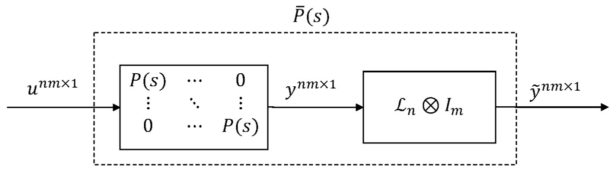

Fig. 1. System setup

with

$$\begin{aligned} \tilde{\boldsymbol{y}}^{nm\times 1} &= \bar{P}(s)\boldsymbol{u}^{nm\times 1} \\ &= (\mathcal{L}_n \otimes I_m)(I_n \otimes P(s))\boldsymbol{u}^{nm\times 1} \\ &= (\mathcal{L}_n \otimes P(s))\boldsymbol{u}^{nm\times 1} \end{aligned} \tag{4}$$

where $\bar{P}(s)$ is the augmented transfer function, $\mathcal{L}_n$ is the Laplacian matrix of the communication topology among the multiple *NI* agents, $\boldsymbol{y}^{nm\times 1} = \left[\boldsymbol{y}_1^T, \cdots, \boldsymbol{y}_n^T\right]^T$ and $\boldsymbol{u}^{nm\times 1} = \left[\boldsymbol{u}_1^T, \cdots, \boldsymbol{u}_n^T\right]^T$. It can be seen that the output $\boldsymbol{y}^{nm\times 1}$ reaches consensus when $\tilde{\boldsymbol{y}}^{nm\times 1} \to \mathbf{0}$ by noticing that the null space of a Laplacian matrix $\mathcal{L}_n \otimes I_m$ for an undirected and connected graph (a directed graph is inapplicable since Lemma 6 further requires positive semi-definiteness of $\mathcal{L}_n \otimes I_m$) has dimension 1 and is characterised by $\mathbf{1}_n \otimes \boldsymbol{e}$, $\forall \boldsymbol{e} \in \mathbb{R}^{m\times 1}$. This formulation actually makes the output consensus problem reduce to an internal stability problem which is usually easier to tackle under the hypothesis of this paper:

Hypothesis 3. $\mathcal{G}$ is undirected and connected.

The following lemmas are needed for this paper:

Lemma 4. [Zhou et al. (1996)] Let λ_j and γ_k, $j = 1, \cdots, n$, $k = 1, \cdots, m$, be eigenvalues of matrices $\Lambda_{n\times n}$ and $\Gamma_{m\times m}$ respectively, the eigenvalues of $\Lambda \otimes \Gamma$ are $\lambda_j \gamma_k$.

Lemma 5. Given $\Lambda \in \mathbb{R}^{n\times n}$ and $\Gamma \in \mathbb{R}^{m\times m}$, then

$$\begin{aligned}\mathcal{N}(\Lambda \otimes \Gamma) =& \{\boldsymbol{a} \otimes \boldsymbol{b} : \boldsymbol{b} \in \mathbb{R}^{m\times 1}, \boldsymbol{a} \in \mathcal{N}(\Lambda)\} \\ &\cup \{\boldsymbol{c} \otimes \boldsymbol{d} : \boldsymbol{c} \in \mathbb{R}^{n\times 1}, \boldsymbol{d} \in \mathcal{N}(\Gamma)\}.\end{aligned}$$

Proof. The proof simply follows from the definition of null space and properties of the Kronecker product. □

With the above knowledge, we can derive the following lemma which will be used to derive the main results of this paper.

Lemma 6. $\bar{P}(s)$ is *NI* if and only if $P(s)$ is *NI*.

