ON UNIQUENESS OF CENTRAL $\mathscr{H}_\infty$ CONTROLLERS IN THE CHAIN-SCATTERING FRAMEWORK

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Abstract: This paper studies the notion of central $\mathscr{H}_\infty$ controllers in the chain-scattering framework. Since in the chain-scattering framework the $\mathscr{H}_\infty$ control problem is equivalent to a factorisation problem that yields non-unique factors, it is important to somehow pin-down these factors so that the central controller is uniquely defined and corresponds to the central and minimum-entropy controller frequently discussed in the literature. In so doing, we device a procedure for the selection of a single uniquely identifiable $\mathscr{H}_\infty$ controller in the chain-scattering framework.

Keywords: $\mathscr{H}_\infty$-control, J-lossless factorisation, chain-scattering, central controller, minimum entropy controller

1. INTRODUCTION

In standard literature (Doyle *et al.*, 1989; Green and Limebeer, 1995; Zhou *et al.*, 1996), the set of all admissible $\mathscr{H}_\infty$ controllers is given by the set of all transfer function matrices $K = \mathcal{F}_l(M, S)$, where M is constructed from the plant state-space matrices and $S \in \mathscr{RH}_\infty$ satisfying $\|S\|_\infty < 1$ is a free parameter that characterises the set. Then, the central controller frequently discussed in the literature is given by $K_c = \mathcal{F}_l(M, 0)$. It has been shown in (Glover and Doyle, 1988; Doyle *et al.*, 1989) that K_c has the same McMillan degree as the plant P and in (Mustafa and Glover, 1988; Glover and Mustafa, 1989; Mustafa and Glover, 1990; Mustafa *et al.*, 1991) that K_c also minimises the entropy function. In fact, this central controller has some interesting interpretations and motivations in the literature. It thus seems natural that if for any reason we wish to select a single uniquely identifiable controller from the set of all admissible $\mathscr{H}_\infty$ controllers, the central (or minimum entropy) controller should be our natural choice.

Since their inception, $\mathscr{H}_\infty$ control problems have been amenable to a variety of solution techniques. In this paper, we shall study the notion of central controller in the chain-scattering approach to $\mathscr{H}_\infty$ control (Kimura, 1997) and we shall show how to select a single uniquely identifiable $\mathscr{H}_\infty$ controller from the admissible controller set in this framework. The chain-scattering operator-theoric framework possesses a distinct advantage over state-space based approaches in that it poses and solves the $\mathscr{H}_\infty$ control problem entirely in the frequency domain (i.e. (Kimura, 1997) shows that the normalised $\mathscr{H}_\infty$ control problem is equivalent to a J-lossless factorisation problem and that the set of all admissible controllers can be completely characterised in terms of one of the resulting factors). This entirely operator-theoric framework permits direct manipulation of the frequency domain symbols which may be very useful in some applications. For example, one may wish to consider changes in the frequency domain symbols that may result in changes in the McMillan degree of the symbols, as in (Bombois and Anderson, 2002; Lanzon *et al.*, 2003). These kind of manipulations would otherwise be more cumbersome and not easily cast in state-space descriptions. It is these kind of arguments that motivate us to study the notion of central controller in the chain-scattering framework.

It is however well known that factorisation problems often do not have unique solutions and hence the admissible controller set is characterised in terms of a non-unique factor in this chain-scattering framework. This is clearly undesirable as the centre of the admissible controller set (i.e. the central controller for the particular parametrisation considered) may be different for every reparametrisation. One has to thus pin-down the factors resulting from the J-lossless factorisation in order to ensure that the central controller is uniquely defined in this chain-scattering framework and to also ensure that it corresponds to the central and minimum-entropy controller so frequently discussed in the literature. This will be done by pinning down one of the factors in the J-lossless factorisation at infinite frequency. In so doing, we shall also demonstrate that the standard assumptions frequently adopted in the literature (Doyle *et al.*, 1989; Green and Limebeer, 1995; Zhou *et al.*, 1996) bury some interesting features of $\mathscr{H}_\infty$ control, particularly associated with the central and minimum-entropy controller. These features will also be exposed and discussed in this paper.