Proof. It will be provided in a journal version. □

Lemma 6 has shown that the networked system $\bar{P}(s)$ is *NI* if and only if $P(s)$ is *NI*. Thus, the output $\tilde{\boldsymbol{y}}^{nm\times 1} \to \mathbf{0}$ if internal stability is achieved for $\bar{P}(s)$ with some controller. From Lanzon and Petersen (2008), Xiong et al. (2010) and Mabrok et al. (2013), the following internal stability results are summarized:

Lemma 7. Given a *NI* transfer function $P(s)$ and an *SNI* function $P_s(s)$, the positive feedback interconnection $[P(s), P_s(s)]$ is internally stable if and only if any of the following conditions is satisfied:

(1) $\bar{\lambda}(P(0)P_s(0)) < 1$ when $P(s)$ has no pole(s) at the origin, $P(\infty)P_s(\infty) = 0$ and $P_s(\infty) \geq 0$;
(2) $J^T P_s(0) J < 0$ when $P(s)$ has pole(s) at the origin and is strictly proper, $P_2 \neq 0, P_1 = 0, \mathcal{N}(P_2) \subseteq \mathcal{N}(P_0^T)$, where $P_2 = \lim_{s\to 0} s^2 P(s) = JJ^T$ with J having full column rank, $P_1 = \lim_{s\to 0} s(P(s) - \frac{P_2}{s^2})$ and $P_0 = \lim_{s\to 0}(P(s) - \frac{P_2}{s^2} - \frac{P_1}{s})$;
(3) $F_1^T P_s(0) F_1 < 0$ when $P(s)$ has pole(s) at the origin and is strictly proper, $P_2 = \lim_{s\to 0} s^2 P(s) = 0$, $P_1 = \lim_{s\to 0} s(P(s) - \frac{P_2}{s^2}) \neq 0$, $\mathcal{N}(P_1^T) \subseteq \mathcal{N}(P_0^T)$, where $P_1 = [\tilde{F}_1\ \tilde{F}_2] \begin{bmatrix} S_2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} = F_1 V_1^T$ and $P_0 = \lim_{s\to 0}(P(s) - \frac{P_2}{s^2} - \frac{P_1}{s})$.

Now, we are ready to state the first main result of this paper:

Theorem 8. Given a graph $\mathcal{G}$ which satisfies Hypothesis 3 and models the communication links for networked homogeneous *NI* systems $P(s)$ with any *SNI* control law $P_s(s)$, the robust output feedback consensus is achieved via the protocol

$$\begin{aligned}\boldsymbol{U}_{cs} &= \bar{P}_s(s)\tilde{\boldsymbol{y}}^{nm\times 1} = C_{cs}(s)\boldsymbol{y}^{nm\times 1} \\ &= (I_n \otimes P_s(s))(\mathcal{L}_n \otimes I_m)\boldsymbol{y}^{nm\times 1} \\ &= (\mathcal{L}_n \otimes P_s(s))\boldsymbol{y}^{nm\times 1}\end{aligned} \tag{5}$$

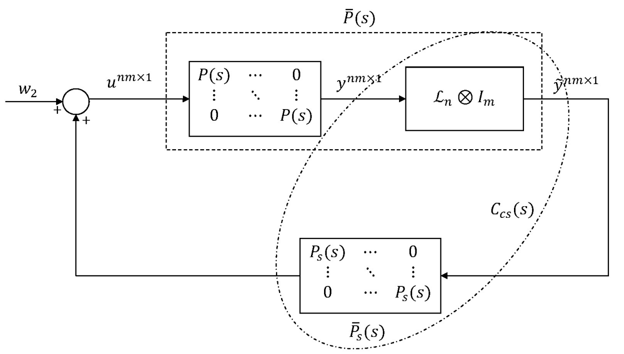

Fig. 2. Closed-loop system with *SNI* controller

shown in Fig. 2, or in a distributed manner, for each agent i by

$$\boldsymbol{u}_i = P_s(s)\sum_{j=1}^{n} a_{ij}(\boldsymbol{y}_i - \boldsymbol{y}_j), \tag{6}$$

under L_2 disturbance if and only if $P(s)$ and $P_s(s)$ satisfy the conditions listed in Lemma 7 except that

$$\bar{\lambda}(P(0)P_s(0)) < \frac{1}{\bar{\lambda}(\mathcal{L}_n)} \tag{7}$$

replaces $\bar{\lambda}(P(0)P_s(0)) < 1$ in case (1).