2. BACKGROUND MATERIAL

2.1 Chain-Scattering Representation of Plant

Consider a plant P with two kinds of inputs (w, u) and two kinds of outputs (z, y) represented as

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix},$$

where z represents the errors to be reduced [$\dim(z) = m$], y denotes the measured outputs [$\dim(y) = q$], w represents the exogenous signals [$\dim(w) = r$], and u denotes the control inputs [$\dim(u) = p$]. Let the state-space realisation of P be given by

$$P := \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A^P & B_1^P & B_2^P \\ \hline C_1^P & D_{11}^P & D_{12}^P \\ C_2^P & D_{21}^P & D_{22}^P \end{array}\right], \qquad (1)$$

and assume that it satisfies the following assumption:

Assumption (A1): $q \le r$, $p \le m$ and $\text{rank}[P_{21}(j\omega)] = q$, $\text{rank}[P_{12}(j\omega)] = p$ for all $\omega \in \mathbb{R} \cup \{\infty\}$.

This assumption is generally fulfilled in practice. If P_{21}^{-1} exists (i.e. if $r = q$), then the plant P can be alternatively represented by

$$\begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix},$$

where

$$G := \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \begin{bmatrix} P_{12} - P_{11}P_{21}^{-1}P_{22} & P_{11}P_{21}^{-1} \\ -P_{21}^{-1}P_{22} & P_{21}^{-1} \end{bmatrix}. \quad (2)$$

This type of representation is usually referred to as a *chain-scattering representation* of P. The transformation in equation (2) mapping P to G is denoted by

$$G := \text{CHAIN}(P)$$

and exists if P_{21} is invertible (Kimura, 1997).

Now, let the plant P or its chain-scattering equivalent G be controlled by a controller $u = Ky$ as shown in Figure 1. Then the closed-loop transfer function matrix T_{zw} mapping

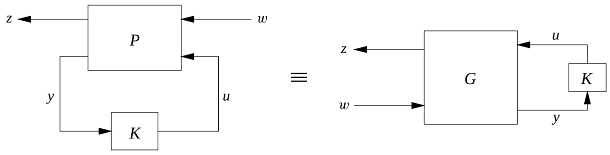

Fig. 1. Linear fractional and Homographic transformations

exogenous inputs w to errors z is given by

$$\begin{aligned} T_{zw} &= \mathcal{F}_l\,(P, K) := P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} \\ &= \text{HM}(G, K) := (G_{11}K + G_{12})(G_{21}K + G_{22})^{-1}, \end{aligned}$$

where $\mathcal{F}_l\,(\cdot,\cdot)$ denotes the “lower Linear Fractional Transformation” frequently used in control theory and $\text{HM}(\cdot,\cdot)$ denotes the “Homographic Transformation” frequently used in classical circuit theory.

2.2 Plant Augmentations

Problems where neither $r = q$ nor $m = p$ hold are harder because (a) the plant needs to be augmented in order to derive a chain-scattering representation, and (b) the results obtained need to be independent of the particular augmentation chosen. Such problems are usually referred to as four-block problems in the literature.

In four-block problems, the plant P is augmented by a fictitious output y' of dimension $(r - q)$ given by

$$y' = P_{21}' w + P_{22}' u \qquad (3)$$

to give

$$\begin{bmatrix} z \\ \begin{pmatrix} y \\ y' \end{pmatrix} \end{bmatrix} = \overbrace{\begin{bmatrix} P_{11} & P_{12} \\ \begin{pmatrix} P_{21} \\ P_{21}' \end{pmatrix} & \begin{pmatrix} P_{22} \\ P_{22}' \end{pmatrix} \end{bmatrix}}^{P_{aug}} \begin{bmatrix} w \\ u \end{bmatrix}. \qquad (4)$$

Assumption (A2): P_{21}' is such that $\text{rank}\left[\begin{pmatrix} P_{21}(j\omega) \\ P_{21}'(j\omega) \end{pmatrix}\right] = r$ for all $\omega \in \mathbb{R} \cup \{\infty\}$.