Proof. It will be provided in a journal version. □

Claim 9. It can be seen that the condition in inequality (7) is stricter than that in the inequality of case 1 of Lemma 7 due to the network interconnection. If originally $P_s(0)$ was such that $0 < \bar{\lambda}(P(0)P_s(0)) < 1$, the controller $P_s(0)$ needs to be tuned for smaller eigenvalues in order to satisfy inequality (7). On the other hand, if $\bar{\lambda}(P(0)P_s(0)) < 0$, there is no need to tune further.

4. CONVERGENCE SET

In the previous section, Theorem 8 has provided a general robust output feedback consensus protocol that guarantees the convergence of the *NI* systems' outputs $\boldsymbol{y}_i$. However, the convergence set is still unspecified. This section investigates the steady state nominal values of $\boldsymbol{y}_{ss}$ under the proposed output feedback consensus protocol. In order to specify the exact convergence set, the disturbance in the input and output channel will not be considered in this section.

Given a minimal realization of the ith *NI* plant $P(s)$,

$$\begin{cases} \dot{\boldsymbol{x}}_i^{p\times 1} = A^{p\times p}\boldsymbol{x}_i^{p\times 1} + B^{p\times m}\boldsymbol{u}_i^{m\times 1} \\ \boldsymbol{y}_i^{m\times 1} = C^{m\times p}\boldsymbol{x}_i^{p\times 1} + D^{m\times m}\boldsymbol{u}_i^{m\times 1} \end{cases}, \ i = 1, \cdots, n, \tag{8}$$

and a minimal realization of the ith *SNI* controller $P_s(s)$,

$$\begin{cases} \dot{\bar{\boldsymbol{x}}}_i^{q\times 1} = \bar{A}^{q\times q}\bar{\boldsymbol{x}}_i^{q\times 1} + \bar{B}^{q\times m}\bar{\boldsymbol{u}}_i^{m\times 1} \\ \bar{\boldsymbol{y}}_i^{m\times 1} = \bar{C}^{m\times q}\bar{\boldsymbol{x}}_i^{q\times 1} + \bar{D}^{m\times m}\bar{\boldsymbol{u}}_i^{m\times 1} \end{cases}, \ i = 1, \cdots, n, \tag{9}$$

where p and q are the dimensions of the states of the *NI* plant and the *SNI* controller, respectively. The closed-loop system of Fig. 2 is given as

$$\begin{bmatrix}\dot{\bar{\boldsymbol{x}}}\\ \dot{\boldsymbol{x}}\end{bmatrix} = \begin{bmatrix} I_n \otimes \bar{A} + \mathcal{L}_n \otimes \bar{B}D\bar{C} & \mathcal{L}_n \otimes \bar{B}C \\ I_n \otimes B\bar{C} & I_n \otimes A + \mathcal{L}_n \otimes B\bar{D}C \end{bmatrix} \begin{bmatrix}\bar{\boldsymbol{x}}\\ \boldsymbol{x}\end{bmatrix}$$
$$\triangleq \Psi^{n(p+q)\times n(p+q)} \begin{bmatrix}\bar{\boldsymbol{x}}\\ \boldsymbol{x}\end{bmatrix}. \tag{10}$$

The eigenvalues of Ψ are of importance since they will determine the equilibria. In particular, in this paper, the eigenvalues on the imaginary axis of Ψ will determine the steady-state behaviour. The following lemma is given to characterise the eigenvalues of Ψ.