If assumption (A2) holds, a chain-scattering representation G of the augmented plant P_{aug} exists. Then, applying the control law $u = Ky$, the closed-loop transfer function matrix T_{zw} mapping exogenous inputs w to errors z is given by

$$T_{zw} = \text{HM}\big(G, \begin{bmatrix} K & 0 \end{bmatrix}\big)$$

and is depicted in Figure 2.

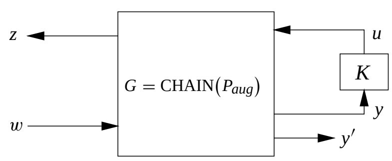

Fig. 2. Closed-loop system with output augmentation

2.3 J-Lossless Factorisations

Definition 1. J_{mr} denotes the signature matrix and is defined by $J_{mr} = \text{diag}(I_m, -I_r)$.

Definition 2. A rational matrix Θ is said to be (J_{mr}, J_{pr})-lossless if $\Theta^{\sim}(s) J_{mr} \Theta(s) = J_{pr} \quad \forall s \in \mathbb{C}$ and $\Theta^*(s) J_{mr} \Theta(s) \leq J_{pr}$ $\forall s \in \overline{\mathbb{C}}_+$. Here, $\Theta^{\sim}(s) := \Theta(-s)^T$ denotes the $\mathscr{L}_2$-adjoint of $\Theta(s)$ whereas $\Theta^*(s) := \Theta(\bar{s})^T$ denotes the complex conjugate transpose of $\Theta(s)$.

Definition 3. A rational matrix Π is said to be unimodular in $\mathscr{RH}_\infty$ if Π^{-1} exists and $\Pi, \Pi^{-1} \in \mathscr{RH}_\infty$.

The (J_{mr}, J_{pr})-lossless factorisation defined below is a generalisation of the well known inner-outer factorisation for stable systems and the well known spectral factorisation for positive hermitian systems.

Definition 4. The rational matrix $G \in \mathscr{R}^{(m+r)\times(p+r)}$ is said to have a (J_{mr}, J_{pr})-lossless factorisation if G is represented as the product $G = \Theta\Pi$, where Θ is (J_{mr}, J_{pr})-lossless and Π is unimodular in $\mathscr{RH}_\infty$.

Necessary and sufficient conditions for the existence of a (J_{mr}, J_{pr})-lossless factorisation of G and the construction of factors Θ and Π can be found in (Kimura, 1997). It should be clear that if factors Θ and Π exist, then they are not unique. In fact, any two solution pairs Θ_1, Π_1 and Θ_2, Π_2 to the (J_{mr}, J_{pr})-lossless factorisation of G must be related by

$$\Theta_2 = \Theta_1 \Psi^{-1} \text{ and } \Pi_2 = \Psi \Pi_1, \tag{5}$$

where Ψ is a real nonsingular matrix satisfying $\Psi^T J_{pr} \Psi = J_{pr}$.

2.4 Normalised $\mathscr{H}_\infty$ control problems

The following theorem reduces a normalised $\mathscr{H}_\infty$-control problem into a (J_{mr}, J_{pr})-lossless factorisation problem.

Theorem 1. (Kimura, 1997) Suppose that a plant $P \in \mathscr{RL}_\infty$ given by equation (1) satisfies assumption (A1) and is such that $q < r$. Then, the normalised $\mathscr{H}_\infty$ control problem is solvable for P if and only if there exists an output augmentation (3) with $P'_{21}, P'_{22} \in \mathscr{RL}_\infty$ such that the augmented plant P_{aug} in equation (4) satisfies assumption (A2) and $G = \text{CHAIN}\big(P_{aug}\big)$ has a (J_{mr}, J_{pr})-lossless factorisation

$$G = \Theta\Pi$$

with Π of the form

$$\Pi = \begin{bmatrix} \Pi_a & 0 \\ \Pi_{b1} & \Pi_{b2} \end{bmatrix} \begin{matrix} \}p+q \\ \}r-q \end{matrix}. \quad \begin{matrix} \underbrace{}_{p+q} & \underbrace{}_{r-q} \end{matrix} \tag{6}$$

In that case, K is an admissible controller if and only if

$$K = \text{HM}\Big(\Pi_a^{-1}, S\Big)$$

for an $S \in \mathscr{RH}_\infty$ satisfying $\|S\|_\infty < 1$.

As indicated earlier in this section, the particular choice of the plant augmentation (3) ends up playing no role in the final solution of the normalised $\mathscr{H}_\infty$ control problem. This can be seen from the following argument: Let the chain-scattering plant $G = \text{CHAIN}\big(P_{aug}\big)$ be written as

$$G = \begin{bmatrix} P_{12} & P_{11} \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ \begin{pmatrix} P_{22} \\ P'_{22} \end{pmatrix} & \begin{pmatrix} P_{21} \\ P'_{21} \end{pmatrix} \end{bmatrix}^{-1}.$$

If the normalised $\mathscr{H}_\infty$ control problem is solvable, then there exists a Π of the form of equation (6) such that $G^{\sim} J_{mr} G = \Pi^{\sim} J_{pr} \Pi$. We must show that Π_a (the only sub-block of Π that is used in characterising the controller set) is independent of the particular augmentation chosen. Towards this end, note that

$$\begin{aligned}
&(G^{\sim} J_{mr} G)^{-1} \\
&= \begin{bmatrix} I & 0 \\ P_{22} & P_{21} \\ P'_{22} & P'_{21} \end{bmatrix} \begin{bmatrix} P_{12}^{\sim} P_{12} & P_{12}^{\sim} P_{11} \\ P_{11}^{\sim} P_{12} & P_{11}^{\sim} P_{11} - I \end{bmatrix}^{-1} \begin{bmatrix} I & P_{22}^{\sim} & P_{22}^{\prime\sim} \\ 0 & P_{21}^{\sim} & P_{21}^{\prime\sim} \end{bmatrix} \\
&= \left[\begin{array}{cc|c} \surd & \surd & \times \\ \surd & \surd & \times \\ \hline \times & \times & \times \end{array}\right].
\end{aligned}$$

Here, $\surd$ denotes terms that do not depend on the augmentation and $\times$ denotes terms that do depend of the particular augmentation chosen. Since

$$(G^{\sim} J_{mr} G)^{-1} = \Pi^{-1} J_{pr} \Pi^{-\sim} = \begin{bmatrix} \Pi_a^{-1} J_{pq} \Pi_a^{-\sim} & \blacklozenge \\ \blacklozenge & \blacklozenge \end{bmatrix},$$

it follows that Π_a is independent of the particular augmentation chosen.

Furthermore, since Π in Theorem 1 is required to be of the lower triangular form of equation (6), the real nonsingular matrix Ψ that characterises the non-uniqueness in the factors Θ and Π satisfies further structural properties besides $\Psi^T J_{pr} \Psi = J_{pr}$. It is in fact easy to show that Ψ has the following block diagonal form

$$\Psi = \begin{bmatrix} \Psi_a & 0 \\ 0 & \Psi_b \end{bmatrix} \tag{7}$$

with $\Psi_a \in \mathbb{R}^{(p+q)\times(p+q)}$ satisfying $\Psi_a^T J_{pq} \Psi_a = J_{pq}$ and $\Psi_b \in \mathbb{R}^{(r-q)\times(r-q)}$ satisfying $\Psi_b^T \Psi_b = I_{(r-q)}$. Note furthermore that Ψ_a expresses the non-uniqueness in the unimodular matrix Π_a.

3. QUESTIONS OF INTEREST

Since the unimodular matrix Π_a is unique up to left multiplication by a constant J-unitary real nonsingular matrix, the questions of interest that will be addressed in this paper are:

A. Is $\text{HM}\big(\Pi_a^{-1}, 0\big)$ a single controller or is there a family of such controllers obtained by considering all possible Π_a in the factorisation of G?

B. If there is a family of such controllers:
   a. Does the central controller advanced by the literature belong to this family?

b. Do all such controllers possess the same properties as the central controller given in the literature? For instance, do they all minimise the entropy function and are they all strictly proper when certain conditions are fulfilled?
c. What properties need to be enforced in order to pinpoint just one member (i.e. select a single uniquely identifiable member) of this family?