Lemma 10. Let $\lambda^i_{\mathcal{L}}$ be the ith eigenvalue of $\mathcal{L}_n$ associated with eigenvector $\boldsymbol{v}^i_{\mathcal{L}}$. The eigenvalues of Ψ are given by the eigenvalues of the following matrices:

$$\psi_i = \begin{bmatrix} \bar{A} + \lambda^i_{\mathcal{L}}\bar{B}D\bar{C} & \lambda^i_{\mathcal{L}}\bar{B}C \\ B\bar{C} & A + \lambda^i_{\mathcal{L}}B\bar{D}C \end{bmatrix},\ i = 1, \cdots, n. \tag{11}$$

Furthermore, let $\begin{bmatrix}\boldsymbol{v}^i_1\\ \boldsymbol{v}^i_2\end{bmatrix}$ be an eigenvector of ψ_i. Then, the corresponding eigenvector of Ψ is $\begin{bmatrix}\boldsymbol{v}^i_{\mathcal{L}} \otimes \boldsymbol{v}^i_1\\ \boldsymbol{v}^i_{\mathcal{L}} \otimes \boldsymbol{v}^i_2\end{bmatrix}$.

Proof. It will be provided in a journal version. □

It is known that there is only one zero eigenvalue, $\lambda^i_{\mathcal{L}} = 0$, when the graph $\mathcal{G}$ satisfies Hypothesis 3. In this case, ψ_i becomes $\begin{bmatrix}\bar{A} & 0\\ B\bar{C} & A\end{bmatrix}$ which has eigenvalues λ_A and $\lambda_{\bar{A}}$ associated with eigenvectors $\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{v}_A\end{bmatrix}$ and $\begin{bmatrix}\boldsymbol{v}_{\bar{A}}\\ (\lambda_{\bar{A}}I_n - A)^{-1}B\bar{C}\boldsymbol{v}_{\bar{A}}\end{bmatrix}$ respectively, where λ_A and $\lambda_{\bar{A}}$ are the eigenvalues of A and $\bar{A}$, $\boldsymbol{v}_A$ and $\boldsymbol{v}_{\bar{A}}$ are the corresponding eigenvectors of A and $\bar{A}$, respectively. This also shows that λ_A and $\lambda_{\bar{A}}$, with the vectors $\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{1}\otimes\boldsymbol{v}_A\end{bmatrix}$ and $\begin{bmatrix}\boldsymbol{1}\otimes\boldsymbol{v}_{\bar{A}}\\ \boldsymbol{1}\otimes(\lambda_{\bar{A}}I_n - A)^{-1}B\bar{C}\boldsymbol{v}_{\bar{A}}\end{bmatrix}$, are also the eigenvalues and eigenvectors of Ψ. It is worth noting that the invertibility of $A - \lambda_{\bar{A}}I_n$ follows since an *SNI* controller can always be chosen such that $\lambda_{\bar{A}} \neq \lambda_A$.

In the case of $\lambda^i_{\mathcal{L}} > 0$ and $\det(A) \neq 0$, it can be shown in a similar manner as Theorem 5 of Lanzon and Petersen (2008) that

$$\begin{aligned}\psi_i &= \begin{bmatrix} \bar{A} + \lambda^i_{\mathcal{L}}\bar{B}D\bar{C} & \lambda^i_{\mathcal{L}}\bar{B}C \\ B\bar{C} & A + \lambda^i_{\mathcal{L}}B\bar{D}C \end{bmatrix} \\ &= \begin{bmatrix}\bar{A} & 0\\ B\bar{C} & A\end{bmatrix} + \lambda^i_{\mathcal{L}}\begin{bmatrix}\bar{B}\\ B\bar{D}\end{bmatrix}\begin{bmatrix}D\bar{C} & C\end{bmatrix} = \Phi T\end{aligned} \tag{12}$$

where $\Phi = \begin{bmatrix}\bar{A}\bar{Y} & 0\\ 0 & AY\end{bmatrix}$ and

$$T = \begin{bmatrix} \bar{Y}^{-1} - \lambda^i_{\mathcal{L}}\bar{C}^* D\bar{C} & -\lambda^i_{\mathcal{L}}\bar{C}^* C \\ -C^*\bar{C} & Y^{-1} - \lambda^i_{\mathcal{L}}C^*\bar{D}C \end{bmatrix}.$$

ψ_i is Hurwitz if $\bar{\lambda}(P(0)P_s(0)) < \frac{1}{\lambda^i_{\mathcal{L}}}$ holds, which coincides with the condition in Theorem 8 when $\lambda^i_{\mathcal{L}} = \bar{\lambda}(\mathcal{L}_n)$.