4. ADDRESSING THE POSED QUESTIONS IN A FOUR-BLOCK SETTING

4.1 *Reparametrisation of controller set*

The first result presented here states that the extra freedom associated with the non-uniqueness of Π simply reparametrises the same controller set.

Lemma 2. Given any two (J_{mr}, J_{pr})-lossless factorisations of $G = \Theta_1\Pi_1 = \Theta_2\Pi_2$ with Π_1 and Π_2 of the form in equation (6), the following two sets are identical:

$$\left\{\mathrm{HM}\left(\Pi_{a,1}^{-1}, S\right) : S \in \mathscr{RH}_\infty, \|S\|_\infty < 1\right\} \equiv \left\{\mathrm{HM}\left(\Pi_{a,2}^{-1}, S\right) : S \in \mathscr{RH}_\infty, \|S\|_\infty < 1\right\}.$$

4.2 *Uniqueness of strictly proper central controllers*

As pointed out at the end of Section 2.4, the unimodular matrix Π_a is unique up to left multiplication by a constant real nonsingular matrix Ψ_a that satisfies $\Psi_a^T J_{pq} \Psi_a = J_{pq}$. Consequently, there is evidently a family of central controllers described by $\mathrm{HM}\left(\Pi_a^{-1}, 0\right)$ for all Π_a arising in the (J_{mr}, J_{pr})-lossless factorisation of G.

The following theorem gives a condition under which the central controller is uniquely defined in the chain-scattering framework. The corresponding existence question will be discussed in the next subsection.

Theorem 3. Suppose that a plant $P \in \mathscr{RL}_\infty$ given by equation (1) satisfies assumption (A1) and is such that $q < r$. Suppose furthermore that there exists an output augmentation (3) with $P'_{21}, P'_{22} \in \mathscr{RL}_\infty$ such that the augmented plant P_{aug} in equation (4) satisfies assumption (A2) and $G = \mathrm{CHAIN}\left(P_{aug}\right)$ admits a (J_{mr}, J_{pr})-lossless factorisation

$$G = \Theta\Pi$$

with Π of the form

$$\Pi = \underbrace{\begin{bmatrix} \Pi_a & 0 \\ \Pi_{b1} & \Pi_{b2} \end{bmatrix}}_{p+q \quad r-q} \begin{matrix} \}p+q \\ \}r-q \end{matrix}.$$

Then the central controller

$$K_c = \mathrm{HM}\left(\Pi_a^{-1}, 0\right)$$

is uniquely defined if K_c is strictly proper.

Consequently, if there exists a strictly proper central controller, then (with the strict properness constraint imposed) it is unique. It should be pointed out that if we impose the same simplifying assumptions as in the literature (Doyle *et al.*, 1989; Green and Limebeer, 1995; Zhou *et al.*, 1996), then the strictly proper central controller discussed in this subsection (which was proved to be unique here) is identical to the central controller given in the literature. However, the simplifying assumptions of the literature bury the existential questions as there always exists a strictly proper central controller under the assumptions, as will be shown in the next subsection.

4.3 *Existence of strictly proper central controllers*

The next lemma gives a necessary and sufficient condition for a central controller to be strictly proper.

Lemma 4. Given a (J_{mr}, J_{pr})-lossless factorisation of $G = \Theta\Pi$ with Π of the form in equation (6), then

$$K_c := \mathrm{HM}\left(\Pi_a^{-1}, 0\right) \text{ is strictly proper}$$

if and only if

$$E := \Pi(j\infty) = \begin{bmatrix} \begin{pmatrix} E_{a11} & 0 \\ E_{a21} & E_{a22} \end{pmatrix} & 0 \\ E_{b1} & E_{b2} \end{bmatrix} \quad (8)$$

with $E_{a11} \in \mathbb{R}^{p\times p}$, $E_{a22} \in \mathbb{R}^{q\times q}$ and $E_{b2} \in \mathbb{R}^{(r-q)\times(r-q)}$.

There are plants P with $\overline{\sigma}(D_{11}^P) \geq 1$ for which the normalised $\mathscr{H}_\infty$ control problem is solvable. For such plants, there cannot exist an admissible strictly proper controller. The following theorem gives a necessary and sufficient condition for the existence of a lower triangular matrix E. As we now well know from Lemma 4, the existence of such a lower triangular matrix E is intimately related to the existence of a strictly proper central controller.