In the case of $\lambda^i_{\mathcal{L}} > 0$ and $\det(A) = 0$, it can be verified in a similar manner as Mabrok et al. (2013) that

$$\psi_i = \begin{bmatrix}\bar{A} & \lambda^i_{\mathcal{L}}\bar{B}C\\ B\bar{C} & A + \lambda^i_{\mathcal{L}}B\bar{D}C\end{bmatrix} \tag{13}$$

due to $D = 0$. ψ_i is also Hurwitz when the conditions (2) and (3) in Lemma 7 hold. Detailed proof is omitted due to page limitations.

One direct observation about the above analysis is that the number of eigenvalues on the imaginary axis of Ψ is equal to the number of eigenvalues on the imaginary axis of A and all of the other eigenvalues lie in the OLHP since $\bar{A}$ is always Hurtwiz (Xiong et al. (2010)). Thus, the steady state of the closed-loop system (10) is in general dependent on the eigenvalues on the imaginary axis of A as shown in the following theorem:

Theorem 11. Given the closed-loop system in (10), the steady state can be expressed in the general form

$$\begin{bmatrix}\bar{\boldsymbol{x}}(t)\\ \boldsymbol{x}(t)\end{bmatrix} \xrightarrow{t\to\infty} [\boldsymbol{w}_j,\ \cdots,\ \boldsymbol{w}^g_k]\, e^{J't} \begin{bmatrix}\boldsymbol{v}_j^T\\ \vdots\\ \boldsymbol{v}^{g\,T}_k\end{bmatrix} \begin{bmatrix}\bar{\boldsymbol{x}}(0)\\ \boldsymbol{x}(0)\end{bmatrix}, \tag{14}$$

where J' is the Jordan block associated with n_0 eigenvalues of Ψ on the imaginary axis, $\boldsymbol{w}_j$ and $\boldsymbol{v}_j$ are the left and right eigenvector of Ψ associated with eigenvalues on the imaginary axis given by

$$\boldsymbol{w}_j = \begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{1}\otimes\boldsymbol{v}^r_A\end{bmatrix} \tag{15}$$

and

$$\boldsymbol{v}_j = \begin{bmatrix}\boldsymbol{1}\otimes(\frac{1}{n}(\lambda_A I_q - \bar{A})^{-1}\bar{C}^T B^T \boldsymbol{v}^l_A)\\ \boldsymbol{1}\otimes\frac{1}{n}\boldsymbol{v}^l_A\end{bmatrix} \tag{16}$$

respectively for $j = 1, \cdots, n_0 - (n_a - n_g)$, where n_a and n_g denote the algebraic and geometric multiplicity respectively. $\boldsymbol{v}^r_A, \boldsymbol{v}^l_A$ are the right and left eigenvectors of A associated with eigenvalues on the imaginary axis. Moreover, in the case that $n_a > n_g$, $\boldsymbol{w}^g_k$ and $\boldsymbol{v}^g_k$ are the generalized left and right eigenvectors given by

$$\boldsymbol{w}^g_k = \begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{1}\otimes\boldsymbol{v}^{r_g}_A\end{bmatrix} \tag{17}$$

and

$$\boldsymbol{v}^g_k = \begin{bmatrix}\boldsymbol{1}\otimes(\frac{1}{n}(\lambda_A I_q - \bar{A})^{-1}\bar{C}^T B^T \boldsymbol{v}^{l_g}_A)\\ \boldsymbol{1}\otimes\frac{1}{n}\boldsymbol{v}^{l_g}_A\end{bmatrix}, \tag{18}$$

where $k = 0, \cdots, n_a - n_g$, $\boldsymbol{v}^{r_g}_A$ and $\boldsymbol{v}^{l_g}_A$ are the generalized eigenvectors of A associated with eigenvalues on the imaginary axis.

Proof. It will be provided in a journal version. □

Next, convergence sets of several special cases of *NI* systems are given in detail:

Corollary 12. In the case that the *NI* plant is a single-integrator, such as $\dot{x}_i = u_i,\ y_i = x_i$, the convergence set of (10) is $y_{ss} = -\bar{C}\bar{A}^{-T}\cdot \mathrm{ave}(\bar{\boldsymbol{x}}(0)) + \mathrm{ave}(\boldsymbol{x}(0))$.

Proof. It is straightforward by finding the eigenvectors and applying Theorem 11. □

Corollary 13. In the case that the *NI* plant is a double-integrator, such as $\dot{\xi}_i = \zeta_i,\ \dot{\zeta}_i = u_i,\ y_i = \xi_i$, the convergence set of (10) is $y_{ss} = -\bar{C}\bar{A}^{-T}\cdot\mathrm{ave}(\bar{\boldsymbol{x}}(0))t + \mathrm{ave}(\boldsymbol{\xi}(0)) + \mathrm{ave}(\boldsymbol{\zeta}(0))t$.

Proof. It is straightforward by finding the eigenvectors and applying Theorem 11. □

Corollary 14. In the case that the *NI* plant is a stable flexible structure, the convergence set of (10) is $y_{ss} = 0$.

Proof. This is straightforward and thus omitted. □

5. ILLUSTRATIVE EXAMPLES

In this section, several numerical examples of typical *NI* systems are given to illustrate the main results of this paper. A scenario of 3 *NI* systems is considered and the communication graph $\mathcal{G}$ is simply given as in Fig. 3.

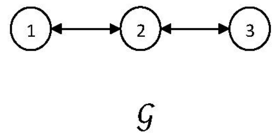

Fig. 3. Communication topology $\mathcal{G}$

Thus, the Laplacian matrix of $\mathcal{G}$ is derived according to the definition in Section 2 as $\mathcal{L}_3 = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$.

5.1 Multiple Single-Integrator Systems

Suppose that the *NI* systems have identical single-integrator dynamics as given in Corollary 12 with the initial conditions being $\boldsymbol{x}(0) = [1; 2; 3]$. The *SNI* controller is designed as indicated in Theorem 8 to be $\bar{A} = -2, \bar{B} = 1, \bar{C} = 1, \bar{D} = -1$, with the initial condition being $\bar{\boldsymbol{x}}(0) = [0.1; 0.2; 0.3]$. Without considering disturbances firstly, it can be verified as Corollary 12 that $y_{ss} = -\bar{C}\bar{A}^{-T} \cdot \text{ave}(\bar{\boldsymbol{x}}(0)) + \text{ave}(\boldsymbol{x}(0)) = \frac{1}{2} * 0.2 + 2 = 2.1$, which is exactly shown at the top of Fig. 4. If the disturbances are inserted, the output consensus is also achieved as shown at the bottom of Fig. 4 with disturbance level of 10^{-3}. Sensitivity of the loop to disturbances will depend on the choice of the *SNI* controller $P_s(s)$ which can be chosen to adjust.

One may notice that when the initial condition of the controller $\bar{\boldsymbol{x}}(0)$ is set to $\mathbf{0}$ (a reasonable choice as the controller is set by the designer), the convergence set reduces to $y_{ss} = \text{ave}(\boldsymbol{x}(0))$ which in turn implies that the results for the average consensus protocol in Ren and Beard (2008) is a special case of the proposed result. Alternatively, the desired final convergence point can be chosen by properly initialising the *SNI* controller, which can be seen as a more general result.