Theorem 5. Let the suppositions of Theorem 3 hold and define $D := G(j\infty)$. Then there exists a unique real nonsingular matrix E satisfying $D^T J_{mr} D = E^T J_{pr} E$ of the form

$$E = \begin{bmatrix} \begin{pmatrix} E_{a11} & 0 \\ E_{a21} & E_{a22} \end{pmatrix} & 0 \\ E_{b1} & E_{b2} \end{bmatrix}$$

with $0 < E_{a11} \in \mathbb{R}^{p\times p}$, $0 < E_{a22} \in \mathbb{R}^{q\times q}$ and $0 < E_{b2} \in \mathbb{R}^{(r-q)\times(r-q)}$ if and only if $\overline{\sigma}(D_{11}^P) < 1$.

Any lower triangular matrix E would have been sufficient in the above theorem to guarantee the existence of a strictly proper central controller K_c. However, it should be noticed that even though K_c is unique when it is strictly proper, the unimodular (in $\mathscr{RH}_\infty$) transfer function matrix Π is not uniquely defined. In some particular situations when explicit use of Π is made, it is desirable to have a unique selection of Π. An example of this situation is (Bombois and Anderson, 2002; Lanzon *et al.*, 2003), where the authors analyse how small changes in weights map to changes in Π and subsequently to changes in K_c. In this case, Π has to be selected always in the same way. This was enforced in the above theorem by selecting a unique lower triangular matrix E.

4.4 Minimum entropy and central controllers

Let T be a transfer function matrix such that $\|T\|_\infty < \gamma$. Then the entropy of $T(s)$ is defined by

$$I(T,\gamma) := -\frac{\gamma^2}{2\pi}\int_{-\infty}^{\infty}\sum_i \ln\left|1-\gamma^{-2}\sigma_i(T(j\omega))^2\right| d\omega$$

where $\sigma_i(T(j\omega))$ is the i-th singular value of $T(j\omega)$.

The entropy function $I(T,\gamma)$ has been studied in great detail (Glover and Mustafa, 1989; Mustafa and Glover, 1990; Mustafa *et al.*, 1991) in the late 1980s and early 1990s. It is not difficult to see that the entropy function $I(T,\gamma)$ is finite if and only if $T(j\infty)=0$. Letting $T(s)$ be our closed-loop transfer function matrix $T_{zw}(s)$ and $\gamma=1$ for a normalised $\mathscr{H}_\infty$ control problem, we have $I(T_{zw},1)$ is finite if and only if $T_{zw}(j\infty)=0$. Since the simplifying assumptions in the literature (Doyle *et al.*, 1989; Green and Limebeer, 1995; Zhou *et al.*, 1996) always require $D_{11}^P = 0$, we know from the preceeding discussion that in this situation there always exists a strictly proper central controller that is uniquely defined in this chain-scattering framework. This controller obviously achieves $T_{zw}(j\infty)=0$ and hence finite entropy. It was in fact shown in (Glover and Mustafa, 1989; Mustafa and Glover, 1990; Mustafa *et al.*, 1991) that this unique strictly proper central controller also minimises the value of the entropy function. For this reason, when the simplifying assumptions of the literature are enforced, the central controller is often also called the minimum entropy controller.

The question that immediately arises is: Would the central controller (in the sense of this paper in the chain-scattering framework considered) still be the same as the minimum entropy controller if $D_{11}^P = 0$ is not assumed and if we are allowed to choose the characterisation of the set of all admissible controllers as we desire? In general, the answer to this question is no (again it is emphasised that the definition of central controller is taken in the sense of this paper). This is illustrated by the following example.

Example: Consider a plant P with $0 \neq \overline{\sigma}(D_{11}^P) < 1$ for which the normalised $\mathscr{H}_\infty$ control problem is solvable. Assume that there exists an admissible controller that makes the closed-loop transfer function matrix T_{zw} strictly proper (i.e. assume that there exists $D^K \in \mathbb{R}^{p\times q}$ such that $D_{11}^P + D_{12}^P D^K(I - D_{22}^P D^K)^{-1}D_{21}^P = 0$). Then for this controller, we get finite entropy. Thus the minimum entropy controller will certainly achieve finite entropy.

However, since $\overline{\sigma}(D_{11}^P) < 1$, we know from Theorems 3 and 5 that there always exists a unique strictly proper central controller. With this controller, T_{zw} is clearly not strictly proper and hence we get infinite entropy. Consequently, this unique strictly proper central controller does not minimise the entropy function in this situation since there is another admissible controller which achieves a smaller value of entropy. ♠

A loop-shifting argument will be adopted in the next subsection to transform the original problem into one where there is correspondence between the unique strictly proper central controller and the minimum entropy controller (as in (Mustafa and Glover, 1988; Glover and Mustafa, 1989; Mustafa and Glover, 1990; Mustafa *et al.*, 1991) through the application of the simplifying assumptions). This will help us to select a single uniquely identifiable controller from the admissible controller set.

4.5 A loop-shifting argument to select a unique controller

Care must be exercised in the selection of a unique E satisfying $D^T J_{mr} D = E^T J_{pr} E$ with $D = G(j\infty)$ as this will uniquely determine the central controller in this chain-scattering setting and the properties associated with this central controller. Furthermore, in an earlier subsection, it was also shown that if the unique selection of E is obtained by requiring some special property on the central controller (such as strict-properness), then there are admissible situations that do not allow this kind of unique selection of E.

In this subsection, we will construct a unique E, show that this unique E always exists (provided the normalised $\mathscr{H}_\infty$ control problem is solvable), and show that it also reduces to the unique lower triangular matrix E of Theorem 5 when $D_{11}^P = 0$. The derivation of such a unique E relies on the following loop-shifting argument. The reader is referred to (Safonov and Limebeer, 1988; Safonov *et al.*, 1989; Green and Limebeer, 1995) for extensive coverage of this topic.

Consider the feedback interconnection of Figure 3. The

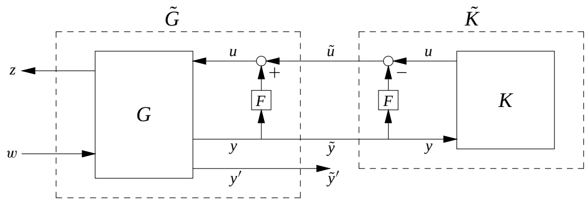

Fig. 3. Loop-shifting transformation

process of loop-shifting can be conceptually viewed as extracting the controller gain at infinite frequency from the controller and putting it into the plant. This is done through the constant gain matrix F. When this is done, the original interconnection of G and K is replaced with an equivalent interconnection of $\tilde{G}$ and $\tilde{K}$. The relations between the original systems and the loop-shifted systems are given below:

$$\tilde{G}(s) = G(s)\begin{bmatrix}\begin{pmatrix} I_p & F \\ 0 & I_q\end{pmatrix} & 0 \\ 0 & I_{(r-q)}\end{bmatrix}$$

and $$\tilde{K}(s) = \mathrm{HM}\left(\begin{bmatrix} I_p & -F \\ 0 & I_r\end{bmatrix}, K(s)\right) = K(s) - F.$$

In the context of this paper, F will be chosen so as to minimise $\overline{\sigma}(\mathrm{HM}(D, F))$, where $D = G(j\infty)$. Note that $\mathrm{HM}(D, F)$ corresponds to the gain at infinite frequency of the transfer function from w to z. Since the normalised $\mathscr{H}_\infty$ control problem is assumed to be solvable, it will always be possible to select an F such that $\overline{\sigma}(\mathrm{HM}(D, F)) < 1$. Then, applying Theorem 5 on the loop-shifted plant $\tilde{G}$, we see that there exists a unique lower triangular matrix $\tilde{E}$ that satisfies $\tilde{D}^T J_{mr} \tilde{D} = \tilde{E}^T J_{pr} \tilde{E}$ where $\tilde{D} = \tilde{G}(j\infty)$. Consequently, the matrix E that needs to be selected will be composed of the unique lower triangular matrix $\tilde{E}$ and the loop-shifting transformation.