5.2 Multiple Double-Integrator Systems

Suppose that the *NI* systems have identical double-integrator dynamics as given in Corollary 13 with the initial conditions being $\boldsymbol{\xi}(0) = [1; 2; 3]$, $\boldsymbol{\zeta}(0) = [0.1; 0.2; 0.3]$. The same *SNI* controller can be adopted as in the previous subsection. Without considering disturbances at first, it can be verified as the Corollary 13 that $y_{ss} = \xi_i(\infty) = -\bar{C}\bar{A}^{-T} \cdot \text{ave}(\bar{\boldsymbol{x}}(0)) + \text{ave}(\boldsymbol{\xi}(0)) + \text{ave}(\boldsymbol{\zeta}(0))t = \frac{1}{2} * 0.2 + 2 + 0.2t = 2.1 + 0.2t$ and $\zeta_i(\infty) = -\bar{C}\bar{A}^{-T} \cdot \text{ave}(\bar{\boldsymbol{x}}(0)) + \text{ave}(\boldsymbol{\zeta}(0)) = \frac{1}{2} * 0.2 + 0.2 = 0.3$, which is exactly as shown at the top of Fig. 5. If disturbances are inserted, output consensus is also achieved as shown at the bottom of Fig. 5 with a disturbance level of 10^{-3}. Again, note that the choice of the dynamics of the *SNI* controller can be made to minimise the effects of unmodelled dynamics and disturbances.

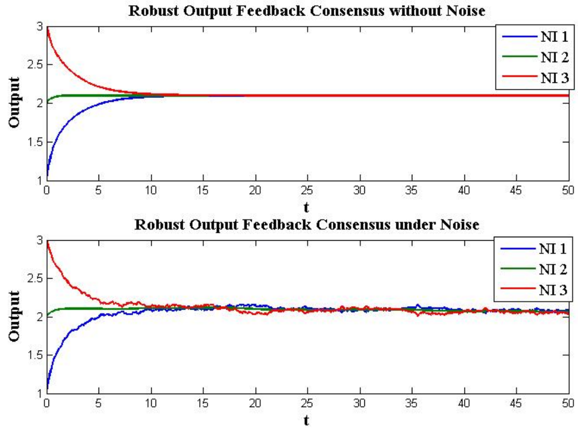

Fig. 4. Robust output feedback consensus for networked single-integrator systems

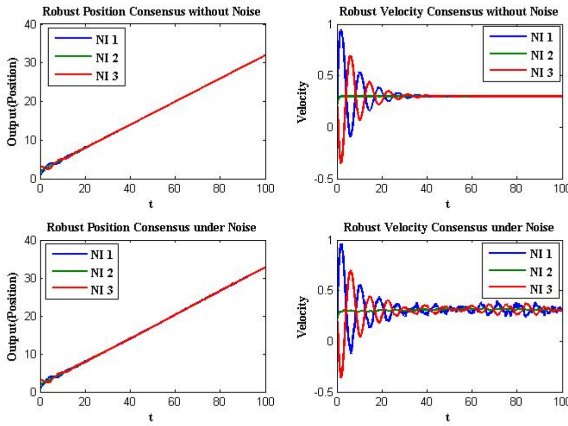

Fig. 5. Robust output feedback consensus for networked double-integrator systems

Similar to Subsection 5.1, one can set the initial condition of the controller to be $\bar{\boldsymbol{x}}(0) = \mathbf{0}$ to obtain the convergence set as $y_{ss} = \xi_{ss} = \text{ave}(\boldsymbol{\xi}(0)) + \text{ave}(\boldsymbol{\zeta}(0))t$ and $\zeta_{ss} = \text{ave}(\boldsymbol{\zeta}(0))$. The same conclusion can hence be drawn as in Subsection 5.1.

5.3 Multiple Undamped Flexible Structures

Suppose that the *NI* systems are undamped flexible structures as shown in Fig. 2 of Xiong et al. (2010) with the parameters of $m_1 = 1$, $m_2 = 0.5$, $k_1 = k_2 = k = 1$. The initial conditions are given as $\boldsymbol{x}(0) = [1; 2; 3; 4; 5; 6]$ and $\dot{\boldsymbol{x}}(0) = [0.1; 0.2; 0.3; 0.4; 0.5; 0.6]$. The *SNI* controller can be designed as indicated in Theorem 8 to be $\bar{A} = -4I_2, \bar{B} = I_2, \bar{C} = I_2, \bar{D} = 0_2$ since $\bar{\lambda}(P(0)) = 1$ and thus $\bar{\lambda}(P(0)P_s(0)) = \frac{1}{4} < \frac{1}{\bar{\lambda}(\mathcal{L}_n)} = \frac{1}{3}$ with the initial condition being $[0; 0; 0; 0; 0; 0]$. With or without disturbances, the output consensus is achieved as shown in Fig. 6.

5.4 Multiple Damped Flexible Structures

Suppose that the *NI* systems are damped flexible structures as shown in Fig.2 of Lanzon and Petersen (2008),

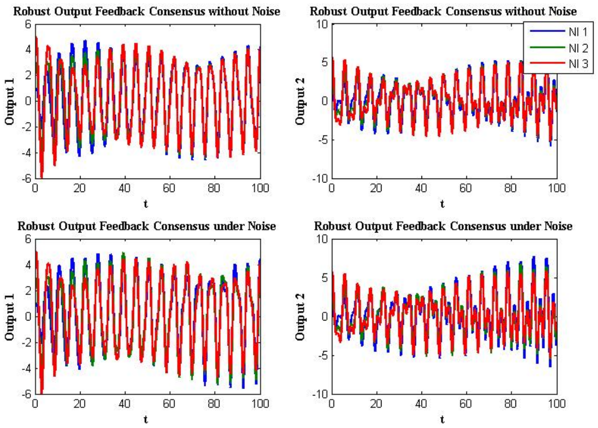

Fig. 6. Robust output feedback consensus for networked undamped flexible structures

where the parameters are exactly the same as Subsection 5.3 and additional damping coefficients are all 0.1. The same *SNI* controller can be adopted as in the previous subsection for simplicity. With or without disturbances, output consensus is achieved as shown in Fig. 7. Note that the outputs of the *NI* systems will reach consensus while the outputs will be damped to zero, which illustrates Corollary 14.

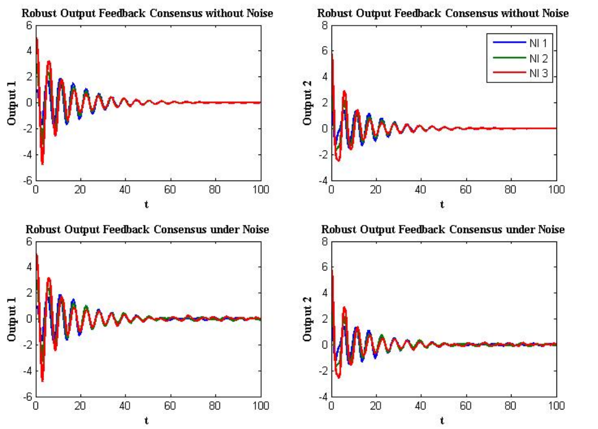

Fig. 7. Robust output feedback consensus for networked damped flexible structures

6. CONCLUSION REMARKS

NI systems include a wide range of LTI systems. As a consequence, the output feedback (as opposed to full state feedback) consensus problem of this class of systems is of interest. The advantage of using *NI* systems theory for solving the consensus problem is four-fold: (a) it only uses output feedback information as opposed to full-state feedback information; (b) it provides robustness guarantees; (c) it allows tuning of a whole class of *SNI* control laws; and (d) it bypasses traditional searches for Lyapunov Candidate functions. Moreover, a class of general output feedback consensus protocols (since a large class of *SNI* control laws is allowed) is available for the cooperative control of networked *NI* systems. The characterised convergence set also makes it possible to initialise the controller to achieve the desired final consensus target.

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