The matrix E is only unique after we fix (a) the choice of plant augmentation, and (b) in the case when finite minimum entropy is not possible (i.e. $\min_F \overline{\sigma}(\mathrm{HM}(D, F)) \neq 0$), the choice of the minimising matrix F. The sub-block E_a in the matrix E (i.e. the only sub-block of interest in constructing the controller set) is however always independent of the choice of plant augmentation. It is also important to point out that if it is possible to achieve finite minimum entropy with an admissible controller, then $\min_F \overline{\sigma}(\mathrm{HM}(D, F)) = 0$ and hence the chosen minimising F is unique (Green and Limebeer, 1995). On the other hand, if it is not possible to achieve finite minimum entropy with an admissible controller, then $0 < \min_F \overline{\sigma}(\mathrm{HM}(D, F)) < 1$ and hence there is a set of matrices F that minimise the quantity $\overline{\sigma}(\mathrm{HM}(D, F))$. We simply pick a single uniquely identifiable member of this set for ease of selection. All this argument is captured by the following theorem.

Theorem 6. Let the suppositions of Theorem 3 hold and let $Q = \arg\min_Q \overline{\sigma}(D_{11}^P + D_{12}^P Q D_{21}^P) \in \mathbb{R}^{p\times q}$. Furthermore, define $D := G(j\infty)$. Then there exists a unique real nonsingular matrix E satisfying $D^T J_{mr} D = E^T J_{pr} E$ of the form

$$E = \begin{bmatrix} \begin{pmatrix} \tilde{E}_{a11} & 0 \\ \tilde{E}_{a21} & \tilde{E}_{a22} \end{pmatrix} & 0 \\ \tilde{E}_{b1} & \tilde{E}_{b2} \end{bmatrix} \begin{bmatrix} \begin{pmatrix} I_p & -F \\ 0 & I_q \end{pmatrix} & 0 \\ 0 & I_{(r-q)} \end{bmatrix} \quad (9)$$

with $0 < \tilde{E}_{a11} \in \mathbb{R}^{p\times p}$, $0 < \tilde{E}_{a22} \in \mathbb{R}^{q\times q}$, $0 < \tilde{E}_{b2} \in \mathbb{R}^{(r-q)\times(r-q)}$ and

$$F = \mathrm{HM}\left(\begin{bmatrix} I_p & 0 \\ D_{22}^P & I_q \end{bmatrix}, Q \right).$$

Note that E in equation (9) reduces to the lower triangular matrix E of Theorem 5 when $D_{11}^P = 0$. Furthermore, with E selected as in Theorem 6 (which is always possible whenever the normalised $\mathscr{H}_\infty$ control problem is solvable), the notions of minimum entropy controller and central control (in the sense of this paper) always coincide as in (Mustafa and Glover, 1988; Glover and Mustafa, 1989; Mustafa and Glover, 1990; Mustafa *et al.*, 1991), as one would expect and desire.

5. CONCLUSIONS

It is well known that in the chain-scattering framework, the $\mathscr{H}_\infty$ control problem can be solved via a J-lossless factorisation and that the admissible controller set is characterised in terms of one of the resulting factors. Since the resulting factors are not uniquely defined, the centre of the parametrised set of admissible controllers (i.e. the central controller for the admissible controller set considered) is also not uniquely defined in this chain-scattering framework. This presents a problem if we wish to pick a single uniquely defined controller from the admissible controller set in the chain-scattering framework.

In this paper, we show how to pin-down the non-unique factors resulting from the J-lossless factorisation, thereby ensuring that there is only one way in which the admissible controller set can be characterised and in turn guaranteeing that the central controller (corresponding to the centre of this admissible controller set) is (a) uniquely defined, and (b) corresponds to the central and minimum entropy controller frequently discussed in the literature.

In the process of ensuring that the centre of the parametrised set of admissible controllers in the chain-scattering framework corresponds to the central and minimum entropy controller frequently discussed in the literature, we discuss and uncover a number of properties associated with the central controller that are buried when $D_{11}^P = 0$ is assumed.

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